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On Narrow Hexagonal Graphs with a 3-Homogeneous Suborbit*

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Abstract. A connected graph of girth $m \geq 3$ is called a polygonal graph if it contains a set of m-gons such that every path of length two is contained in a unique element of the set. In this paper we investigate polygonal graphs of girth 6 or more having automorphism groups which are transitive on the vertices and such that the vertex stabilizers are 3-homogeneous on adjacent vertices. We previously showed that the study of such graphs divides naturally into a number of substantial subcases. Here we analyze one of these cases and characterize the k-valent polygonal graphs of girth 6 which have automorphism groups transitive on vertices, which preserve the set of special hexagons, and which have a suborbit of size $k - 1$ at distance three from a given vertex.

Keywords: polygonal graph, automorphism group, 3-homogeneous suborbit

1. Introduction

A connected graph Γ of girth $m \geq 3$ is called a *polygonal graph* (or *m-gon graph*) if it contains a set Π of m-gons (i.e. simple cycles of length m) such that every path of length two of Γ is contained in a unique element of Π. Such a graph is easily shown to be regular (see, for example, [4]) and, if k denotes its valency, then it is also called a (k, m) *polygonal graph*. The polygonal graph Γ is called *strict* if Π is the set of *all* m-gons of Γ. In this paper we will be dealing mainly with polygonal graphs of girth 6 and we shall refer to these as *hexagonal graphs*. In general, they will not be strict hexagonal graphs. (Note that elsewhere (for example references [4, 5, 7, 8]), what we are calling hexagonal graphs were formerly referred to by the awkward name of "6-gon graphs" and "6-gonal graphs". We believe that this change in name is well-justified.)

It was the scarcity of examples of polygonal graphs with large girth and large valency which motivated the present investigation. In [8] we made a careful theoretical analysis of

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vertex-transitive (k, m) polygonal graphs with $k \geq 4$ and $m \geq 6$ such that the automorphism group preserves the set Π and, on the neighborhood of a vertex, induces either a 3-transitive group or a certain family of projective groups. We identified several types of such graphs according to the value of m and the action of the stabilizer, G_x, of a vertex x, on the set of vertices at distance three from x. Some of the types might be regarded as the "standard" types for the relevant value of m.

However, one type of hexagonal graph Γ which we identified, seemed especially interesting. In this type, the set of vertices antipodal to x in the hexagons (6-gons) of Π containing x, formed a G_x-orbit $\Delta(x)$ of size $k-1$, that is *less than* the valency k of Γ. This type is case (b)(iii) of Theorem 1.1 of [8], and for the reader's convenience, the statement of this theorem has been repeated in Section 2. We call such hexagonal graphs *narrow*, and it is the purpose of this paper to classify the narrow, hexagonal graphs of valency $k \geq 4$ admitting a vertex-transitive group G of automorphisms preserving the set Π of hexagons, such that G_x is 3-homogeneous (that is, transitive on 3-element subsets) on the neighbors of x. It is interesting to note that two of the examples (with $k = 4$ and 8, having G_x 3-transitive on the neighbors of x) are the smallest members of two infinite families of hexagonal graphs constructed and studied in [8]. Also two of the other examples (with $k = 8$ and 32, with G_x not 3-transitive on the neighbors of x) are the second and fourth members of an infinite family of narrow hexagonal graphs constructed in [6].

We denote by $H : K$ a semi-direct product of a group H by a group K. Our main theorem is the following.

Theorem 1 *Let Γ be a narrow hexagonal graph of valency $k \geq 4$ admitting a vertex-transitive group G of automorphisms preserving the set Π of hexagons, such that, for a vertex x of Γ, G_x is 3-homogeneous on $\Gamma(x)$. Then either*
(a) $k = 4$, $\Gamma \cong PP(4)$, $G_x, \cong \mathrm{AGL}(2, 2)$, *and* $G \cong \mathrm{S}_5 \times \mathrm{S}_3$, *or*
(b) $k = 8$, $\Gamma \cong SP(8)$, $G_x \cong \mathrm{AGL}(3, 2)$, *and* $G \cong \mathrm{S}_9$, *or*
(c) $k = 6$, Γ *is known* $G_x \cong \mathrm{PGL}(2, 5)$ *and* $G \cong \mathrm{S}_7 \times \mathrm{Z}_2$, *or*
(d) $k = 8$, $\Gamma \cong AP(8)$, *and either* $G_x \cong \mathrm{AGL}(1, 8)$ *and* $G \cong \mathrm{SL}(2, 8) \times \mathrm{D}_7$, *or* $G_x \cong \mathrm{A\Gamma L}(1, 8)$ *and* $G \cong (\mathrm{SL}(2, 8) \times \mathrm{D}_7) : \mathrm{Z}_3$, *or*
(e) $k = 12$, Γ *is known*, $G_x \cong \mathrm{PSL}(2, 11)$, *and* $G \cong \mathrm{J}_1 \times \mathrm{Z}_2$, *or*
(f) $k = 32$, $\Gamma \cong AP(32)$, $G_x \cong \mathrm{A\Gamma L}(1, 32)$ *and* $G \cong (\mathrm{SL}(2, 32) \times \mathrm{D}_{31}) : \mathrm{Z}_5$.

Here, the *projective-polygonal* graphs $PP(2^{2n})$, for $n \geq 1$, and the *symmetric-polygonal* graphs $SP(2^d)$, for $d \geq 3$, are as constructed and investigated in [8]. In particular, it was shown in [8, Propositions 3.1(b) and 3.2(b)] that $PP(4)$ and $SP(8)$ satisfy the conditions of the theorem. A summary description of these two graphs is given below in Section 2.

As for conclusion (*e*), it is well-known (see, for example [2]) that the graph Δ of degree 12 defined by the rank 5 action of Janko's first simple group J_1 on the cosets of a subgroup isomorphic to $\mathrm{PSL}(2, 11)$, has girth 5 and diameter 3, with the points at distance 3 separating into a suborbit of subdegree 11 and one with subdegree 110. We now "double" Δ as follows. Let $\mathrm{C} = \{0, 1\}$ and let Γ be the graph having vertex set $\mathrm{V}(\Gamma) = \mathrm{V}(\Delta) \times \mathrm{C}$, with two vertices (v, a) and (w, b) adjacent if and only if v is adjacent to w in Δ and $a \neq b$. It is not difficult to show that Γ has girth 6, diameter 5, and is a narrow hexagonal graph admitting $\mathrm{J}_1 \times \mathrm{Z}_2$ as automorphism group.

Regarding conclusion (c), it is easily verified that there is a graph Δ of degree 6 defined by the rank 4 action (with subdegrees $1, 6, 30, 5$) of A_7 on the cosets of a subgroup isomorphic to PSL(2, 5). We proceed to "double" this graph as in the preceding paragraph to obtain Γ, a narrow hexagonal graph of diameter 5. Its full automorphism group is $S_7 \times Z_2$. This group contains two subgroups isomorphic to $A_7 \times Z_2$ and S_7, which have the transitivity properties of Theorem 2 below (see Theorem 2, conclusion $d(i)$).

Finally, in conclusions (d) and (f), the *affine polygonal* graphs $AP(2^d)$, for $d \geq 2$, are as constructed and investigated in [6]. Note that $AP(4) \cong PP(4)$. A summary description of these graphs is also given in Section 2.

Theorem 1 follows from Theorem 2 and Theorem 3, stated below.

Theorem 2 *Let Γ be a narrow hexagonal graph of valency $k \geq 4$ admitting a vertex-transitive group G of automorphisms preserving the set Π of hexagons, such that for a vertex x of Γ, either G_x is 3-transitive on $\Gamma(x)$, or $k - 1$ is a prime power greater than 3 and $G_x^{\Gamma(x)} \geq \mathrm{PSL}(2, k-1)$. Then either*

(a) $k = 4$, $\Gamma \cong PP(4)$, $G_x \cong \mathrm{AGL}(2, 2)$, *and* $G \cong S_5 \times S_3$, *or*

(b) $k = 8$, $\Gamma \cong SP(8)$, $G_x \cong \mathrm{AGL}(3, 2)$, *and* $G \cong S_9$, *or*

(c) $k = 12$, Γ *is known*, $G_x \cong \mathrm{PSL}(2, 11)$, *and* $G \cong J_1 \times Z_2$, *or*

(d) $k = 6$ *and* Γ *is known.*

Further, in case (d), we have two subcases: either (i) $G_x \cong \mathrm{PSL}(2, 5)$, and if y is a vertex adjacent to x in Γ, then either $G_{\{x,y\}} \cong Z_5 : Z_4$ and $G \cong A_7 \times Z_2$, or $G_{\{x,y\}} \cong Z_2 \times D_5$ and $G \cong S_7$, or (ii) $G_x \cong \mathrm{PGL}(2, 5)$ and $G \cong S_7 \times Z_2$.

The permutation groups which are 3-homogeneous but not 3-transitive were determined in [3]. See Section 4 of this paper.

Theorem 3 *Let Γ be a narrow hexagonal graph of valency $k \geq 4$ admitting a vertex-transitive group G of automorphisms preserving the set Π of hexagons, such that for a vertex x of Γ, G_x is 3-homogeneous but not 3-transitive on $\Gamma(x)$. Then either*

(a) $k = 8$, $\Gamma \cong AP(8)$, *and either* $G_x \cong \mathrm{AGL}(1, 8)$ *and* $G \cong \mathrm{SL}(2, 8) \times D_7$, *or* $G_x \cong \mathrm{A\Gamma L}(1, 8)$ *and* $G \cong (\mathrm{SL}(2, 8) \times D_7) : Z_3$, *or*

(b) $k = 12$, Γ *is known*, $G_x \cong \mathrm{PSL}(2, 11)$, *and* $G \cong J_1 \times Z_2$, *or*

(c) $k = 32$, $\Gamma \cong AP(32)$, $G_x \cong \mathrm{A\Gamma L}(1, 32)$ *and* $G \cong (\mathrm{SL}(2, 32) \times D_{31}) : Z_5$.

In Section 2 we collect together the definitions and preliminary results needed in the course of the proofs of the Theorems 2 and 3. Also in this section we give the definitions and brief descriptions of the constructions of the graphs $PP(4)$, $SP(8)$, $AP(8)$ and $AP(32)$.

Section 3 is devoted entirely to the proof of Theorem 2 and Section 4 to the proof of Theorem 3. The proofs use coset enumeration for which we rely heavily on the algebraic computation system MAGMA [1]. We first use case (b)(iii) of Theorem 1.1 of [8] (see Proposition 2.1 of Section 2) to limit the possibilities. We then find a presentation for a vertex stabilizer in the group G and, using the "narrow" hypothesis, find a presentation for the group itself. In some cases the presentation is, in some cases, quite huge, with many dozens of relations and no doubt many of these relations are redundant. However, we have not attempted to eliminate these redundancies as they often help with an understanding of how the relations are derived from the graph.

In some cases, different presentations are found for the same group leading to different presentations for the vertex-stabilizer, or (as in Theorem 2(d)) presentations for groups and subgroups are found. In these cases we used MAGMA to determine whether or not the graphs obtained were isomorphic. For example, during the course of proving Theorem 2(d), there are some choices of relations. One choice (with $G_x \cong \mathrm{PSL}(2, 5)$ and $G_{\{x,y\}} \cong \mathrm{Z}_2 \times \mathrm{D}_5$) defines the group PGL(2, 11) in its rank 4 action of degree 22 (with subdegrees 1, 6, 10, 5), leading to a graph of diameter 3. This graph has girth 4 so is not the hexagonal graph of the theorem but is, however, an example of what is called a *near* hexagonal graph (see [5]).

It also should be observed that in case (b)(iii) of Proposition 2.1, there is also the possibility that $k = 8$ and $G_x \cong \mathrm{PGL}(2, 7)$. In our proof we shall see that in this case, the generators and relations lead to the group S_7 acting on a graph of degree 30, diameter 3 and girth 4 (this is a rank 4 action with subdegrees 1, 8, 14, 7). Again this is an example of a near hexagonal graph. (Note also that if we try the "doubling" procedure on this graph we just get two disjoint copies of the graph.) So we have the following.

Corollary 1 *There is no hexagonal graph satisfying the hypotheses of Theorem* 1 *with* $k = 8$ *and* $G_x \cong \mathrm{PGL}(2, 7)$.

Some computer calculations using MAGMA were used to confirm our presentations for the vertex stabilizers and MAGMA was also used to find non-trivial elements in the centers of some of the groups finally constructed. Also, as mentioned above, we were sometimes led to a number of possible graphs and we used MAGMA's graph isomorphism algorithm to confirm they are, or are not, isomorphic to one another.

An entirely computer-free but somewhat lengthy, purely group-theoretic proof of parts (a) and (b) of Theorem 2, under slightly different hypotheses, is given in [7, Theorem A], available from the authors. There, the case when $k = 16$ and $G_x \cong 2^4 \cdot \mathrm{A}_7$ was left only partially resolved [7, Theorem B] and the other parts covered by the Theorem 2 of this paper were left as "open problems."

2. Definitions and preliminary results

We only consider finite groups and graphs. Since our notation and terminology is standard for the most part, we only note below those definitions and terms used frequently throughout the paper.

In particular, if Γ is a graph and $x \in \mathrm{V}(\Gamma)$, the vertices of Γ, then $\Gamma_n(x)$ will denote the set of vertices at distance n from x, and we use just $\Gamma(x)$ for $\Gamma_1(x)$, the set of vertices adjacent to x. An *s-chain*, or *path of length s*, is a sequence $(x_0, x_1, \ldots, x_s)$ of vertices, each adjacent to its successor, and all distinct save possibly for $x_0 = x_s$. (We note that this is different from the usual definition of an s-arc.) If in fact we do have $x_0 = x_s$, with $s \geq 3$, then the path is called a *simple cycle of length s*, or an *s-gon*. The *girth* of the graph Γ is the length of the smallest s-gon in Γ. As usual, $\mathrm{Aut}(\Gamma)$ will denote the *automorphism group* of Γ. The definition of a polygonal graph was given in the first paragraph of the paper.

Suppose G is a group of automorphisms of an undirected, connected graph Γ, and let $x \in \mathrm{V}(\Gamma)$ and $g \in G$ with $x^g \in \Gamma(x)$. Suppose G is transitive on $\mathrm{V}(\Gamma)$ and that G_x, the stabilizer in G of x, is transitive on $\Gamma(x)$. Then $G = \langle G_x, g\rangle$, the group generated by G_x and g.

Since Theorem 1.1 of [8] forms an important starting off point of our proof, its statement is given in its entirety below.

Proposition 2.1 (*Theorem* 1.1 *of* [8]) *Let Γ be a polygonal graph with vertex set Ω, valency $k \geq 4$, girth $m \geq 6$, and Π the set of "special" m-gons of Γ. Let $x \in \Omega$, and set $\Gamma(x) = \{1, 2, \ldots, k\}$. Suppose that there is a group G of automorphisms of Γ preserving the set Π of m-gons of Γ, such that G is transitive on Ω and either G_x is 3-transitive on $\Gamma(x)$, or $k-1$ is a prime power greater than 3, and $G_x^{\Gamma(x)} \geq \mathrm{PSL}(2, k-1)$. Then we have the following.*

(a) *The group G_x is transitive on $\Gamma_2(x)$ and we may identify $\Gamma_2(x)$ with the set of ordered pairs $\{ij : i, j \in \Gamma(x), i \neq j\}$ such that, for $1 \leq i \leq k$, $\Gamma(i) = \{x\} \cup \{ij : j \neq i\}$.*

(b) *G_x has two or three orbits on $\Gamma_3(x)$. Let $\Gamma_3(x) = \Delta(x) \cup \Sigma(x)$, where if $\pi = (x, i, ij, z, \ldots) \in \Pi$, then $\Delta(x) = z^{G_x}$. Then one of the following is true.*

  (i) *$m \geq 7$. Here we may identify $\Delta(x)$ with $\{\bar{ij} : i, j \in \Gamma(x), i \neq j\}$ (with equivalent G_x actions on $\Delta(x)$ and on $\Gamma_2(x)$) so that $\Gamma(ij) \cap \Delta(x) = \{\bar{ij}\}$ for all $i \neq j$, and if G_x is 3-transitive on $\Gamma(x)$ then $\Sigma(x)$ is a G_x-orbit and we may identify $\Sigma(x)$ with $\{ij\ell : i \neq j \neq \ell \neq i$ in $\Gamma(x)\}$, so that $\Gamma(ij) \cap \Sigma(x) = \{ij\ell :$ for all ℓ with $j \neq \ell \neq i\}$.*

  (ii) *$m = 6$ and we may identify $\Delta(x)$ with the set of unordered pairs $\{\{i, j\} : i, j \in \Gamma(x), i \neq j\}$, so that $\Gamma(\{i, j\}) \cap \Gamma_2(x) = \{ij, ji\}$ for all i, j in $\Gamma(x)$.*

  (iii) *$m = 6$, $|\Delta(x)| = k - 1$ and one of the following occurs:*

    A. *$k = 2^d$, either $G_x \cong \mathrm{AGL}(d, 2)$, $d \geq 2$ and $G_{xz} \cong 2^d \cdot \mathrm{GL}(d-1, 2)$, or $G_x \cong 2^4 \cdot \mathrm{A}_7 < \mathrm{AGL}(4, 2)$ and $G_{xz} \cong 2^4 \cdot \mathrm{PSL}(2, 7)$; also G_{xz} is the stabilizer of a parallel class of affine lines of $\Gamma(x)$ and $\Sigma(x)$ is a G_x-orbit;*

    B. *$k = 12$, $G_x \cong \mathrm{PSL}(2, 11)$ and $G_{xz} \cong \mathrm{A}_5$;*

    C. *$k = 8$, $G_x \cong \mathrm{PSL}(2, 7)$ and $G_{xz} \cong \mathrm{S}_4$; or*

    D. *$k = 6$, $G_x \cong \mathrm{PSL}(2, 5)$ or $\mathrm{PGL}(2, 5)$ and $G_{xz} \cong \mathrm{A}_4$ or S_4.*

  (iv) *$m = 6$, $G_x \geq \mathrm{PSL}(2, k-1)$, $k \equiv 0 \pmod 4$, $|\Delta(x)| = \binom{k}{2}$, $|\Gamma(ij) \cap \Delta(x)| = k/2$, and $|\Gamma(z) \cap \Gamma_2(x)| = k$. Also $\Sigma(x)$ is a G_x-orbit.*

  (v) *$m = 6$, $k = 10$, $G_x \cong \mathrm{PSL}(2, 9)$, $|\Delta(x)| = 15$, $|\Gamma(ij) \cap \Delta(x)| = 1$ and $|\Gamma(z) \cap \Gamma_2(x)| = 6$.*

Note that if G_x is 3-transitive on $\Gamma(x)$ then cases (b)(iv) *and* (b)(v) *of Theorem* 1.1 *do not arise and, in case* (b)(iii), *$G_x \cong \mathrm{AGL}(d, 2)$, $2^4 \cdot \mathrm{A}_7$ or $\mathrm{PGL}(2, 5)$.*

We now give the definitions of the graphs $PP(4)$, $SP(8)$, $AP(8)$ and $AP(32)$ (see [6] and [8] for more details). The definitions are in terms of the so-called coset graph construction, described as follows. If G is a group with core-free subgroup H, and if $g \in G - \mathrm{N}_G(H)$ is such that $g^2 \in H$, then the *coset graph* $\Gamma(G, H, g)$ is the graph with vertex set $\{Hx : x \in G\}$, consisting of right cosets of H in G, and edge set $\{\{Hx, Hgx\} : x \in G\}$. It is easy to verify that $\Gamma(G, H, g)$ is an undirected graph admitting G as a group of automorphisms acting transitively, by right multiplication, on vertices and edges.

Construction 2.1 (The projective-polygonal graph $PP(4)$) Let $G = \Sigma\mathrm{L}(2,4)$, that is $G = \mathrm{SL}(2,4)\cdot\langle\sigma\rangle$, where σ is the Frobenius automorphism of GF(4) given by $\sigma : z \to z^2$ for all z in GF(4), acting component-wise on matrices in SL(2, 4). Let $H = \{(\begin{smallmatrix}1 & \alpha\\ 0 & 1\end{smallmatrix}) : \alpha \in \mathrm{GF}(4)\}$, a subgroup of G of order 4, and let $g = (\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix})\cdot\sigma$, an element of order 2 lying in $G - \mathrm{N}_G(H)$. Then the projective-polygonal graph $PP(4)$ is $\Gamma(G, H, g)$, has 30 vertices, and is a narrow hexagonal graph with automorphism group $\mathrm{S}_5 \times \mathrm{S}_3$.

Construction 2.2 (The symmetric-polygonal graph $SP(8)$) Let $G = \mathrm{S}_9$, the symmetric group of degree 9, acting naturally on the set $X = \{0, 1, \ldots, 8\}$. Let $H = \mathrm{AGL}(3, 2)$ regarded as a subgroup of G by fixing the element 0 of X and acting in its natural 3-transitive representation, of degree 8, on $X - \{0\}$. Let $a = (0, 1) \in G$. Then the symmetric-polygonal graph $SP(8)$ is $\Gamma(G, H, a)$, has 270 vertices, and is a narrow hexagonal graph with automorphism group S_9.

Construction 2.3 (The affine-polygonal graph $AP(8)$ and $AP(32)$) We describe the construction of $AP(8)$. (That of $AP(32)$ follows similarly.) We construct an extension $G \cong (\mathrm{SL}(2,8) \times \mathrm{D}_7):\mathrm{Z}_3$ as follows. Let $S = \mathrm{SL}(2,8)$ and take the group $\Sigma\mathrm{L}(2,8) = S\cdot\langle\sigma\rangle$, where σ is the Frobenius automorphism of GF(8) given by $\sigma : z \to z^2$ for all z in GF(8). This group has a subgroup isomorphic to the affine linear group, $\mathrm{A\Gamma L}(1,8) = VM\cdot\langle\sigma\rangle$, which in turn has a subgroup $M\cdot\langle\sigma\rangle$, where V is an elementary abelian 2-group isomorphic to the additive group of GF(8) and M is a cyclic group of order 7 isomorphic to the multiplicative group of GF(8). There is an involution $\tau \in S$, say, which normalizes M and the group $M\langle\sigma, \tau\rangle$ has order $|M||\sigma||\tau| = 42$. Consider the external direct product $\Sigma\mathrm{L}(2,8) \times M\langle\sigma,\tau\rangle$. This group has a subgroup $G = (S \times M\cdot\langle\tau\rangle)\cdot\langle(\sigma,\sigma)\rangle \cong (\mathrm{SL}(2,8) \times \mathrm{D}_7):\mathrm{Z}_3$ of index 3 with trivial center. Let $H = \{(vm\sigma^i, m\sigma^i) : v \in V, m \in M, 0 \le i \le 2\} \cong (V \times 1)\cdot\mathrm{diag}(M \times M)\cdot\langle(\sigma,\sigma)\rangle \cong \mathrm{A\Gamma L}(1,8)$, where $\mathrm{diag}(M \times M) = \{(m,m) : m \in M\}$. Let $g = (\tau,\tau)$, an involution in $G - \mathrm{N}_G(H)$ which normalizes $\mathrm{diag}(M \times M)$. The affine-polygonal graph $AP(8)$ is $\Gamma(G, H, g)$, has 126 vertices and is a narrow hexagonal graph with automorphism group G. (The graph, $AP(32)$, constructed similarly, has 2,046 vertices and is a narrow hexagonal graph with automorphism group $\cong (\mathrm{SL}(2,32) \times \mathrm{D}_{31}) : \mathrm{Z}_5$.)

3. Proof of Theorem 2

Assume the hypotheses of Theorem 2. Let the notation be as in Proposition 2.1. Let y be a vertex with $\{x, y\}$ an edge of Γ. We have to consider cases (iii) A–D of Proposition 2.1.

Case A. Here we suppose that $k = 2^d$, and either $G_x \cong \mathrm{AGL}(d,2)$, $d \ge 2$ and $G_{xz} \cong 2^d\cdot\mathrm{GL}(d-1,2)$, or $G_x \cong 2^4\cdot\mathrm{A}_7 < \mathrm{AGL}(4,2)$ and $G_{xz} \cong 2^4\cdot\mathrm{PSL}(2,7)$; also G_{xz} is the stabilizer of a parallel class of affine lines of $\Gamma(x)$ and $\Sigma(x)$ is a G_x-orbit. Note as well that $\mathrm{O}_2(G_x)$ acts trivially on $\Delta(x)$.

We first prove the following lemma which allows us to proceed by induction.

Lemma 3.1 *Let Γ and G be as in Proposition* 2.1(iii)A. *Consider $\Gamma(x)$ as a d-dimensional vector space over the field of two elements and let S be an affine subspace of dimension $c \ge 2$. Let $G_{x(S)}$ denote the pointwise stabilizer in G_x of S, let $\Gamma_{x(S)}$ be that connected* *subgraph of* Γ *containing* x *induced by the points of* Γ *fixed by* $G_{x(S)}$*, and let* $\Pi(S)$ *be the subset of* Π *consisting of those hexagons of* Π *contained in* $\Gamma_{x(S)}$*. Then* $\Gamma_{x(S)}$ *is a narrow hexagonal graph with respect to* $\Pi(S)$*, and* $N_G(G_{x(S)})$ *induces on* $\Gamma_{x(S)}$ *a group of automorphisms satisfying the hypotheses of Theorem* 1.

Proof: That $\Gamma_{x(S)}$ is a hexagonal graph, follows from Lemma 1.6 of [5]. The rest will follow when we show that $N_G(G_{x(S)})$ is transitive on the vertices of $\Gamma_{x(S)}$.

Let F denote the set of vertices of $\Gamma_{x(S)}$ so that $\Gamma(x) \cap F = S$. Let $y \in \Gamma(x) \cap F$. Then there is an element $g \in G$ which interchanges x and y so that S^g is an affine subspace of $\Gamma(y)$ of dimension c containing x. On the other hand, by considering the hexagons in Π on the 2-chains (y, x, i), for $y \neq i \in S$, we see that $S^g \subseteq F$. Thus $G_{x(S)} \leq G_{y(S^g)} = (G_{x(S)})^g$ and hence $G_{x(S)} = (G_{x(S)})^g$ so that g normalizes $G_{x(S)}$. The result now follows. □

Subcase (i) $k = 4$ $(d = 2)$.

We first give a presentation for AGL(2, 2) in a form convenient for our analysis. We think of AGL(2, 2) naturally as a split extension of the vector space of dimension 2 over the field of two elements by GL(2, 2). As such,

$$\mathrm{AGL}(2,2) = \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix} \right\rangle.$$

Letting `b` and `c` represent the two vectors and `f` and `g` the two matrices, in the order given above, we have the following presentations.

$$\begin{aligned} G_x &= \langle \mathtt{b, c, f, g} \mid \mathtt{b}^2, \mathtt{c}^2, [\mathtt{b, c}], \mathtt{f}^2, \mathtt{g}^2, (\mathtt{fg})^3, [\mathtt{f,b}], \\ &\quad [\mathtt{g, c}], [\mathtt{f, c}] = \mathtt{b}, [\mathtt{g,b}] = \mathtt{c}\rangle \\ &\cong \mathrm{AGL}(2,2), \quad \text{and} \end{aligned}$$

$$G_{xy} = \langle \mathtt{f, g} \mid \mathtt{f}^2, \mathtt{g}^2, (\mathtt{fg})^3\rangle \cong \mathrm{GL}(2,2).$$

(In this, as in future presentations, we will use the MAGMA convention that the relation $\mathtt{b}^2$ means $\mathtt{b}^2 = 1$, etc. Also, `[b,c]` stands for the commutator $\mathtt{b}^{-1}\mathtt{c}^{-1}\mathtt{bc}$.)

There are three "special hexagons" in Π through the edge $\{x, y\}$. They are permuted among themselves by the subgroup $G_{\{x,y\}}$ which normalizes G_{xy}. Since $G_{xy} \cong S_3$ acts faithfully on this set of hexagons, it follows that there exists an element in $G_{\{x,y\}}$ interchanging x and y, centralizing G_{xy} and acting trivially on the set of three special hexagons through $\{x, y\}$. Hence, $G_{\{x,y\}} \cong \mathrm{Z}_2 \times \mathrm{GL}(2,2)$.

Thus there is an element $\mathtt{a} \in G$ such that $G_{\{x,y\}} = \langle \mathtt{a}, G_{xy}\rangle = \langle \mathtt{a,f,g}\rangle$ and $\mathtt{a}^2 = [\mathtt{a,f}] = [\mathtt{a,g}] = 1$. Further, $G = \langle G_x, \mathtt{a}\rangle$.

Let $v = y^{\mathtt{b}} \in \Gamma(x)$ and let $\pi \in \Pi$ be the hexagon on (y, x, v). Since `f` fixes x, y and v, it acts trivially on π. Since `a` and `b` both map π to itself, their product `ab` rotates π. So $(\mathtt{ab})^6 \in G_{xyv}$. Thus $(\mathtt{ab})^6 = 1$ or $(\mathtt{ab})^6 = \mathtt{f}$.

Let $\mathtt{p} = \mathtt{a}^{\mathtt{ba}} = \mathtt{ababa}$ and let $w = x^{\mathtt{p}}$, so that w is opposite x on π, i.e. $w \in \Delta(x)$. Since $O_2(G_x) = \langle \mathtt{b}, \mathtt{c} \rangle$ acts trivially on $\Delta(x)$, we have $G_{xw} = \langle \mathtt{b}, \mathtt{c}, \mathtt{f} \rangle$. Since $\mathtt{p} \in G_{\{x,w\}}$, $\mathtt{p}$ normalizes $\langle \mathtt{b}, \mathtt{c}, \mathtt{f} \rangle \cong D_4$. Hence $\mathtt{p}$ centralizes $\langle \mathtt{b} \rangle = Z(\langle \mathtt{b}, \mathtt{c}, \mathtt{f} \rangle)$, so $(\mathtt{ab})^6 = (\mathtt{pb})^2 = 1$. Moreover, $\mathtt{c}^{\mathtt{p}}$ must be a non-central involution in $\langle \mathtt{b}, \mathtt{c}, \mathtt{f} \rangle$, so $\mathtt{c}^{\mathtt{p}} \in \{\mathtt{c}, \mathtt{bc}, \mathtt{f}, \mathtt{fb}\}$ or, equivalently, $[\mathtt{a}^{\mathtt{ba}}, \mathtt{c}] \in \{1, \mathtt{b}, \mathtt{fc}, \mathtt{fbc}\}$.

Using MAGMA, with the relations $\mathtt{a}^2 = [\mathtt{a}, \mathtt{f}] = [\mathtt{a}, \mathtt{g}] = 1$, we find that $|\langle \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{f}, \mathtt{g} \rangle : \langle \mathtt{b}, \mathtt{c}, \mathtt{f}, \mathtt{g} \rangle| = 2$ except when $[\mathtt{a}^{\mathtt{ba}}, \mathtt{c}] = \mathtt{b}$, in which case $|\langle \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{f}, \mathtt{g} \rangle : \langle \mathtt{b}, \mathtt{c}, \mathtt{f}, \mathtt{g} \rangle| = 30$. Thus, since $|G : G_x| > 2$, the latter relation holds. Then, since in $S_5 \times S_3$ we can find five elements satisfying the same relations as $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{f}, \mathtt{g}$, and $|S_5 \times S_3| = 30 \cdot |G_x|$, it follows that $G \cong S_5 \times S_3$ and $\Gamma \cong PP(4)$.

Subcase (ii) $k = 8$ $(d = 3)$.

In this case we have

$$\mathrm{AGL}(3,2) = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix} \right\rangle$$

Letting $\mathtt{b}$, $\mathtt{c}$, and $\mathtt{d}$ represent the three vectors and $\mathtt{f}, \mathtt{g}, \mathtt{h}, \mathtt{i}$, and $\mathtt{j}$ the five matrices, in the order given above, we have the following presentations.

$$\begin{aligned}G_x = \langle &\mathtt{b},\mathtt{c},\mathtt{d},\mathtt{f},\mathtt{g},\mathtt{h},\mathtt{i},\mathtt{j} \mid \mathtt{b}^2,\mathtt{c}^2,\mathtt{d}^2,[\mathtt{b},\mathtt{c}],[\mathtt{b},\mathtt{d}],[\mathtt{c},\mathtt{d}],\mathtt{f}^2,\mathtt{g}^2,\mathtt{h}^2,\\ &\mathtt{i}^2,\mathtt{j}^2,(\mathtt{fg})^3,[\mathtt{f},\mathtt{h}],[\mathtt{f},\mathtt{i}] = \mathtt{h},[\mathtt{f},\mathtt{j}],[\mathtt{g},\mathtt{h}] = \mathtt{i},[\mathtt{g},\mathtt{i}],(\mathtt{gj})^4,\\ &[\mathtt{h},\mathtt{i}],[\mathtt{h},\mathtt{j}] = \mathtt{f},(\mathtt{ij})^3,[\mathtt{f},\mathtt{b}],[\mathtt{f},\mathtt{c}] = \mathtt{b},[\mathtt{f},\mathtt{d}],[\mathtt{g},\mathtt{b}]\\ &= \mathtt{c},[\mathtt{g},\mathtt{c}],[\mathtt{g},\mathtt{d}],[\mathtt{i},\mathtt{b}],[\mathtt{i},\mathtt{c}],[\mathtt{i},\mathtt{d}] = \mathtt{c},[\mathtt{j},\mathtt{b}],[\mathtt{j},\mathtt{c}]\\ &= \mathtt{d},[\mathtt{j},\mathtt{d}]\rangle \cong \mathrm{AGL}(3,2), \quad \text{and}\\ G_{xy} = \langle &\mathtt{f},\mathtt{g},\mathtt{h},\mathtt{i},\mathtt{j} \mid \mathtt{f}^2,\mathtt{g}^2,\mathtt{h}^2,\mathtt{i}^2,\mathtt{j}^2,(\mathtt{fg})^3,[\mathtt{f},\mathtt{h}],[\mathtt{f},\mathtt{i}]\\ &= \mathtt{h},[\mathtt{f},\mathtt{j}],[\mathtt{g},\mathtt{h}] = \mathtt{i},[\mathtt{g},\mathtt{i}],(\mathtt{gj})^4,[\mathtt{h},\mathtt{i}],[\mathtt{h},\mathtt{j}]\\ &= \mathtt{f},(\mathtt{ij})^3\rangle \cong \mathrm{GL}(3,2).\end{aligned}$$

There are seven "special hexagons" in Π through the edge $\{x, y\}$ and $G_{\{x,y\}}$ leaves this set invariant. The subgroup $G_{xy} \cong \mathrm{GL}(3,2)$ acts faithfully on this set of hexagons and $G_{\{x,y\}}$ acts as a subgroup of S_7 containing G_{xy} as a subgroup of index 2. Thus $G_{\{x,y\}} \cong Z_2 \times \mathrm{GL}(3,2)$, since $\mathrm{Aut}(\mathrm{GL}(3,2))$ is not a subgroup of S_7, and we can choose an involution $\mathtt{a}$ in $G_{\{x,y\}}$ interchanging x and y, centralizing G_{xy} and acting trivially on the set of special hexagons through $\{x, y\}$.

So we have $\mathtt{a} \in G$ such that $G_{\{x,y\}} = \langle \mathtt{a}, \mathtt{f}, \mathtt{g}, \mathtt{h}, \mathtt{i}, \mathtt{j} \rangle$ and $\mathtt{a}^2 = [\mathtt{a},\mathtt{f}] = [\mathtt{a},\mathtt{g}] = [\mathtt{a},\mathtt{h}] = [\mathtt{a},\mathtt{i}] = [\mathtt{a},\mathtt{j}] = 1$. Further, $G = \langle G_x, \mathtt{a} \rangle$.

Let Λ denote the connected component of the fixed point graph of $\langle \mathtt{h,i} \rangle$ containing x. The centralizer of $\langle \mathtt{h,i} \rangle$ in $\langle \mathtt{b,c,d} \rangle$ is $\langle \mathtt{b,c} \rangle$. We can thus apply Lemma 3.1 with $S = y^{\langle \mathtt{b,c} \rangle}$. By induction we have $\Lambda \cong PP(4)$, and $N_G(\langle \mathtt{h,i} \rangle)$ induces $S_5 \times S_3$ on Λ.

From subcase (i), we can thus assume that $\mathtt{(ab)}^6 = \mathtt{h}^q\mathtt{i}^r$, and $\mathtt{[a^{ba},c]} = \mathtt{b}\mathtt{h}^s\mathtt{i}^t$, where each of q, r, s, t is equal to 0 or 1.

Using MAGMA, with the relations $\mathtt{a}^2 = \mathtt{[a,f]} = \mathtt{[a,g]} = \mathtt{[a,h]} = \mathtt{[a,i]} = \mathtt{[a,j]} = 1$, we find that $|\langle \mathtt{a,b,c,d,f,g,h,i,j} \rangle : \langle \mathtt{b,c,d,f,g,h,i,j} \rangle| = 2$ in all cases except when $\mathtt{(ab)}^6 = 1$ and $\mathtt{[a^{ba},c]} = \mathtt{bi}$, in which case the index is 270. Thus the latter relations hold. Then, since in S_9 we can find nine elements satisfying the same relations as $\mathtt{a,b,c,d,f,g,h,i,j}$, and $|S_9| = 270 \cdot |G_x|$, it follows that $G \cong S_9$ and $\Gamma \cong SP(8)$.

Alternatively, as another approach, we can define $N = N_G(\langle \mathtt{h,i} \rangle) = \langle \mathtt{a,b,c,f,g,h,i} \rangle$, a semi-direct product of $\langle \mathtt{h,i} \rangle$ with $\langle \mathtt{a,b,c,f,g} \rangle$. Using MAGMA, we can verify that $|N| = 2$ in all cases except when either $\mathtt{(ab)}^6 = 1$ and $\mathtt{[a^{ba},c]} = \mathtt{b}$, in which case $|N| = 48 = 4 \cdot 12 = |\langle \mathtt{h,i} \rangle| \cdot |D_6|$, or $\mathtt{(ab)}^6 = 1$ and $\mathtt{[a^{ba},c]} = \mathtt{bi}$, in which case $|N| = 2880 = 4 \cdot 720 = |\langle \mathtt{h,i} \rangle| \cdot |S_5 \times S_3|$ as expected. (Recall that the automorphism group of a hexagon is D_6.) This approach will prove to be important in the next subcase.

Subcase (iii) $\quad k \geq 16$ ($d \geq 4$) and $G_x \cong \mathrm{AGL}(d, 2)$.

We will show that, with $k = 16$, there are no narrow hexagonal graphs having $G_x \cong \mathrm{AGL}(4, 2)$, whence by Lemma 3.1, there are no narrow hexagonal graphs with $k > 16$ either. Hence assume $k = 16$ and $G_x \cong \mathrm{AGL}(4, 2)$

$$= \left\langle \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \right.$$
$$\begin{pmatrix}1&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},$$
$$\left. \begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix} \right\rangle$$

Letting $\mathtt{b,c,d}$ and $\mathtt{e}$ represent the four vectors and $\mathtt{f,g,h,i,j,k,l,m}$ and $\mathtt{n}$ the nine matrices, in the order given above, we have the following presentations.

$$G_x = \langle \mathtt{b,c,d,e,f,g,h,i,j,k,l,m,n,} \mid \mathtt{b^2,c^2,d^2,e^2,[b,c],[b,d],}$$
$$\mathtt{[b,e],[c,d],[c,e],[d,e],f^2,g^2,h^2,i^2,j^2,k^2,l^2,m^2,n^2,(fg)^3,}$$

$$
\begin{aligned}
&\quad \mathtt{[f,h],[f,i]=h,[f,j],[f,k],[f,l]=k,[f,m],[f,n],[g,h]}\\
&= \mathtt{i,[g,i],(gj)^4,[g,k]=l,[g,l],[g,m],[g,n],[h,i],[h,j]}\\
&= \mathtt{f,[h,k],[h,l],[h,m]=k,[h,n],(ij)^3,\ [i,k],[i,l],\ [i,m]}\\
&= \mathtt{l,[i,n],[j,k],[j,l]=m,[j,m],(jn)^4,[k,l],[k,m],[k,n]}\\
&= \mathtt{h,[l,m],[l,n]=i,(mn)^3,[f,b],[f,c]=b,[f,d],[f,e],[g,b]}\\
&= \mathtt{c,[g,c],[g,d],[g,e],[i,b],[i,c],[i,d]}\\
&= \mathtt{c,[i,e],[j,b],[j,c]=d,[j,d],[j,e],[m,b],[m,c],[m,d],}\\
&\quad \mathtt{[m,e]=d,[n,b],[n,c],[n,d]=e,[n,e]}\rangle\\
&\cong \mathrm{AGL}(4,2), \quad \text{and}\\
G_{xy} &= \langle \mathtt{f,g,h,i,j,k,l,m,n} \mid \mathtt{f^2,g^2,h^2,i^2,j^2,k^2,l^2,m^2,n^2,(fg)^3,}\\
&\quad \mathtt{[f,h],[f,i]=h,[f,j],[f,k],[f,l]=k,[f,m],[f,n],[g,h]}\\
&= \mathtt{i,\ [g,i],(gj)^4,[g,k]=l,[g,l],[g,m],[g,n],[h,i],[h,j]}\\
&= \mathtt{f,[h,k],[h,l],[h,m]=k,[h,n],(ij)^3,[i,k],[i,l],[i,m]}\\
&= \mathtt{l,[i,n],[j,k],[j,l]=m,[j,m],(jn)^4,[k,l],[k,m],[k,n]}\\
&= \mathtt{h,[l,m],[l,n]=i,(mn)^3}\rangle \cong \mathrm{GL}(4,2).
\end{aligned}
$$

Here there are fifteen “special hexagons” in Π through the edge $\{x, y\}$ and $G_{\{x,y\}}$ acts on these. The subgroup $G_{xy} \cong \mathrm{GL}(4, 2)$ acts faithfully on this set of hexagons. So $G_{\{x,y\}}$ acts as a subgroup of S_{15} containing G_{xy} as a subgroup of index 2. It follows that $G_{\{x,y\}} \cong Z_2 \times \mathrm{GL}(4, 2)$ and we can choose an element $\mathtt{a}$ in $G_{\{x,y\}}$ exchanging x and y, centralizing G_{xy} and acting trivially on the set of special hexagons through $\{x, y\}$.

So we have $\mathtt{a} \in G$ such that $G_{\{x,y\}} = \langle \mathtt{a,f,g,h,i,j,k,l,m,n} \rangle$ and $\mathtt{a}^2 = [\mathtt{a,f}] = [\mathtt{a,g}] = [\mathtt{a,h}] = [\mathtt{a,i}] = [\mathtt{a,j}] = [\mathtt{a,k}] = [\mathtt{a,l}] = [\mathtt{a,m}] = [\mathtt{a,n}] = 1$. Further, $G = \langle G_x, \mathtt{a} \rangle$.

Let Λ denote the connected component of the fixed point graph of $\langle \mathtt{k,l,m} \rangle$ containing x. The centralizer of $\langle \mathtt{k,l,m} \rangle$ in $\langle \mathtt{b,c,d,e} \rangle$ is $\langle \mathtt{b,c,d} \rangle$. We can thus apply Lemma 3.1 with $S = y^{\langle \mathtt{b,c,d} \rangle}$. By induction we have $\Lambda \cong SP(8)$ and $N_G(\langle \mathtt{k,l,m} \rangle)$ induces S_9 on Λ.

From subcase (*ii*), we can thus assume that $(\mathtt{ab})^6 = \mathtt{k}^q \mathtt{l}^r \mathtt{m}^s$ and $[\mathtt{a}^{\mathtt{ba}}, \mathtt{c}] = \mathtt{b}\ \mathtt{i} \cdot \mathtt{k}^t \mathtt{l}^u \mathtt{m}^v$, where each of q, r, s, t, u, v is equal to 0 or 1.

In this case, when MAGMA attempted to compute the index $|\langle \mathtt{a,b,c,d,e,f,g,h,i,j,k,l,m,n} \rangle : \langle \mathtt{b,c,d,e,f,g,h,i,j,k,l,m,n} \rangle|$, the Todd-Coxeter algorithm overflowed the memory and available storage. We therefore adopted the alternative approach mentioned at the end of subcase (*ii*).

We define $N = N_G(\langle \mathtt{k,l,m} \rangle) = \langle \mathtt{a,b,c,d,f,g,h,i,j,k,l,m} \rangle$, a semi-direct product of $\langle \mathtt{k,l,m} \rangle$ with $\langle \mathtt{a,b,c,d,f,g,h,i,j} \rangle$. Using MAGMA, we find that $|N| = 2$ in all cases. In no case do we get $|N| = |\langle \mathtt{k,l,m} \rangle| \cdot |S_9|$, as we would expect if the graph Γ exists. We conclude that no such graph exists with $k = 16$.

Subcase (iv) $\quad k = 16$ $(d = 4)$ and $G_x \cong Z_2^4 \cdot A_7$.

As a subgroup of GL(4, 2), a subgroup isomorphic to A_7 is

$$\left\langle \begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&1\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&1\\0&1&0&0\\0&1&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \right\rangle$$

Letting $\mathtt{b,c,d}$ and $\mathtt{e}$ represent the four vectors in subcase (iii) and $\mathtt{f,g,h,i}$ and $\mathtt{j}$ the five matrices, in the order given above, we have the following presentations.

$$\begin{aligned} G_x = \langle &\mathtt{b,c,d,e,f,g,h,i,j} \,|\, \mathtt{b}^2,\ \mathtt{c}^2, \mathtt{d}^2, \mathtt{e}^2, [\mathtt{b,c}], [\mathtt{b,d}], [\mathtt{b,e}], \\ &[\mathtt{c,d}], [\mathtt{c,e}], [\mathtt{d,e}], \mathtt{f}^2, \mathtt{g}^2, \mathtt{h}^2, \mathtt{i}^2, \mathtt{j}^3, (\mathtt{fg})^3, [\mathtt{f,h}], [\mathtt{f,i}] \\ &= \mathtt{h}, [\mathtt{f,j}] = [\mathtt{f,g}], (\mathtt{fj})^3, \mathtt{g}^{\mathtt{h}} = \mathtt{g}^{-1}, (\mathtt{gi})^3, [\mathtt{g,j}], [\mathtt{h,i}], \mathtt{j}^{\mathtt{h}} \\ &= \mathtt{j}^{-1}, (\mathtt{ij})^4, [\mathtt{f,b}], [\mathtt{f,c}], [\mathtt{f,d}] = \mathtt{b}, [\mathtt{f,e}] \\ &= \mathtt{c}, [\mathtt{g,b}], [\mathtt{g,c}], [\mathtt{g,d}] = \mathtt{ed}, [\mathtt{g,e}] = \mathtt{d}, [\mathtt{h,b}], [\mathtt{h,c}] \\ &= \mathtt{b}, [\mathtt{h,d}], [\mathtt{h,e}] = \mathtt{d}, [\mathtt{i,b}], [\mathtt{i,c}] = \mathtt{bd}, [\mathtt{i,d}], [\mathtt{i,e}] \\ &= \mathtt{b}, [\mathtt{j,b}] = \mathtt{c}, [\mathtt{j,c}] = \mathtt{bc}, [\mathtt{j,d}], [\mathtt{j,e}]\rangle \cong Z_2^4 \cdot A_7, \quad \text{and} \end{aligned}$$

$$\begin{aligned} G_{xy} = \langle &\mathtt{f,g,h,i,j} \,|\, \mathtt{b}^2, \mathtt{c}^2, \mathtt{d}^2, \mathtt{e}^2, [\mathtt{b,c}], [\mathtt{b,d}], [\mathtt{b,e}], [\mathtt{c,d}], [\mathtt{c,e}], \\ &[\mathtt{d,e}], \mathtt{f}^2, \mathtt{g}^3, \mathtt{h}^2, \mathtt{i}^2, \mathtt{j}^3, (\mathtt{fg})^3, [\mathtt{f,h}], [\mathtt{f,i}], = \mathtt{h}, [\mathtt{f,j}] \\ &= [\mathtt{f,g}], (\mathtt{fj})^3, \mathtt{g}^{\mathtt{h}} = \mathtt{g}^{-1}, (\mathtt{gi})^3, [\mathtt{g,j}], [\mathtt{h,i}], \mathtt{j}^{\mathtt{h}} = \mathtt{j}^{-1}, (\mathtt{ij})^4\rangle \\ &\cong A_7. \end{aligned}$$

Again there are fifteen "special hexagons" in Π through the edge $\{x, y\}$, $G_{\{x,y\}}$ acts on these, with the subgroup G_{xy} acting faithfully as A_7 on this set. Hence $G_{\{x,y\}}$ acts as a subgroup of S_{15} containing G_{xy} as a subgroup of index 2. This forces $G_{\{x,y\}} \cong Z_2 \times A_7$. Thus there is an element $\mathtt{a} \in G$ such that $G_{\{x,y\}} = \langle \mathtt{a,f,g,h,i,j}\rangle$ and $\mathtt{a}^2 = [\mathtt{a,f}] = [\mathtt{a,g}] = [\mathtt{a,h}] = [\mathtt{a,i}] = [\mathtt{a,j}] = 1$. Further, $G = \langle G_x, \mathtt{a}\rangle$.

Let $\mathtt{p} = \mathtt{a}^{\mathtt{ba}}$. Then $w = x^{\mathtt{p}} \in \Delta(x)$ and w is opposite x on the special hexagon $\pi \in \Pi$ containing x, y and $y^{\mathtt{b}}$. So $\mathtt{p}$ normalizes G_{xw} and hence also $O_2(G_{xw})$. Since G_{xw}, of index 15 in G_x, is the stabilizer of a parallel class of affine lines of $\Gamma(x)$, we must have $O_2(G_{xw}) = \langle \mathtt{b,c,d,e}\rangle$ of order 2^4.

It follows that $[\mathtt{a}^{\mathtt{ba}}, \mathtt{c}] = \mathtt{b}^q\mathtt{c}^r\mathtt{d}^s\mathtt{e}^t$, where each of q, r, s, t is equal to 0 or 1.

Using MAGMA, with the relations $\mathtt{a}^2 = [\mathtt{a,f}] = [\mathtt{a,g}] = [\mathtt{a,h}] = [\mathtt{a,i}] = [\mathtt{a,j}] = 1$, we find that $|\langle \mathtt{a,b,c,d,f,g,h,i,j}\rangle : \langle \mathtt{b,c,d,f,g,h,i,j}\rangle| = 2$ in all cases. Hence this subcase cannot occur.

Next we consider Case B, so $k = 12$, $G_x \cong \mathrm{PSL}(2, 11)$ and $G_{xz} \cong \mathrm{A}_5$.

As a presentation of PSL(2, 11) we have $\langle \mathtt{b}, \mathtt{c}, \mathtt{d} \,|\, \mathtt{b}^{11}, \mathtt{c}^5, \mathtt{d}^2, \mathtt{b}^{\mathtt{c}} = \mathtt{b}^3, (\mathtt{bd})^3, (\mathtt{cd})^2 \rangle$. In this case, $G_{xy} = \langle \mathtt{b}, \mathtt{c} \rangle$, an extension of a cyclic group of order 11 by a cyclic group of order 5. Thus $G_{\{x,y\}}$ contains an involution $\mathtt{a}$ such that $\mathtt{b}^{\mathtt{a}} = \mathtt{b}$ or $\mathtt{b}^{\mathtt{a}} = \mathtt{b}^{-1}$. We can thus assume that

$$\mathtt{a}^2 = [\mathtt{a}, \mathtt{c}] = 1 \quad \text{and} \quad \mathtt{b}^{\mathtt{a}} = \mathtt{b}^i \quad \text{for } i = 1 \text{ or } -1.$$

The element $\mathtt{c}$ fixes two points in $\Gamma(x)$, one of which is y and the other y', say. Let π be the special hexagon in Π through (y, x, y'). Then $G_{xyy'} = \langle \mathtt{c} \rangle$ and $G_{x\{y,y'\}} = \langle \mathtt{c}, \mathtt{d} \rangle$. Since $[\mathtt{a}, \mathtt{c}] = 1$, $\mathtt{a}$ must map π to itself. Thus $\mathtt{ad}$ rotates π so $(\mathtt{ad})^6 \in \langle \mathtt{c} \rangle$. Since $\mathtt{c}^{\mathtt{d}} = \mathtt{c}^{-1}$ and $\mathtt{c}^{\mathtt{a}} = \mathtt{c}$ we have that $[\mathtt{ad}, \mathtt{c}] = \mathtt{c}^2 \neq 1$. So $(\mathtt{ad})^6$ cannot be a non-trivial power of $\mathtt{c}$. Hence

$$(\mathtt{ad})^6 = 1.$$

Let $\mathtt{p} = \mathtt{a}^{\mathtt{da}}$ and let $w = x^{\mathtt{p}} \in \Delta(x)$. We have $G_{xw} \cong \mathrm{A}_5$. Since $\mathtt{p} \in G_{\{x,w\}}$, it follows that $\mathtt{p}$ normalizes G_{xw}. Now $\langle \mathtt{c}, \mathtt{d} \rangle \cong \mathrm{D}_5$, a dihedral group of order 10, and $\langle \mathtt{c}, \mathtt{d} \rangle \leq G_{xw}$. Further, $\mathtt{p}$ centralizes $\langle \mathtt{c}, \mathtt{d} \rangle$. It follows that $\mathtt{p}$ centralizes G_{xw}. Finally, $\langle \mathtt{c}, \mathtt{d} \rangle$ lies in exactly two subgroups of G_x isomorphic to A_5, viz. $\langle \mathtt{c}, \mathtt{d}, \mathtt{d}^{\mathtt{b}} \rangle$ and $\langle \mathtt{c}, \mathtt{d}, \mathtt{d}^{\mathtt{b}^3} \rangle$. Thus,

$$[\mathtt{a}^{\mathtt{da}}, \mathtt{d}^{\mathtt{b}^j}] = 1, \quad \text{for } j = 1 \text{ or } 3.$$

Using MAGMA we find that the group $G = \langle G_x, \mathtt{a} \rangle$ collapses if $i = 1$. Thus $i = -1$ (so $\mathtt{b}^{\mathtt{a}} = \mathtt{b}^{-1}$) and $j = 1$ or 3. In both these cases, by MAGMA, we find that $G \cong \mathrm{J}_1 \times \mathrm{Z}_2$ and Γ is as described in Section 1, proving (c) of the Theorem.

The next case to consider is Case C, where we have $k = 8$, $G_x \cong \mathrm{PSL}(2, 7)$ and $G_{xz} \cong \mathrm{S}_4$.

As a presentation of PSL(2, 7) we have $\langle \mathtt{b}, \mathtt{c}, \mathtt{d} \,|\, \mathtt{b}^7, \mathtt{c}^3, \mathtt{d}^2, \mathtt{b}^{\mathtt{c}} = \mathtt{b}^2, (\mathtt{bd})^3, (\mathtt{cd})^2 \rangle$. In this case, $G_{xy} = \langle \mathtt{b}, \mathtt{c} \rangle$, an extension of a cyclic group of order 7 by a cyclic group of order 3. Thus $G_{\{x,y\}}$ contains an involution $\mathtt{a}$ such that $\mathtt{b}^{\mathtt{a}} = \mathtt{b}$ or $\mathtt{b}^{\mathtt{a}} = \mathtt{b}^{-1}$. We can thus assume that

$$\mathtt{a}^2 = [\mathtt{a}, \mathtt{c}] = 1 \quad \text{and} \quad \mathtt{b}^{\mathtt{a}} = \mathtt{b}^i \quad \text{for } i = 1 \text{ or } -1.$$

The element $\mathtt{c}$ fixes two points in $\Gamma(x)$, one of which is y and the other y', say. Let π be the special hexagon in Π through (y, x, y'). Then $G_{xyy'} = \langle \mathtt{c} \rangle$ and $G_{x\{y,y'\}} = \langle \mathtt{c}, \mathtt{d} \rangle$. Since $[\mathtt{a}, \mathtt{c}] = 1$, $\mathtt{a}$ must map π to itself. Thus $\mathtt{ad}$ rotates π so $(\mathtt{ad})^6 \in \langle \mathtt{c} \rangle$. Since $\mathtt{c}^{\mathtt{d}} = \mathtt{c}^{-1}$ and $\mathtt{c}^{\mathtt{a}} = \mathtt{c}$ we have that $[\mathtt{ad}, \mathtt{c}] = \mathtt{c}^2 \neq 1$. So $(\mathtt{ad})^6$ cannot be a non-trivial power of $\mathtt{c}$. Hence

$$(\mathtt{ad})^6 = 1.$$

Let $\mathtt{p} = \mathtt{a}^{\mathtt{da}}$ and let $w = x^{\mathtt{p}} \in \Delta(x)$. We have $G_{xw} \cong \mathrm{S}_4$. Since $\mathtt{p} \in G_{\{x,w\}}$, it follows that $\mathtt{p}$ normalizes G_{xw}. Now $\langle \mathtt{c}, \mathtt{d} \rangle \cong \mathrm{S}_3$ and $\langle \mathtt{c}, \mathtt{d} \rangle \leq G_{xw}$. Further, $\mathtt{p}$ centralizes $\langle \mathtt{c}, \mathtt{d} \rangle$. It follows that $\mathtt{p}$ centralizes G_{xw}. Finally, $\langle \mathtt{c}, \mathtt{d}\rangle$ lies in exactly two subgroups of G_x isomorphic to S_4, viz. $\langle \mathtt{c}, \mathtt{d}, \mathtt{d}^{\mathtt{b}^2}\rangle$ and $\langle \mathtt{c}, \mathtt{d}, \mathtt{d}^{\mathtt{b}^4}\rangle$. Thus,

$$[\mathtt{a}^{\mathtt{da}}, \mathtt{d}^{\mathtt{b}^j}] = 1, \quad \text{for } j = 2 \text{ or } 4.$$

Using MAGMA we find that the group $G = \langle G_x, \mathtt{a}\rangle$ collapses if $i = 1$. Thus $i = -1$ (so $\mathtt{b}^{\mathtt{a}} = \mathtt{b}^{-1}$) and $j = 2$ or 4. In both these cases, we find that $G \cong S_7$. However, the representation of S_7 on the cosets of a PSL(2, 7) subgroup leads to a graph of valency 8 and girth 4, so that we get a near hexagonal graph in this case, proving the Corollary.

Finally, we consider Case D, so $k = 6$, $G_x \cong$ PSL(2, 5) or PGL(2, 5) and $G_{xz} \cong A_4$ or S_4.

Subcase (i) $k = 6$, $G_x \cong$ PSL(2, 5) and $G_{xz} \cong A_4$.

As a presentation of PSL(2, 5) we have $\langle \mathtt{b}, \mathtt{c}, \mathtt{d} \mid \mathtt{b}^5, \mathtt{c}^2, \mathtt{d}^2, \mathtt{b}^{\mathtt{c}} = \mathtt{b}^{-1}, (\mathtt{bd})^3, (\mathtt{cd})^2, (\mathtt{bcd})^5\rangle$. (For example, $\mathtt{b}$, $\mathtt{c}$ and $\mathtt{d}$ can represent the permutations (1 2 3 4 5), (2 5)(3 4) and (2 3)(4 5) in $A_5 \cong$ PSL(2, 5)). In this case, $G_{xy} = \langle \mathtt{b}, \mathtt{c}\rangle \cong D_5$ so that either $G_{\{x,y\}} \cong Z_5 : Z_4$, or $G_{\{x,y\}} \cong Z_2 \times D_5$.

In the first of these (when $G_{\{x,y\}} \cong Z_5 : Z_4$) we have an element $\mathtt{a}$ such that $\mathtt{a}^2 = \mathtt{c}$ and $\mathtt{b}^{\mathtt{a}} = \mathtt{b}^2$. This time we see that $(\mathtt{ad})^6 = \mathtt{c}^i$, for $i = 0$ or 1.

Again let $\mathtt{p} = \mathtt{a}^{\mathtt{da}}$ and let $w = x^{\mathtt{p}} \in \Delta(x)$. We have $G_{xw} \cong A_4$. Again, $\mathtt{p}$ normalizes G_{xw}. Since $\langle \mathtt{c}, \mathtt{d}\rangle \leq G_{xw}$, we must have $G_{xw} = N_{G_x}(\langle \mathtt{c}, \mathtt{d}\rangle) = \langle \mathtt{c}, \mathtt{d}, \mathtt{q}\rangle$, where $\mathtt{q} = (\mathtt{bd})^{\mathtt{b}^2}$. Thus,

$$\mathtt{q}^{\mathtt{p}} = \mathtt{q}^j \mathtt{c}^k \mathtt{d}^l \quad \text{for } j = 1 \text{ or} -1, \quad \text{and} \quad k, l = 0 \text{ or } 1.$$

Using MAGMA we find that $(\mathtt{ad})^6 = 1$, $\mathtt{q}^{\mathtt{p}} = \mathtt{q}^{-1}\mathtt{c}$, and $G \cong A_7 \times Z_2$ of order 5040. Again using MAGMA we find that the graph is the "doubling" of the valency 6 graph as described in Section 1.

In the second case, when $G_{\{x,y\}} \cong Z_2 \times D_5$, there is an involution $\mathtt{a} \in G_{\{x,y\}}$ with

$$\mathtt{a}^2 = [\mathtt{a}, \mathtt{c}] = 1 \quad \text{and} \quad \mathtt{b}^{\mathtt{a}} = \mathtt{b}^h \quad \text{for } h = 1 \text{ or } -1.$$

Again we have $(\mathtt{ad})^6 = \mathtt{c}^i$, for $i = 0$ or 1, and, with $\mathtt{q}$ as above,

$$\mathtt{q}^{\mathtt{p}} = \mathtt{q}^j \mathtt{c}^k \mathtt{d}^l \quad \text{for } j = 1 \text{ or } -1, \quad \text{and} \quad k, l = 0 \text{ or } 1.$$

This time we have only the following possibilities. Either $\mathtt{b}^{\mathtt{a}} = \mathtt{b}$, $(\mathtt{ad})^6 = 1$ and $\mathtt{q}^{\mathtt{p}} = \mathtt{qcd}$, or $\mathtt{b}^{\mathtt{a}} = \mathtt{b}^{-1}$, $(\mathtt{ad})^6 = 1$ and $\mathtt{q}^{\mathtt{p}} = \mathtt{q}$, both of these giving $G \cong S_7$ of order 5040, and the graph as above. The other possibilities are that either $\mathtt{b}^{\mathtt{a}} = \mathtt{b}$, $(\mathtt{ad})^6 = \mathtt{c}$ and $\mathtt{q}^{\mathtt{p}} = \mathtt{q}^{-1}\mathtt{c}$ or $\mathtt{q}^{-1}$, or $\mathtt{b}^{\mathtt{a}} = \mathtt{b}^{-1}$, $(\mathtt{ad})^6 = \mathtt{c}$ and $\mathtt{q}^{\mathtt{p}} = \mathtt{q}^{-1}\mathtt{c}$ or $\mathtt{q}^{-1}$. All four of these lead to $G \cong$ PGL(2, 11) of order 1320 in its rank 4 action with subdegrees 1, 6, 10, 5, on a graph of girth 4. This graph is a near hexagonal graph of girth 4.

Subcase (ii) $k = 6$, $G_x \cong \mathrm{PGL}(2,5)$ and $G_{xz} \cong S_4$.

As a presentation of PGL(2, 5) we have $\langle \mathtt{b}, \mathtt{c}, \mathtt{d} \mid \mathtt{b}^5, \mathtt{c}^4, \mathtt{d}^2, \mathtt{b}^{\mathtt{c}} = \mathtt{b}^2, (\mathtt{bd})^3, (\mathtt{cd})^2 \rangle$. (For example, b, c and d can represent the permutations (1 2 3 4 5), (2 3 5 4) and (2 3)(4 5) in $S_5 \cong \mathrm{PGL}(2,5)$). In this case, $G_{xy} = \langle \mathtt{b}, \mathtt{c} \rangle$ is a semi-direct product of a cyclic group of order 5 with one of order 4 and $G_{\{x,y\}} \cong Z_2 \times G_{xy}$.

Thus there is an involution $\mathtt{a} \in G_{\{x,y\}}$ with

$$\mathtt{a}^2 = [\mathtt{a}, \mathtt{b}] = [\mathtt{a}, \mathtt{c}] = 1.$$

Since $\mathtt{a}^2 = \mathtt{d}^2 = 1$, a inverts ad. Thus $(\mathtt{ad})^6 = \mathtt{c}^{2i}$, for $i = 0$ or 1.

Again let $\mathtt{p} = \mathtt{a}^{\mathtt{da}}$ and let $w = x^{\mathtt{p}} \in \Delta(x)$. We have $G_{xw} \cong S_4$, $O_2(G_{xw}) = \langle \mathtt{c}^2, \mathtt{d} \rangle$ and $G_{xw} = N_{G_x}(\langle \mathtt{c}^2, \mathtt{d} \rangle) = \langle \mathtt{c}^2, \mathtt{d}, \mathtt{q} \rangle$, where $\mathtt{q} = (\mathtt{bd})\mathtt{b}^2$. Thus,

$$\mathtt{q}^{\mathtt{p}} = \mathtt{q}^j \mathtt{c}^{2k} \mathtt{d}^l \quad \text{for } j = 1 \text{ or } -1, \quad \text{and} \quad k, l = 0 \text{ or } 1.$$

Using MAGMA we find that $(\mathtt{ad})^6 = 1$, $\mathtt{q}^{\mathtt{p}} = \mathtt{qc}^2\mathtt{d}$, and $G \cong S_7 \times Z_2$ of order 10080. The graph is again the "doubling" of the valency 6 graph as described in Section 1.

This proves the theorem.

4. Proof of Theorem 3

By [3, Theorem 1], if G is a permutation group which is 3-homogeneous of degree n on a finite set Ω, but not 3-transitive on Ω, then, up to permutation isomorphism, one of the following holds.

(i) $\mathrm{PSL}(2,q) \leq G^{\Omega} \leq \mathrm{P\Gamma L}(2,q)$, where $n - 1 = q \equiv 3 \pmod 4$;
(ii) $G^{\Omega} = \mathrm{AGL}(1,8)$, $\mathrm{A\Gamma L}(1,8)$ or $\mathrm{A\Gamma L}(1,32)$; or
(iii) $n = 5$ and G^{Ω} is a (doubly transitive) subgroup of order 20 of S_5.

Theorem 2 dealt with the case $G_x^{\Gamma(x)} \geq \mathrm{PSL}(2,q)$. If $k = 5$ and $G_x^{\Gamma(x)}$ is a (doubly transitive) subgroup of order 20 of S_5, then $G_x^{\Gamma(x)}$ contains a regular normal subgroup of order 5, which cannot occur by the following.

Lemma 4.2 *Let Γ be a narrow hexagonal graph of valency k admitting a vertex-transitive group G of automorphisms preserving the set Π of "special" hexagons. Suppose that, for a vertex x of Γ, G_x contains a regular normal subgroup N. Then N is an elementary abelian 2-group.*

Proof: We know that G_x is faithful on $\Gamma(x)$ (as in Lemma 2.6 of [4]). Since N is transitive and regular on $\Gamma(x)$, it follows that $|N| = k$, the valency of Γ. Since Γ is a narrow hexagonal graph, the set of vertices antipodal to x in the hexagons of Π containing x form a G_x-orbit $\Delta(x)$ of size $k-1$. Let $w \in \Delta(x)$. Then $|G_x : G_{xw}| = k - 1$, so that $N \leq G_{xw}$. Hence N fixes every vertex in $\Delta(x)$.

Now suppose there is an element $n \in N$ with $|n| > 2$. Then there is $y \in \Gamma(x)$ with $y \neq y^n = v$, say, and $v^n \neq y$. Let $\pi \in \Pi$ be the hexagon on (v, x, y), say $\pi = (v, x, y, z, w, u)$ for some $w \in \Delta(x)$. Since n fixes w and the girth of Γ is 6, we have $z^n = u$, whence (x, v, u) is on both π and π^n and hence $\pi = \pi^n$, a contradiction. □

Hence we have only to consider $G_x^{\Gamma(x)} = \mathrm{AGL}(1, 8)$, $\mathrm{A\Gamma L}(1, 8)$ or $\mathrm{A\Gamma L}(1, 32)$. Assume the hypotheses of Theorem 3. Let y be a vertex with $\{x, y\}$ an edge of Γ. Let z be antipodal to x in a special hexagon through x and let $\Delta(x) = z^{G_x}$ (so $\Delta(x)$ is the "narrow" suborbit of size $k - 1$).

Case A. $k = 8$, $G_x \cong \mathrm{AGL}(1, 8)$ and $G_{xz} \cong Z_2 \times Z_2 \times Z_2$.

The group AGL(1, 8) can be thought of as an extension (semi-direct product) of the additive group of a field of order 8, by its multiplicative group of order 7. We have the following presentations.

$$\begin{aligned} G_x &= \langle \mathtt{b,c,d,} \,|\, \mathtt{b}^2, \mathtt{c}^2, \mathtt{d}^2, \mathtt{g}^7, [\mathtt{b,c}], [\mathtt{c,d}], [\mathtt{b,d}], \mathtt{b}^{\mathtt{g}} = \mathtt{c}, \mathtt{c}^{\mathtt{g}} = \mathtt{d}, \mathtt{d}^{\mathtt{g}} \\ &\quad = \mathtt{bc}\rangle \cong \mathrm{AGL}(1, 8), \\ G_{xy} &= \langle \mathtt{g} \,|\, \mathtt{g}^7 \rangle \cong Z_7, \quad \text{and} \\ G_{xz} &= \langle \mathtt{b,c,d} \,|\, \mathtt{b}^2, \mathtt{c}^2, \mathtt{d}^2, [\mathtt{b,c}], [\mathtt{c,d}], [\mathtt{b,d}] \rangle \cong Z_2 \times Z_2 \times Z_2. \end{aligned}$$

In this case, $G_{\{x,y\}}$ contains an involution $\mathtt{a}$ such that $\mathtt{g}^{\mathtt{a}} = \mathtt{g}$ or $\mathtt{g}^{\mathtt{a}} = \mathtt{g}^{-1}$. We can thus assume that

$$\mathtt{a}^2 = 1 \quad \text{and} \quad \mathtt{g}^{\mathtt{a}} = \mathtt{g}^i \quad \text{for } i = 1 \text{ or } -1.$$

Let $y' = x^{\mathtt{aba}}$ so that y' is at distance two from x and adjacent to y. Let π be the special hexagon in Π through (x, y, y') and let $w \in \Delta(x)$ be the vertex in π adjacent to y'. Then $w = x^{\mathtt{amaba}}$ for some non-identity element m in $\langle \mathtt{b,c,d} \rangle < G_x$. Since $G_{xw} = \langle \mathtt{b,c,d} \rangle$, we have $w^{\mathtt{n}} = w$, where $\mathtt{n} = \mathtt{b}$, $\mathtt{c}$, or $\mathtt{d}$. Hence $\mathtt{n}^{\mathtt{abama}}$ is an involution in G_x, so we have $\mathtt{n}^{\mathtt{aba}} = (\mathtt{b}^t \mathtt{c}^u \mathtt{d}^v)^{\mathtt{am}}$, where each of t, u, v is equal to 0 or 1, they are not all 0, and they depend on $\mathtt{n} \in \{\mathtt{b,c,d}\}$.

The latter three relations (one for each $\mathtt{n}$), together with the former one, leads to approximately 2^{13} different sets of relations for such a group $G = \langle G_x, \mathtt{a} \rangle$. Using MAGMA we find that if $i = 1$, the group collapses in all the remaining cases. Surprisingly, when $i = -1$, in all but seven of the nearly 2^{12} remaining possibilities, the group also collapses. Each of the seven remaining cases leads to $|G| = 7,056$ and, with some work using $\mathtt{g}^{\mathtt{a}} = \mathtt{g}^{-1}$, we have identities of the form $(\mathtt{ap})^6 = 1$ and $[\mathtt{a}^{\mathtt{pa}}, \mathtt{p}'] = 1$, where $\mathtt{p} \neq \mathtt{p}'$ are involutions in $\langle \mathtt{b,c,d} \rangle$ (one such $\mathtt{p}$ for each of the seven cases). These two simple identities can then be used as relations to replace the three more complex relations defining the same group. Again, using MAGMA, we are then able to conclude that $\Gamma \cong AP(8)$ and $G \cong \mathrm{SL}(2, 8) \times D_7$.

Case B. $k = 8$, $G_x \cong \mathrm{A\Gamma L}(1, 8)$ and $G_{xz} \cong (Z_2)^3 : Z_3$.

The group $\mathrm{A\Gamma L}(1, 8)$ can be thought of as an extension (semi-direct product) of $\mathrm{AGL}(1, 8)$ by the automorphism group of the field of order 8 (which is cyclic of order 3). We have the following presentations.

$$
\begin{aligned}
G_x &= \langle \mathtt{b,c,d,g,h} \mid \mathtt{b}^2, \mathtt{c}^2, \mathtt{d}^2, \mathtt{g}^7, \mathtt{h}^3, [\mathtt{b,c}], [\mathtt{c,d}], [\mathtt{b,d}], \mathtt{b}^{\mathtt{g}} = \mathtt{c}, \mathtt{c}^{\mathtt{g}} \\
&\quad = \mathtt{d}, \mathtt{d}^{\mathtt{g}} = \mathtt{bc},\ \mathtt{b}^{\mathtt{h}} = \mathtt{b}, \mathtt{c}^{\mathtt{h}} = \mathtt{d}, \mathtt{d}^{\mathtt{h}} = \mathtt{dc}, \mathtt{g}^{\mathtt{h}} = \mathtt{g}^2 \rangle \cong \mathrm{A\Gamma L}(1,8), \\
G_{xy} &= \langle \mathtt{g,h} \mid \mathtt{g}^7, \mathtt{h}^3, \mathtt{g}^{\mathtt{h}} = \mathtt{g}^2 \rangle \cong \mathrm{Z}_7 : \mathrm{Z}_3, \text{ and} \\
G_{xz} &= \langle \mathtt{b,c,d,h} \mid \mathtt{b}^2, \mathtt{c}^2, \mathtt{d}^2, \mathtt{h}^3, [\mathtt{b,c}], [\mathtt{c,d}], [\mathtt{b,d}], \mathtt{b}^{\mathtt{h}} = \mathtt{b}, \mathtt{c}^{\mathtt{h}} = \mathtt{d}, \mathtt{d}^{\mathtt{h}} \\
&\quad = \mathtt{dc} \rangle \cong (\mathrm{Z}_2) : \mathrm{Z}_3.
\end{aligned}
$$

In this case, $G_{xy} = \langle \mathtt{g,h} \rangle$ is an extension of a cyclic group of order 7 by a cyclic group of order 3. Thus $G_{\{x,y\}}$ contains an involution $\mathtt{a}$ such that $\mathtt{g}^{\mathtt{a}} = \mathtt{g}$ or $\mathtt{g}^{\mathtt{a}} = \mathtt{g}^{-1}$. We can thus assume that

$$\mathtt{a}^2 = [\mathtt{a,h}] = 1 \quad \text{and} \quad \mathtt{g}^{\mathtt{a}} = \mathtt{g}^i \quad \text{for } i = 1 \text{ or } -1.$$

The element $\mathtt{h}$ fixes two points in $\Gamma(x)$, one of which is y and the other y', say. Let π be the special hexagon in Π through (y, x, y'). Then $G_{xyy'} = \langle \mathtt{h} \rangle$ and $G_{x\{y,y'\}} = \langle \mathtt{b}, \mathtt{h} \rangle$. Since $[\mathtt{a}, \mathtt{h}] = 1$, $\mathtt{a}$ must map π to itself. Thus $\mathtt{ab}$ rotates π so $(\mathtt{ab})^6 \in \langle \mathtt{h} \rangle$. Since $\mathtt{a}^2 = \mathtt{b}^2 = 1$, $(\mathtt{ab})^{\mathtt{a}} = (\mathtt{ab})^{-1}$. Since $\mathtt{h}^{\mathtt{a}} \neq \mathtt{h}^{-1}$, we conclude that

$$(\mathtt{ab})^6 = 1.$$

Let $\mathtt{p} = \mathtt{a}^{\mathtt{ba}}$ and let $w = x^{\mathtt{p}} \in \Delta x$. We have $G_{xw} \cong (\mathrm{Z}_2)^3 : \mathrm{Z}_3$. Since $\mathtt{p} \in G_{\{x,w\}}$, it follows that $\mathtt{p}$ normalizes $\langle \mathtt{b,c,d} \rangle = \mathrm{O}_2(G_{xw})$. Thus,

$$[\mathtt{a}^{\mathtt{ba}}, \mathtt{c}] = \mathtt{b}^t \mathtt{c}^u \mathtt{d}^v, \text{ where each of } t, u, v = 0 \text{ or } 1.$$

Using MAGMA we find that the group $G = \langle G_x, \mathtt{a} \rangle$ collapses except when $i = -1$, $\mathtt{g}^{\mathtt{a}} = \mathtt{g}^{-1}$ and $[\mathtt{a}^{\mathtt{ba}}, \mathtt{c}] = 1$, in which case $|G| = 21{,}168$. Again, using MAGMA, we are then able to conclude that $\Gamma \cong AP(8)$ and $G \cong (\mathrm{SL}(2, 8) \times \mathrm{D}_7) : \mathrm{Z}_3$.

Case C. $k = 32$, $G_x \cong \mathrm{A\Gamma L}(1, 32)$ and $G_{xz} \cong (\mathrm{Z}_2)^5 : \mathrm{Z}_5$.

The group $\mathrm{A\Gamma L}(1, 32)$ can be thought of as an extension (semi-direct product) of $\mathrm{AGL}(1, 32)$ by the automorphism group of the field of order 32 (which is cyclic of order 5). We have the following presentations.

$$
\begin{aligned}
G_x = \langle &\mathtt{b,c,d,e,f,g,h} \mid \mathtt{b}^2, \mathtt{c}^2, \mathtt{d}^2, \mathtt{e}^2, \mathtt{f}^2, \mathtt{g}^{31}, \mathtt{h}^5, [\mathtt{b,c}], [\mathtt{b,d}], [\mathtt{b,e}], \\
&[\mathtt{b,f}], [\mathtt{c,d}], [\mathtt{c,e}], [\mathtt{c,f}], [\mathtt{d,e}], [\mathtt{d,f}], [\mathtt{e,f}], \mathtt{b}^{\mathtt{g}} = \mathtt{c}, \mathtt{c}^{\mathtt{g}} = \mathtt{d}, \mathtt{d}^{\mathtt{g}} \\
&= \mathtt{e}, \mathtt{e}^{\mathtt{g}} = \mathtt{f}, \mathtt{f}^{\mathtt{g}} \\
&= \mathtt{bcde}, \mathtt{b}^{\mathtt{h}} = \mathtt{b}, \mathtt{c}^{\mathtt{h}} = \mathtt{d}, \mathtt{d}^{\mathtt{h}} = \mathtt{f}, \mathtt{e}^{\mathtt{h}} = \mathtt{cdef}, \mathtt{f}^{\mathtt{h}} = \mathtt{be}, \mathtt{g}^{\mathtt{h}} = \mathtt{g}^2 \rangle \\
&\cong \mathrm{A\Gamma L}(1, 32),
\end{aligned}
$$

$$G_{xy} = \langle \mathtt{g}, \mathtt{h} \mid \mathtt{g}^{31}, \mathtt{h}^5, \mathtt{g}^{\mathtt{h}} = \mathtt{g}^2 \rangle \cong \mathrm{Z}_{31} : \mathrm{Z}_5, \text{ and}$$

$$G_{xz} = \langle \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}, \mathtt{h} \mid \mathtt{b}^2, \mathtt{c}^2, \mathtt{d}^2, \mathtt{e}^2, \mathtt{f}^2, \mathtt{h}^5, [\mathtt{b}, \mathtt{c}], [\mathtt{b}, \mathtt{d}], [\mathtt{b}, \mathtt{e}], [\mathtt{b}, \mathtt{f}], [\mathtt{c}, \mathtt{d}], [\mathtt{c}, \mathtt{e}], [\mathtt{c}, \mathtt{f}], [\mathtt{d}, \mathtt{e}], [\mathtt{d}, \mathtt{f}], [\mathtt{e}, \mathtt{f}], \mathtt{b}^{\mathtt{h}} = \mathtt{b}, \mathtt{c}^{\mathtt{h}} = \mathtt{d}, \mathtt{d}^{\mathtt{h}} = \mathtt{f}, \mathtt{e}^{\mathtt{h}} = \mathtt{cdef}, \mathtt{f}^{\mathtt{h}} = \mathtt{be} \rangle \cong (\mathrm{Z}_2)^5 : \mathrm{Z}_5.$$

In this case, $G_{xy} = \langle \mathtt{g}, \mathtt{h} \rangle$, an extension of a cyclic group of order 31 by a cyclic group of order 5. As in case B above, there is an involution a such that $\mathtt{a}^2 = [\mathtt{a}, \mathtt{h}] = 1$ and $\mathtt{g}^{\mathtt{a}} = \mathtt{g}^i$ for $i = 1$ or -1. Further, we again have $(\mathtt{ab})^6 = 1$. Here, if $\mathtt{p} = \mathtt{a}^{\mathtt{ba}}$ and $w = x^{\mathtt{p}} \in \Delta(x)$, then $G_{xw} \cong (\mathrm{Z}_2)^5 : \mathrm{Z}_5$, and it follows that p normalizes $\langle \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f} \rangle = \mathrm{O}_2(G_{xw})$. Thus,

$$[\mathtt{a}^{\mathtt{ba}}, \mathtt{c}] = \mathtt{b}^r \mathtt{c}^s \mathtt{d}^t \mathtt{e}^u \mathtt{f}^v, \quad \text{where each of } r, s, t, u, v \text{ is equal to 0 or 1.}$$

Using MAGMA we find that the group $G = \langle G_x, \mathtt{a} \rangle$ collapses except when $i = -1$, $\mathtt{g}^{\mathtt{a}} = \mathtt{g}^{-1}$ and $[\mathtt{a}^{\mathtt{ba}}, \mathtt{c}] = 1$, in which case $|G| = 10, 148, 160$. Again, using MAGMA, we are then able to conclude that $\Gamma \cong AP(32)$ and $G \cong (\mathrm{SL}(2, 32) \times \mathrm{D}_{31}) : \mathrm{Z}_5$.

This proves the theorem.

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