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A Chromatic Symmetric Function in Noncommuting Variables

DAVID D. GEBHARD David.Gebhard@wlc.edu
Department of Mathematics, Wisconsin Lutheran College, 8800 W. Bluemont Rd., Milwaukee, WI 53226, USA

BRUCE E. SAGAN sagan@math.msu.edu
Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA

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Abstract. Stanley (*Advances in Math.* **111**, 1995, 166–194) associated with a graph G a symmetric function X_G which reduces to G's chromatic polynomial $\mathcal{X}_G(n)$ under a certain specialization of variables. He then proved various theorems generalizing results about $\mathcal{X}_G(n)$, as well as new ones that cannot be interpreted on the level of the chromatic polynomial. Unfortunately, X_G does not satisfy a Deletion-Contraction Law which makes it difficult to apply the useful technique of induction. We introduce a symmetric function Y_G in noncommuting variables which does have such a law and specializes to X_G when the variables are allowed to commute. This permits us to further generalize some of Stanley's theorems and prove them in a uniform and straightforward manner. Furthermore, we make some progress on the $(\mathbf{3}+\mathbf{1})$-free Conjecture of Stanley and Stembridge (*J. Combin Theory* (*A*) J. **62**, 1993, 261–279).

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1. Introduction

Let G be a finite graph with verticies $V = V(G) = \{v_1, v_2, \ldots, v_d\}$ and edge set $E = E(G)$. We permit our graphs to have loops and multiple edges. Let $\mathcal{X}_G(n)$ be the *chromatic polynomial* of G, i.e., the number of proper colorings $\kappa : V \rightarrow \{1, 2, \ldots, n\}$. (*Proper* means that $vw \in E$ implies $\kappa(v) \neq \kappa(w)$.)

In [11, 12], R. P. Stanley introduced a symmetric function, X_G, which generalizes $\mathcal{X}_G(n)$ as follows. Let $x = \{x_1, x_2, \ldots\}$ be a countably infinite set of commuting indeterminates. Now define

$$X_G = X_G(x_1, x_2, \ldots) = \sum_{\kappa} x_{\kappa(v_1)} \ldots x_{\kappa(v_d)}$$

where the sum ranges over all proper colorings, $\kappa : V(G) \rightarrow \{1, 2, \ldots\}$. It is clear from the definition that X_G is a symmetric function, since permuting the colors of a proper coloring leaves it proper, and is homogeneous of degree $d = |V|$. Also the specialization $X_G(1^n)$ obtained by setting $x_1 = x_2 = \cdots = x_n = 1$, and $x_i = 0$ for all $i > n$ yields $\mathcal{X}_G(n)$.

Stanley used X_G to generalize various results about the chromatic polynomial as well as proving new theorems that only apply to the symmetric function. However, there is a problem when trying to find a deletion-contraction law for X_G. To see what goes wrong, suppose that for $e \in E$ we let $G\backslash e$ and G/e denote G with the e deleted and contracted, respectively. Then X_G and $X_{G\backslash e}$ are homogeneous of degree d while $X_{G/e}$ is homogeneous of degree $d-1$ so there can be no linear relation involving all three. We should note that Noble and Welsh [14] have a deletion contraction method for computing X_G equivalent to [11, Theorem 2.5]. However, it only works in the larger category of vertex-weighted graphs and only for the expansion of X_G in terms of the power sum symmetric functions. Since we are interested in other bases as well, we take a different approach.

In this paper we define an analogue, Y_G, of X_G which is a symmetric function in *noncommuting* variables. (Note that these symmetric functions are different from the noncommutative symmetric functions studied by Gelfand and others, see [7] for example.) The reason for not letting the variables commute is so that we can keep track of the color which κ assigns to each vertex. This permits us to prove a Deletion-Contraction Theorem for Y_G and use it to derive generalizations of results about X_G in a straightforward manner by induction, as well as make progress on a conjecture.

The rest of this paper is organized as follows. In the next section we begin with some basic background about symmetric functions in noncommuting variables (see also [5]). In Section 3 we define Y_G and derive some of its basic properties, including the Deletion-Contraction Law. Connections with acyclic orientations are explored in Section 4. The next three sections are devoted to making some progress on the $(\mathbf{3}+\mathbf{1})$-free Conjecture of Stanley and Stembridge [13]. Finally we end with some comments and open questions.

2. Symmetric functions in noncommuting variables

Our symmetric functions in noncommuting variables will be indexed by elements of the partition lattice. We let Π_d denote the lattice of set partitions π of $\{1, 2, \ldots, d\} := [d]$, ordered by refinement. We write $\pi = B_1/B_2 \ldots /B_k$ if $\uplus_i B_i = [d]$ and call B_i a *block* of π. The meet (greatest lower bound) of the elements π and σ is denoted by $\pi \wedge \sigma$. We use $\hat{0}$ to denote the unique minimal element, and $\hat{1}$ for the unique maximal element.

For $\pi \in \prod_d$ we define $\lambda(\pi)$ to be the integer partition of d whose parts are the block sizes of π. Also, if $\lambda(\pi) = (1^{r_1}, 2^{r_2}, \ldots, d^{r_d})$, we will need the constants

$$|\pi| = r_1!r_2!\cdots r_d! \quad \text{and}$$
$$\pi! = 1!^{r_1}2!^{r_2}\cdots d!^{r_d}.$$

We now introduce the vector space for our symmetric functions. Let $\{x_1, x_2, x_3, \ldots\}$ be a set of *noncommuting* variables. We define our *monomial symmetric functions*, m_π, by

$$m_\pi = \sum_{i_1, i_2, \ldots, i_d} x_{i_1} x_{i_2} \cdots x_{i_d}, \tag{1}$$

where the sum is over all sequences $i_1, i_2, \ldots, i_d$ of positive integers such that $i_j = i_k$ if and only if j and k are in the same block of π. For example, we get

$$m_{13/24} = x_1x_2x_1x_2 + x_2x_1x_2x_1 + x_1x_3x_1x_3 + x_3x_1x_3x_1 + \cdots$$

for the partition $\pi = 13/24$.

From the definition it is easy to see that letting the x_i commute transforms m_π into $|\pi| m_{\lambda(\pi)}$, a multiple of the ordinary monomial symmetric function. The monomial symmetric functions, $\{m_\pi : \pi \in \prod_d, d \in \mathbb{N}\}$, are linearly independent over $\mathbb{C}$, and we call their span the algebra of *symmetric functions in noncommuting variables*.

There are two other bases of this algebra that will interest us. One of them consists of the *power sum symmetric functions* given by

$$p_\pi \stackrel{\text{def}}{=} \sum_{\sigma \geq \pi} m_\sigma = \sum_{i_1, i_2, \ldots, i_d} x_{i_1} x_{i_2} \cdots x_{i_d}, \tag{2}$$

where the second sum is over all positive integer sequences $i_1, i_2, \ldots, i_d$ such that $i_j = i_k$ if j and k are both in the same block of π. The other basis contains the *elementary symmetric functions* defined by

$$e_\pi \stackrel{\text{def}}{=} \sum_{\sigma : \sigma \wedge \pi = \hat{0}} m_\sigma = \sum_{i_1, i_2, \ldots, i_d} x_{i_1} x_{i_2} \cdots x_{i_d}, \tag{3}$$

where the second sum is over all sequences $i_1, i_2, \ldots, i_d$ of positive integers such that $i_j \neq i_k$ if j and k are both in the same block of π. As an illustration of these definitions, we see that

$$\begin{aligned} p_{13/24} &= x_1 x_2 x_1 x_2 + x_2 x_1 x_2 x_1 + \cdots + x_1^4 + x_2^4 + \cdots \\ &= m_{13/24} + m_{1234} \end{aligned}$$

and that

$$\begin{aligned} e_{13/24} &= x_1^2 x_2^2 + \cdots + x_1 x_2^2 x_1 + \cdots + x_1^2 x_2 x_3 + \cdots + x_1 x_2^2 x_3 + \cdots + x_1 x_2 x_3^2 + \cdots \\ &\quad + x_1 x_2 x_3 x_1 + \cdots + x_1 x_2 x_3 x_4 \cdots \\ &= m_{12/34} + m_{14/23} + m_{12/3/4} + m_{1/23/4} + m_{1/2/34} + m_{14/2/3} + m_{1/2/3/4}. \end{aligned}$$

Allowing the variables to commute transforms p_π into $p_{\lambda(\pi)}$ and e_π into $\pi ! e_{\lambda(\pi)}$. We may also use these definitions to derive the change-of-basis formulae found in the appendix of Doubilet's paper [3] which show

$$m_\pi = \sum_{\sigma \geq \pi} \mu(\pi, \sigma) p_\sigma, \tag{4}$$

$$m_\pi = \sum_{\tau \geq \pi} \frac{\mu(\pi, \tau)}{\mu(\hat{0}, \tau)} \sum_{\sigma \leq \tau} \mu(\sigma, \tau) e_\sigma, \tag{5}$$

$$e_\pi = \sum_{\sigma \leq \pi} \mu(\hat{0}, \sigma) p_\sigma, \text{ and} \tag{6}$$

$$p_\pi = \frac{1}{\mu(\hat{0}, \pi)} \sum_{\sigma \leq \pi} \mu(\sigma, \pi) e_\sigma, \tag{7}$$

where $\mu(\pi, \sigma)$ is the Möbius function of Π_n.

It should be clear that these functions are symmetric in the usual sense, i.e., they are invariant under the usual symmetric group action on the variables. However, it will be useful to define a new action of the symmetric group on the symmetric functions in noncommuting variables which permutes the *positions* of the variables. For $\delta \in \mathcal{S}_d$, we define

$$\delta \circ m_\pi \stackrel{\text{def}}{=} m_{\delta(\pi)},$$

where the action of $\delta \in S_d$ on a set partition of $[d]$ is the obvious one acting on the elements of the blocks. It follows that for any δ this action induces a vector space isomorphism, since it merely produces a permutation of the basis elements. Alternatively we can consider this action to be defined on the monomials so that

$$\delta \circ \left(x_{i_1} x_{i_2} \cdots x_{i_k}\right) \stackrel{\text{def}}{=} x_{i_{\delta^{-1}(1)}} x_{i_{\delta^{-1}(2)}} \cdots x_{i_{\delta^{-1}(k)}}$$

and extend linearly.

Utilizing the first characterization of this action, it follows straight from definitions (2) and (3) that $\delta \circ p_\pi = p_{\delta(\pi)}$ and $\delta \circ e_\pi = e_{\delta(\pi)}$.

3. Y_G, The noncommutative version

We begin by defining our main object of study, Y_G.

Definition 3.1 For any graph G with vertices labeled $v_1, v_2, \ldots, v_d$ in a fixed order, define

$$Y_G = \sum_\kappa x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)} = \sum_\kappa x_\kappa,$$

where again the sum is over all proper colorings κ of G, but the x_i are now *noncommuting* variables.

As an example, for P_3, the path on three vertices with edge set $\{v_1v_2, v_2v_3\}$, we can calculate

$$\begin{aligned} Y_{P_3} &= x_1x_2x_1 + x_2x_1x_2 + x_1x_3x_1 + \cdots + x_1x_2x_3 + x_1x_3x_2 + \cdots + x_3x_2x_1 + \cdots \\ &= m_{13/2} + m_{1/2/3}. \end{aligned}$$

Note that if G has loops then this sum is empty and $Y_G = 0$. Furthermore, Y_G depends not only on G, but also on the *labeling* of its vertices.

In this section we will prove some results about the expansion of Y_G in various bases for the symmetric functions in noncummuting variables and show that it satisfies a Deletion-Contraction Recursion. To obtain the expansion in terms of monomial symmetric functions, note that any partition P of V induces a set partition $\pi(P)$ of $[d]$ corresponding to the subscripts of the vertices. A partition P of V is *stable* if any two adjacent vertices are in different blocks of P. (If G has a loop, there are no stable partitions.) The next result follows directly from the definitions.

Proposition 3.2 *We have*

$$Y_G = \sum_P m_{\pi(P)}$$

where the sum is over all stable partitions, P, *of* V.

In order to show that Y_G satisfies a Deletion-Contraction Recurrence it is necessary to have a distinguished edge. Most of the time we will want this edge to be between the last two vertices in the fixed order, but to permit an arbitrary edge choice we will define an action of the symmetric group $\mathcal{S}_d$ on a graph. For all $\delta \in \mathcal{S}_d$ we let δ act on the vertices of G by $\delta(v_i) = v_{\delta(i)}$. This creates an action on graphs given by $\delta(G) = H$, where H is just a relabeling of G.

Proposition 3.3 (Relabeling Proposition) *For any graph* G, *we have*

$$\delta \circ Y_G = Y_{\delta(G)},$$

where the vertex order $v_1, v_2, \ldots, v_d$ *is used in both* Y_G *and* $Y_{\delta(G)}$.

Proof: Let $\delta(G) = H$. We note that the action of δ produces a bijection between the stable partitions of G and H. Utilizing the previous proposition and denoting the stable partitions of G and H by P_G and P_H, respectively, we have

$$Y_H = \sum_{P_H} m_{\pi(P_H)} = \sum_{P_G} m_{\delta(\pi(P_G))} = \sum_{P_G} \delta \circ m_{\pi(P_G)} = \delta \circ \sum_{P_G} m_{\pi(P_G)} = \delta \circ Y_G. \qquad \square$$

Using the Relabeling Proposition allows us, without loss of generality, to choose a labeling of G with the distinguished edge for deletion-contraction being $e = v_{d-1}v_d$. It is this edge for which we will derive the basic recurrence for Y_G.

Definition 3.4 We define an operation called *induction*, $\uparrow$, on the monomial $x_{i_1}x_{i_2}\cdots x_{i_{d-2}}x_{i_{d-1}}$, by

$$\left(x_{i_1}x_{i_2}\cdots x_{i_{d-2}}x_{i_{d-1}}\right)\uparrow = x_{i_1}x_{i_2}\cdots x_{i_{d-2}}x_{i_{d-1}}^2$$

and extend this operation linearly.

Note that this function takes a symmetric function in noncommuting variables which is homogeneous of degree $d-1$ to one which is homogeneous of degree d. Context will make it clear whether the word induction refers to this operation or to the proof technique.

Sometimes we will also need to use induction on an edge $e = v_k v_l$ so we extend the definition as follows. For $k < l$, define an operation $\uparrow_k^l$ on symmetric functions in noncommuting variables which simply repeats the variable in the kth position again in the lth. That is, for a monomial $x_{i_1} \cdots x_{i_k} \cdots x_{i_{d-1}}$, define

$$\left(x_{i_1} \cdots x_{i_k} \cdots x_{i_{l-1}} x_{i_l} \cdots x_{i_{d-1}}\right) \uparrow_k^l = x_{i_1} \cdots x_{i_k} \cdots x_{i_{l-1}} x_{i_k} x_{i_l} \cdots x_{i_{d-1}}$$

and extend linearly.

Provided G has an edge which is not a loop, we will usually start by choosing a labeling such that $e = v_{d-1}v_d$. We also note here that if there is no such edge, then

$$Y_G = \begin{cases} p_{1/2/\cdots/d} = e_{1/2/\cdots/d} & \text{if } G = \bar{K}_d \\ 0 & \text{if } G \text{ has a loop,} \end{cases} \tag{8}$$

where $\bar{K}_d$ is the completely disconnected graph on d vertices. We note that contracting an edge e can create multiple edges (if there are vertices adjacent to both of e's endpoints) or loops (if e is part of a multiple edge), while contracting a loop deletes it.

Proposition 3.5 (Deletion-Contraction Proposition) *For $e = v_{d-1}v_d$, we have*

$$Y_G = Y_{G\backslash e} - Y_{G/e}\uparrow,$$

where the contraction of $e = v_{d-1}v_d$ is labeled v_{d-1}.

Proof: The proof is very similar to that of the Deletion-Contraction Property for $\mathcal{X}_G$. We consider the proper colorings of $G\backslash e$, which can be split disjointly into two types:

1. proper colorings of $G\backslash e$ with vertices v_{d-1} and v_d different colors;
2. proper colorings of $G\backslash e$ with vertices v_{d-1} and v_d the same color.

Those of the first type clearly correspond to proper colorings of G. If κ is a coloring of $G\backslash e$ of the second type then (since the vertices v_{d-1} and v_d are the same color) we have

$$x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_{d-1})} x_{\kappa(v_d)} = \left(x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_{d-1})}\right)\uparrow = x_{\tilde{\kappa}}\uparrow$$

where $\tilde{\kappa}$ is a proper coloring of G/e. Thus we have $Y_{G\backslash e} = Y_G + Y_{G/e}\uparrow$. □

We note that if e is a repeated edge, then the proper colorings of $G\backslash e$ are exactly the same as those of G. The fact that there are no proper colorings of the second type corresponds to the fact that G/e has at least one loop, and so it has no proper colorings. Also note that because of our conventions for contraction we always have

$$|E(G \setminus e)| = |E(G/e)| = |E(G)| - 1$$

where $|\cdot|$ denotes cardinality.

It is easy to see how the operation of induction affects the monomial and power sum symmetric functions. For $\pi \in \Pi_{d-1}$ we let $\pi + (d) \in \Pi_d$ denote the partition π with d inserted into the block containing $d-1$. From Eqs. (1) and (2) it is easy to see that

$$m_\pi\uparrow = m_{\pi+(d)} \quad \text{and} \quad p_\pi\uparrow = p_{\pi+(d)}.$$

With this notation we can now provide an example of the Deletion-Contraction Proposition for P_3, where the vertices are labeled sequentially, and the distinguished edge is $e = v_2v_3$:

$$Y_{P_3} = Y_{P_2 \uplus \{v_3\}} - Y_{P_2}\uparrow .$$

It is not difficult to compute

$$\begin{aligned} Y_{P_2} &= m_{1/2}, \\ Y_{P_2}\uparrow &= m_{1/23}, \\ Y_{P_2 \uplus \{v_3\}} &= m_{1/2/3} + m_{1/23} + m_{13/2}. \end{aligned}$$

This gives us

$$\begin{aligned} Y_{P_3} &= m_{1/2/3} + m_{1/23} + m_{13/2} - m_{1/23} \\ &= m_{1/2/3} + m_{13/2}, \end{aligned}$$

which agrees with our previous calculation.

We may use the Deletion-Contraction Proposition to provide inductive proofs for non-commutative analogues of some results of Stanley [11].

Theorem 3.6 *For any graph G,*

$$Y_G = \sum_{S \subseteq E} (-1)^{|S|} p_{\pi(S)},$$

where $\pi(S)$ denotes the partition of $[d]$ associated with the vertex partition for the connected components of the spanning subgraph of G induced by S.

Proof: We induct on the number of non-loops in E. If E consists only of n loops, for $n \geq 0$, then for all $S \subseteq E(G)$, we will have $\pi(S) = 1/2/\cdots/d$. So

$$\sum_{S \subseteq E} (-1)^{|S|} p_{\pi(S)} = \sum_{S \subseteq E} (-1)^{|S|} p_{1/2/\ldots/d} = \begin{cases} p_{1/2/\ldots/d} & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

This agrees with Eq. (8).

Now, if G has edges which are not loops, we use the Relabeling Proposition to obtain a labeling for G with $e = v_{d-1}v_d$. From the Deletion-Contraction Proposition we know that

$Y_G = Y_{G\backslash e} - Y_{G/e}\uparrow$ and by induction

$$Y_G = \sum_{S \subseteq E(G\backslash e)} (-1)^{|S|} p_{\pi(S)} - \sum_{\tilde{S} \subseteq E(G/e)} (-1)^{|\tilde{S}|} p_{\pi(\tilde{S})}\uparrow .$$

It should be clear that

$$\sum_{S \subseteq E(G\backslash e)} (-1)^{|S|} p_{\pi(S)} = \sum_{\substack{S \subseteq E(G) \\ e \notin S}} (-1)^{|S|} p_{\pi(S)}.$$

Hence it suffices to show that

$$- \sum_{\tilde{S} \subseteq E(G/e)} (-1)^{|\tilde{S}|} p_{\pi(\tilde{S})}\uparrow \;=\; \sum_{\substack{S \subseteq E(G) \\ e \in S}} (-1)^{|S|} p_{\pi(S)}. \tag{9}$$

To do so, we define a map $\Theta : \{\tilde{S} \subseteq E(G/e)\} \to \{S \subseteq E(G) : e \in S\}$ by

$$\Theta(\tilde{S}) = S, \text{ where } S = \tilde{S} \cup e.$$

Then, because of our conventions for contraction, Θ is a bijection. Clearly $\pi(S) = \pi(\tilde{S}) + (d)$ giving $p_{\pi(S)} = p_{\pi(\tilde{S})}\uparrow$. Furthermore $|S| = |\tilde{S}| + 1$ so Eq. (9) follows and this completes the proof. □

By letting the x_i commute, we get Stanley's Theorem 2.5 [11] as a corollary. Another result which we can obtain by this method is Stanley's generalization of Whitney's Broken Circuit Theorem.

A *circuit* is a closed walk, $v_1, v_2, \ldots, v_m, v_1$, with distinct vertices and edges. Note that since we permit loops and multiple edges, we can have $m = 1$ or 2. If we fix a total order on $E(G)$, a *broken circuit* is a circuit with its largest edge (with respect to the total order) removed. Let B_G denote the *broken circuit complex* of G, which is the set of all $S \subseteq E(G)$ which do *not* contain a broken circuit. Whitney's Broken Circuit Theorem states that the chromatic polynomial of a graph can be determined from its broken circuit complex. Before we prove our version of this theorem, however, we will need the following lemma, which appeared in the work of Blass and Sagan [1].

Lemma 3.7 *For any non-loop e, there is a bijection between B_G and $B_{G\backslash e} \cup B_{G/e}$ given by*

$$S \longrightarrow \begin{cases} \tilde{S} = S - e \in B_{G/e} & \text{if } e \in S \\ \tilde{S} = S \in B_{G\backslash e} & \text{if } e \notin S, \end{cases}$$

where we take e to be the first edge of G in the total order on the edges. ■

Using this lemma, we can now obtain a characterization of Y_G in terms of the broken circuit complex of G for any fixed total ordering on the edges.

Theorem 3.8 *We have*

$$Y_G = \sum_{S \in B_G} (-1)^{|S|} p_{\pi(S)},$$

where $\pi(S)$ *is as in Theorem* 3.6.

Proof: We again induct on the number of non-loops in $E(G)$. If the edge set consists only of n loops, it should be clear that for $n > 0$ we will have every edge being a circuit, and so the empty set is a broken circuit. Thus we have

$$Y_G = \begin{cases} \sum_{S \in \{\phi\}} (-1)^{|S|} p_{\pi(S)} = p_{1/2/\ldots/d} & \text{if } n = 0, \\ \sum_{S \in \phi} (-1)^{|S|} p_{\pi(S)} = 0 & \text{if } n > 0, \end{cases}$$

which matches Eq. (8).

For $n > 0$ and e a non-loop, we consider $Y_G = Y_{G\backslash e} - Y_{G/e}\uparrow$, and again apply induction. From Lemma 3.7 and arguments as in Proposition 3.6, we have

$$\sum_{\substack{S \in B_G \\ e \notin S}} (-1)^S p_{\pi(S)} = \sum_{S \in B_{G\backslash e}} (-1)^S p_{\pi(S)}$$

and

$$\sum_{\substack{S \in B_G \\ e \in S}} (-1)^{|S|} p_{\pi(S)} = -\sum_{\tilde{S} \in B_{G/e}} (-1)^{|\tilde{S}|} p_{\pi(\tilde{S})}\uparrow,$$

which gives the result. □

4. Acyclic orientations

An *orientation* of G is a digraph obtained by assigning a direction to each of its edges. The orientation is *acyclic* if it contains no circuits. A *sink* of an orientation is a vertex v_0 such that every edge of G containing it is oriented towards v_0. There are some interesting results which relate the chromatic polynomial of a graph to the number of acyclic orientations of the graph and the sinks of these orientations. The one which is the main motivation for this section is the following theorem of Greene and Zaszlavsky [8]. To state it, we adopt the convention that the coefficient of n^i in $\mathcal{X}$ is a_i.

Theorem 4.1 *For any fixed vertex* v_0, *the number of acyclic orientations of* G *with a unique sink at* v_0 *is* $|a_1|$.

The original proof of this theorem uses the theory of hyperplane arrangements. For elementary bijective proofs, see [6]. Stanley [11] has a related result.

Theorem 4.2 *If $X_G = \sum_\lambda c_\lambda e_\lambda$, then the number of acyclic orientations of G with j sinks is given by $\sum_{l(\lambda)=j} c_\lambda$.*

We can prove an analogue of this theorem in the noncommutative setting by using techniques similar to his, but have not been able to do so using induction. We can, however, inductively demonstrate a related result which, unlike Theorem 4.2 implies Theorem 4.1 For this result we need a lemma from [6]. To state it, we denote the set of acyclic orientations of G by $\mathcal{A}(G)$, and the set of acyclic orientations of G with a unique sink at v_0 by $\mathcal{A}(G, v_0)$. For completeness, we provide a proof.

Lemma 4.3 *For any fixed vertex v_0, and any edge $e = uv_0$, $u \neq v_0$, the map*

$$D \longrightarrow \begin{cases} D\backslash e \in \mathcal{A}(G\backslash e, v_0) & \text{if } D\backslash e \in \mathcal{A}(G\backslash e, v_0) \\ D/e \in \mathcal{A}(G/e, v_0) & \text{if } D\backslash e \notin \mathcal{A}(G\backslash e, v_0), \end{cases}$$

is a bijection between $\mathcal{A}(G, v_0)$ and $\mathcal{A}(G\backslash e, v_0) \uplus \mathcal{A}(G/e, v_0)$, where the vertex of G/e formed by contracting e is labeled v_0.

Proof: We must first show that this map is well-defined, i.e., that in both cases we obtain an acyclic orientation with unique sink at v_0. This is true in the first case by definition. In case two, where $D\backslash e \notin \mathcal{A}(G\backslash e, v_0)$, it must be true that $D\backslash e$ has sinks both at u and at v_0 (since deleting a directed edge of D will neither disturb the acyclicity of the orientation nor cause the sink at v_0 to be lost). Since u and v_0 are the only sinks in D/uv_0 the contraction must have a unique sink at v_0, and there will be no cycles formed. Thus the orientation D/e will be in $\mathcal{A}(G/e, v_0)$ and this map is well-defined.

To see that this is a bijection, we exhibit the inverse. This is obtained by simply orienting all edges of G as in $D\backslash uv_0$ or D/uv_0 as appropriate, and then adding in the oriented edge $\overrightarrow{uv_0}$. Clearly this map is also well-defined. □

We can now apply deletion-contraction to obtain a noncommutative version of Theorem 4.1.

Theorem 4.4 *Let $Y_G = \sum_{\pi \in \Pi_d} c_\pi e_\pi$. Then for any fixed vertex, v_0, the number of acyclic orientations of G with a unique sink at v_0 is $(d-1)!c_{[d]}$.*

Proof: We again induct on the number of non-loops in G. In the base case, if all the edges of G are loops, then

$$Y_G = \begin{cases} e_{1/2/\ldots/d} & \text{if } G \text{ has no edges} \\ 0 & \text{if } G \text{ has loops.} \end{cases}$$

So

$$c_{[d]} = \left\{ \begin{matrix} 1 & \text{if } G = K_1 \\ 0 & \text{if } d > 1 \text{ or } G \text{ has loops} \end{matrix} \right\} = |\mathcal{A}(G, v_0)|.$$

If G has non-loops, then by the Relabeling Proposition we may let $e = v_{d-1}v_d$. We know that $Y_G = Y_{G\backslash e} - Y_{G/e}\uparrow$. Since we will only be interested in the leading coefficient, let

$$\begin{aligned} Y_G &= ae_{[d]} + \sum_{\sigma<[d]} a_\sigma e_\sigma, \\ Y_{G\backslash e} &= be_{[d]} + \sum_{\sigma<[d]} b_\sigma e_\sigma, \end{aligned}$$

and

$$Y_{G/e} = ce_{[d-1]} + \sum_{\sigma<[d-1]} c_\sigma e_\sigma$$

where $\leq$ is the partial order on set partitions. By induction and Lemma 4.3, it is enough to show that $(d-1)!a = (d-1)!b + (d-2)!c$.

Utilizing the change-of-basis formulae (6) and (7) as well as the fact that for $\pi \in \Pi_{d-1}$ we have $p_\pi\uparrow = p_{\pi+(d)}$, we obtain

$$e_\pi\uparrow = \sum_{\sigma\leq\pi} \frac{\mu(\hat{0}, \sigma)}{\mu(\hat{0}, \sigma + (d))} \sum_{\tau\leq\sigma+(d)} \mu(\tau, \sigma + (d))e_\tau. \tag{10}$$

With this formula, we compute the coefficient of $e_{[d]}$ from $Y_{G/e}\uparrow$. The only term which contributes comes from $ce_{[d-1]}\uparrow$, which gives us

$$\begin{aligned} ce_{[d-1]}\uparrow &= c \sum_{\sigma\in\Pi_{d-1}} \frac{\mu(\hat{0}, \sigma)}{\mu(\hat{0}, \sigma + (d))} \sum_{\tau\leq\sigma+(d)} \mu(\tau, \sigma + (d))e_\tau \\ &= c\frac{\mu(\hat{0}, [d-1])}{\mu(\hat{0}, [d])} e_{[d]} + \sum_{\tau<[d]} d_\tau e_\tau \\ &= \frac{-c}{d-1} e_{[d]} + \sum_{\tau<[d]} d_\tau e_\tau \end{aligned}$$

Thus, from $Y_G = Y_{G\backslash e} - Y_{G/e}\uparrow$ we have that

$$\begin{aligned} (d-1)!a &= (d-1)!b + (d-1)!\frac{c}{d-1} \\ &= (d-1)!b + (d-2)!c, \end{aligned}$$

which completes the proof. $\square$

This result implies Theorem [4.1] since under the specialization $x_1 = x_2 = \cdots = x_n = 1$, and $x_i = 0$ for $i > n$, e_π becomes

$$\prod_{i=1}^{k} n(n-1)(n-2)\cdots(n-|B_i|+1)$$

where $\pi = B_1/B_2/\ldots/B_k$. So if $k \geq 2$ this polynomial in n is divisible by n^2. Thus the only summand contributing to the linear term of $\mathcal{X}_G(n)$ is when $\pi = [d]$ and in that case the coefficient has absolute value $(d-1)!c_{[d]}$.

The next corollary follows easily from the previous result.

Corollary 4.5 *If $Y_G = \sum_{\pi\in\Pi_d} c_\pi e_\pi$, then the number of acyclic orientations of G with one sink is $d!c_{[d]}$.* ∎

5. Inducing e_π

We now turn our attention to the expansion of Y_G in terms of the elementary symmetric function basis. We recall that for any fixed $\pi \in \Pi_d$ we use $\pi + (d+1)$ to denote the partition of $[d+1]$ formed by inserting the element $(d+1)$ into the block of π which contains d. We will denote the block of π which contains d by B_π. We also let $\pi/d+1$ be the partition of $[d+1]$ formed by adding the block $\{d+1\}$ to π.

It is necessary for us to understand the coefficients arising in $e_\pi\uparrow$ if we want to understand the coefficients of Y_G which occur in its expansion in terms of the elementary symmetric function basis. We have seen in Eq. (10) that the expression for $e_\pi\uparrow$ is rather complicated. However, if the terms in the expression of $e_\pi\uparrow$ are grouped properly, the coefficients in many of the groups will sum to zero. Specifically, we need to combine the coefficients from set partitions which are of the same type (as integer partitions), and whose block containing $d+1$ have the same size. Keeping track of the size of the block containing $d+1$ will allow us to use deletion-contraction repeatedly. To do this formally, we introduce a bit of notation. Suppose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ is a *composition*, i.e., an ordered integer partition. Let $P(\alpha)$ be the set of all partitions $\tau = B_1/B_2/\ldots/B_l$ of $[d+1]$ such that

1. $\tau \leq \pi + (d+1)$,
2. $|B_i| = \alpha_i$ for $1 \leq i \leq l$, and
3. $d+1 \in B_1$.

The proper grouping for the terms of $e_\pi\uparrow$ is given by the following lemma.

Lemma 5.1 *If $e_\pi\uparrow = \sum_{\tau\in\Pi_{d+1}} c_\tau e_\tau$, then $c_\tau = 0$ unless $\tau \leq \pi + (d+1)$, and for any composition α, we have*

$$\sum_{\tau\in P(\alpha)} c_\tau = \begin{cases} 1/|B_\pi| & \text{if } P(\alpha) = \{\pi/d+1\}, \\ -1/|B_\pi| & \text{if } P(\alpha) = \{\pi+(d+1)\}, \\ 0 & \text{else.} \end{cases}$$

Proof: Fix $\pi \in \Pi_d$. By Eq. (10)

$$e_\pi\uparrow = \sum_{\sigma \le \pi} \frac{\mu(\hat{0}, \sigma)}{\mu(\hat{0}, \sigma + (d+1))} \sum_{\tau \le \sigma + (d+1)} \mu(\tau, \sigma + (d+1)) e_\tau.$$

Hence we may express

$$e_\pi\uparrow = \sum_{\tau \le \pi + (d+1)} c_\tau e_\tau,$$

where for any fixed $\tau \le \pi + (d+1)$ we have

$$c_\tau = \sum_{\substack{\sigma \le \pi \\ \sigma + (d+1) \ge \tau}} \frac{-1}{|B_\sigma|} \mu\left(\tau, \sigma + (d+1)\right). \tag{11}$$

We first note that if $\tau = \pi/d+1 \in P(\alpha)$, then $|P(\alpha)| = 1$ and we have the interval $[\tau, \pi + (d+1)] \cong \Pi_2$. A simple computation shows that $c_{\pi/d+1} = 1/|B_\pi|$. Similarly, if $\tau = \pi + (d+1) \in P(\alpha)$, then again $|P(\alpha)| = 1$ and we can easily compute $c_{\pi+(d+1)} = -1/|B_\pi|$.

We now fix $\tau = B_1/B_2/\cdots/B_{q+2}/\cdots/B_l \in P(\alpha)$ and without loss of generality we can let $B_1, B_2, \cdots, B_{q+2}$ where $q \ge -1$ be the blocks of τ which are contained in $B_{\pi+(d+1)}$. For notational convenience, we will also let $|B_{\pi+(d+1)}| = m+1$, where $m \ge 1$. Finally, let β denote the partition obtained from τ by merging the blocks of τ which contain d and $d+1$, allowing $\beta = \tau$ if d and $d+1$ are in the same block of τ. Replacing $\sigma + (d+1)$ by $\sigma \in \Pi_{d+1}$ in Eq. (11), we see that

$$c_\tau = \sum_{\beta \le \sigma \le \pi + (d+1)} \frac{-1}{|B_\sigma| - 1} \mu(\tau, \sigma).$$

Now for any $B \subseteq [d+1]$ we will consider the sets

$$L(B) = \{\sigma \in \Pi_{d+1} : \{d, d+1\} \subseteq B \in \sigma, \text{ where } \beta \le \sigma \le \pi + (d+1)\}.$$

The nonempty $L(B)$ partition the interval $[\beta, \pi + (d+1)]$ according to the content of the block containing $\{d, d+1\}$ and so we may express

$$c_\tau = \sum_B \frac{-1}{|B| - 1} \sum_{\sigma \in L(B)} \mu(\tau, \sigma).$$

To compute the inner sum, we need to consider the following 2 cases.

Case (1) For some $k > q+2$, B_k is strictly contained in a block of $\pi + (d+1)$. In this case, we see that each non-empty $L(B)$ forms a non-trivial cross-section of a product of partition lattices, and so for this case

$$\sum_{\sigma \in L(B)} \mu(\tau, \sigma) = 0.$$

Thus these τ will not contribute to $\displaystyle\sum_{\tau \in P(\alpha)} c_\tau$.

Case (2) For all $k > q + 2$, B_k is a block of $\pi + (d + 1)$. So, by abuse of notation, we can write $\tau = B_1/\cdots/B_{q+2}$ and $P(\alpha) = P(\alpha_1, \ldots, \alpha_{q+2})$. Also in this case, we can assume $q \geq 0$, since we have already computed this sum when $\tau = \pi + (d + 1)$. Then we will show

$$\frac{1}{|B| - 1} \sum_{\sigma \in L(B)} \mu(\tau, \sigma) = \begin{cases} \dfrac{(-1)^{q+1}(q+1)!}{m} & \text{if } B = B_{\pi+(d+1)}, \\ \dfrac{(-1)^q q!}{m - \alpha_i} & \text{if } B = B_{\pi+(d+1)} \backslash B_i,\, 2 \leq i \leq q+2, \\ 0 & \text{else.} \end{cases} \tag{12}$$

Indeed, it is easy to see that if $B = B_{\pi+(d+1)}$ then $L(B) = \{\pi + (d + 1)\}$ and so this part is clear. Also, if $B = B_{\pi+(d+1)} \setminus B_i$ for some $2 \leq i \leq q + 2$, then we have $|L(B)| = 1$ again and $\sum_{\sigma \in L(B)} \mu(\tau, \sigma) = (-1)^q q!$. Otherwise, $L(B)$ again forms a non-trivial cross-section of a product of partition lattices, and again gives us no net contribution to the sum.

We notice that since $\{d, d + 1\} \subseteq B$, the second case in (12) will only occur if $d \in B_j$ for $j \neq i$. Adding up all these contributing terms gives us

$$-c_\tau = (-1)^q q! \left(\sum_{\substack{i=2 \\ i \neq j}}^{q+2} \frac{1}{m - \alpha_i} - \frac{q+1}{m} \right).$$

In order to compute the sum over all $\tau \in P(\alpha)$, it will be convenient to consider all possible orderings for the block of τ containing d. So for $1 \leq j \leq q + 2$, let

$$P(\alpha, j) = \{(B_1, B_2, \ldots, B_{q+2}) \mid B_1/B_2/\cdots/B_{q+2} \in P(\alpha),\ d \in B_j\}.$$

The sequence $(B_1, B_2, \ldots, B_{q+2})$ forms the *ordered* set partition τ. We also define

$$\delta_i = \begin{cases} \alpha_i - 1 & \text{if } i = 1 \\ \alpha_i & \text{else,} \end{cases}$$

so

$$|P(\alpha, j)| = \binom{m - 1}{\delta_1, \ldots, \delta_j - 1, \ldots, \delta_{q+2}}.$$

Thus we can see that

$$-\sum_{\tau\in P(\alpha,j)} c_\tau = \binom{m-1}{\delta_1,\ldots,\delta_j-1,\ldots,\delta_{q+2}}(-1)^q q!\left(\sum_{\substack{i=2\\ i\neq j}}^{q+2}\frac{1}{m-\alpha_i}-\frac{q+1}{m}\right).$$

To obtain the sum over all $\tau \in P(\alpha)$ we need to sum over all $P(\alpha, j)$ for $1 \leq j \leq q+2$. However, if we let k_r be the number of blocks B_i, $1 \leq i \leq q+2$, which have size r, then in the sum over all $P(\alpha, j)$, each $\tau \in P(\alpha)$ appears $\Pi_{r=1}^{m+1} k_r!$ times. Combining all this information, we see that

$$-\sum_{\tau\in P(\alpha)} c_\tau = \frac{(-1)^q q!}{\prod_{r=1}^{m+1} k_r!}\sum_{j=1}^{q+2}\binom{m-1}{\delta_1,\ldots,\delta_j-1,\ldots,\delta_{q+2}}\left(\sum_{\substack{i=2\\ i\neq j}}^{q+2}\frac{1}{m-\delta_i}-\frac{q+1}{m}\right).$$

Hence it suffices to show that

$$\sum_{j=1}^{q+2}\binom{m-1}{\delta_1,\ldots,\delta_j-1,\ldots,\delta_{q+2}}\left(\sum_{\substack{i=2\\ i\neq j}}^{q+2}\frac{1}{m-\delta_i}-\frac{q+1}{m}\right)=0.$$

Using the multinomial recurrence we have,

$$\sum_{j=1}^{q+2}\binom{m-1}{\delta_1,\ldots,\delta_j-1,\ldots,\delta_{q+2}}=\binom{m}{\delta_1,\ldots,\delta_j,\ldots,\delta_{q+2}}$$

and so we need only show that

$$\sum_{j=1}^{q+2}\binom{m-1}{\delta_1,\ldots,\delta_j-1,\ldots,\delta_{q+2}}\sum_{\substack{i=2\\ i\neq j}}^{q+2}\frac{1}{m-\delta_i}=\frac{q+1}{m}\binom{m}{\delta_1,\ldots,\delta_j,\ldots,\delta_{q+2}}.$$

However, we may express

$$\sum_{j=1}^{q+2}\binom{m-1}{\delta_1,\ldots,\delta_j-1,\ldots,\delta_{q+2}}\sum_{\substack{i=2\\ i\neq j}}^{q+2}\frac{1}{m-\delta_i}$$
$$=\sum_{j=1}^{q+2}\frac{\binom{m}{\delta_1,\ldots,\delta_j,\ldots,\delta_{q+2}}\delta_j}{m}\sum_{\substack{i=2\\ i\neq j}}^{q+2}\frac{1}{m-\delta_i}=\frac{\binom{m}{\delta_1,\ldots,\delta_j,\ldots,\delta_{q+2}}}{m}\sum_{j=1}^{q+2}\sum_{\substack{i=2\\ i\neq j}}^{q+2}\frac{\delta_j}{m-\delta_i}$$

$$= \frac{\binom{m}{\delta_1,\ldots,\delta_j,\ldots,\delta_{q+2}}}{m} \sum_{i=2}^{q+2} \frac{1}{m-\delta_i} \sum_{\substack{j=1\\ j\neq i}}^{q+2} \delta_j = \frac{\binom{m}{\delta_1,\ldots,\delta_j,\ldots,\delta_{q+2}}}{m} \sum_{i=2}^{q+2} \frac{1}{m-\delta_i}(m-\delta_i)$$

$$= \frac{q+1}{m}\binom{m}{\delta_1,\ldots,\delta_j,\ldots,\delta_{q+2}}. \qquad \square$$

6. Some *e*-positivity results

We wish to use Lemma 5.1 to prove some positivity theorems about Y_G's expansion in the elementary symmetric function basis. If the coefficients of the elementary symmetric functions in this expansion are all non-negative, then we say that Y_G is *e-positive*. Unfortunately, even for some of the simplest graphs, Y_G is usually not $e-$positive. The only graphs which are obviously $e-$positive are the complete graphs on n vertices and their complements, for which we have $Y_{K_n} = e_{[n]}$ and $Y_{\overline{K_n}} = e_{1/2/\cdots/n}$. Even paths, with the vertices labeled sequentially, are not $e-$positive, for we can compute that $Y_{P_3} = \frac{1}{2}e_{12/3} - \frac{1}{2}e_{13/2} + \frac{1}{2}e_{1/23} + \frac{1}{2}e_{123}$. However, in this example we can see that while Y_{P_3} is not $e-$positive, if we identify all the terms having the same type and the same size block containing 3, the sum will be non-negative for each of these sets.

This observation along with the proof of the previous lemma inspires us to define equivalence classes reflecting the sets $P(\alpha)$. If the block of σ containing i is $B_{\sigma,i}$ and the block of τ containing i is $B_{\tau,i}$, we define

$$\sigma \equiv_i \tau \text{ iff } \lambda(\sigma) = \lambda(\tau) \quad \text{and} \quad |B_{\sigma,i}| = |B_{\tau,i}|$$

and extend this definition so that

$$e_\sigma \equiv_i e_\tau \quad \text{iff } \sigma \equiv_i \tau.$$

We let (τ) and $e_{(\tau)}$ denote the equivalence classes of τ and e_τ, respectively. Taking formal sums of these equivalence classes allows us to write expressions such as

$$\sum_{\sigma\in\Pi_d} c_\sigma e_\sigma \equiv_i \sum_{(\tau)\subseteq\Pi_d} c_{(\tau)} e_{(\tau)} \quad \text{where } c_{(\tau)} = \sum_{\sigma\in(\tau)} c_\sigma.$$

We will refer to this equivalence relation as *congruence modulo i*.

Using this notation, we have $Y_{P_3} \equiv_3 \frac{1}{2}e_{(12/3)} + \frac{1}{2}e_{(123)}$, since $e_{13/2} \equiv_3 e_{1/23}$. We will say that a labeled graph G (and similarly Y_G) is $(e)-$*positive* if all the $c_{(\tau)}$ are non-negative for some labeling of G and suitably chosen congruence. We notice that the expansion of Y_G for a labeled graph may have all non-negative amalgamated coefficients for congruence modulo i, but not for congruence modulo j. However, if a different labeling for an (e)-positive graph is chosen, then we can always find a corresponding congruence class to again see (e)-positivity. This should be clear from the Relabeling Proposition.

We now turn our attention to showing that paths, cycles, and complete graphs with one edge deleted are all (e)-positive. We begin with a few more preliminary results about this congruence relation and how it affects our induction of e_π.

We note that in the proof of Lemma 5.1, the roles played by the elements d and $d+1$ are essentially interchangeable. That is, if we let $\tilde{P}(\alpha)$ be the set of all partitions $\tau = B_1/B_2/\cdots/B_l$ of $[d+1]$ such that

1. $\tau \leq \pi + (d+1)$,
2. $|B_i| = \alpha_i$ for $1 \leq i \leq l$, and
3. $d \in B_1$,

and let $\tilde{\pi}$ be the partition $\pi \in \Pi_d$ with d replaced by $d+1$, then the same proof will show that

$$\sum_{\tau \in \tilde{P}(\alpha)} c_\tau = \begin{cases} 1/|B_\pi| & \text{if } \tilde{P}(\alpha) = \{\tilde{\pi}/d\}, \\ -1/|B_\pi| & \text{if } \tilde{P}(\alpha) = \{\tilde{\pi} + (d)\}, \\ 0 & \quad\text{otherwise.} \end{cases}$$

Note that here $\tilde{\pi} + (d)$ is the partition obtained from $\tilde{\pi}$ by inserting the element d into the block of $\tilde{\pi}$ containing $d+1$. This allows us to state a corollary in terms of the congruence relationship just defined.

Corollary 6.1 *If $b = |B_\pi|$, then for any $\pi \in \Pi_d$, we have*

$$e_\pi\uparrow\ \equiv_{d+1} \frac{1}{b} e_{(\pi/d+1)} - \frac{1}{b} e_{(\pi+(d+1))}$$

and

$$e_\pi\uparrow\ \equiv_d \frac{1}{b} e_{(\tilde{\pi}/d)} - \frac{1}{b} e_{(\tilde{\pi}+(d))}.$$

The next lemma simply verifies that the induction operation respects the congruence relation and follows immediately from Eq. (10) or the previous corollary.

Lemma 6.2 *If $e_\gamma \equiv_d e_\tau$, then $e_\gamma\uparrow\ \equiv_{d+1} e_\tau\uparrow$.*

From this we can extend induction to congruence classes in a well-defined manner:

$$\text{if}\quad e_\pi\uparrow\ = \sum_{\tau \in \Pi_{d+1}} c_\tau e_\tau \quad\text{then}\quad e_{(\pi)}\uparrow\ \equiv_{d+1} \sum_{(\tau) \subseteq \Pi_{d+1}} c_{(\tau)} e_{(\tau)}.$$

In order to use induction to prove the (e)-positivity of a graph G, we will usually try to delete a set of edges which will isolate either a single vertex or a complete graph from G in the hope of obtaining a simpler (e)-positive graph. In order to see how this procedure will affect Y_G, we use the following lemma.

Lemma 6.3 *Given a graph, G on d vertices let $H = G \uplus K_m$ where the vertices in K_m are labeled $v_{d+1}, v_{d+2}, \ldots, v_{d+m}$. If $Y_G = \sum_{\sigma \in \Pi_d} c_\sigma e_\sigma$, then $Y_H = \sum_{\sigma \in \Pi_d} c_\sigma e_{\sigma/d+1,d+2,\ldots,d+m}$.*

Proof: From the labeling of H we have

$$\begin{aligned} Y_H &= Y_G e_{[m]} \\ &= \sum_{\sigma \in \Pi_d} c_\sigma e_\sigma e_{[m]} \\ &= \sum_{\sigma \in \Pi_d} c_\sigma e_{\sigma/d+1,d+2,\ldots,d+m}. \end{aligned}$$ □

This result suggests we use the natural notation G/v_{d+1} for the graph $G \uplus \{v_{d+1}\}$. We are now in a position to prove the $(e)-$positivity of paths.

Proposition 6.4 *For all $d \geq 1$, Y_{P_d} is $(e)-$positive.*

Proof: We proceed by induction, having labeled P_d so that the edge set is $E(P_d) = \{v_1 v_2, v_2 v_3, \ldots, v_{d-1} v_d\}$. If $d = 1$, then we have $Y_{P_1} = e_1$ and the proposition is clearly true.

So we assume by induction that

$$Y_{P_d} \equiv_d \sum_{(\tau) \subseteq \Pi_d} c_{(\tau)} e_{(\tau)},$$

where $c_{(\tau)} \geq 0$ for all $(\tau) \in \Pi_d$. From the Deletion-Contraction Recurrence applied to $e = v_d v_{d+1}$, Corollary 6.1 and Lemma 6.3, we see that

$$\begin{aligned} Y_{P_{d+1}} &= Y_{P_d/v_{d+1}} - Y_{P_d}\uparrow \\ &\equiv_{d+1} \sum_{(\tau) \subseteq \Pi_d} c_{(\tau)} e_{(\tau/d+1)} - \sum_{(\tau) \subseteq \Pi_d} c_{(\tau)} e_{(\tau)}\uparrow \\ &\equiv_{d+1} \sum_{(\tau) \subseteq \Pi_d} c_{(\tau)} \left(1 - \frac{1}{|B_\tau|}\right) e_{(\tau/d+1)} + \sum_{(\tau) \subseteq \Pi_d} \frac{c_{(\tau)}}{|B_\tau|} e_{(\tau+(d+1))}. \end{aligned}$$

Since we know that $c_{(\tau)} \geq 0$, and $|B_\tau| \geq 1$ for all τ, this completes the induction step and the proof. □

In the commutative context we will say that the symmetric function X_G is *e-positive* if all the coefficients in the expansion of the elementary symmetric functions are non-negative. Clearly (e)-positivity results for Y_G specialize to e-positivity results for X_G.

Corollary 6.5 *X_{P_d} is $e-$positive.*

One would expect the (e)-expansions for cycles and paths to be related as is shown by the next proposition. For labeling purposes, however, we first need a lemma which follows easily from the Relabeling Proposition.

Lemma 6.6 *If $\gamma \in \mathcal{S}_d$ fixes d, then $Y_{\gamma(G)} \equiv_d Y_G$.*

Proposition 6.7 *For all $d \geq 1$, if*

$$Y_{P_d} \equiv_d \sum_{(\tau)} c_{(\tau)} e_{(\tau)}, \text{ then } Y_{C_{d+1}} \equiv_{d+1} \sum_{(\tau)} c_{(\tau)} e_{(\tau+(d+1))},$$

where we have labeled the graphs so $E(P_d) = \{v_1v_2, v_2v_3, \ldots, v_{d-1}v_d\}$ and $E(C_{d+1}) = \{v_1v_2, v_2v_3, \ldots, v_{d-1}v_d, v_dv_{d+1}, v_{d+1}v_1\}$.

Proof: We proceed by induction on d. If $d = 1$, then $Y_{P_1} = e_{[1]}$ and $Y_{C_2} = e_{[2]}$, so the proposition holds for $d = 1$.

For the induction step, we assume that

$$Y_{P_{d-1}} \equiv_{d-1} \sum_{(\tau)} c_{(\tau)} e_{(\tau)}$$

and also that

$$Y_{C_d} \equiv_d \sum_{(\tau)} c_{(\tau)} e_{(\tau+(d))}.$$

We notice that if $e = v_dv_{d+1}$, then $C_{d+1} \backslash e$ does not have the standard labeling for paths. But if we let $\gamma = (d+1)(1, d)(2, d-1) \cdots (\lfloor \frac{d+1}{2} \rfloor, \lceil \frac{d+1}{2} \rceil)$ then we can use the Deletion-Contraction Recurrence to get

$$Y_{C_{d+1}} = Y_{\gamma(P_{d+1})} - Y_{C_d}\uparrow .$$

However, since $d + 1$ is a fixed point for γ, Lemma 6.6 allows us to deduce that

$$Y_{C_{d+1}} \equiv_{d+1} Y_{P_{d+1}} - Y_{C_d}\uparrow .$$

In the proof of Proposition 6.4 we saw that

$$Y_{P_{d+1}} = Y_{P_d/v_{d+1}} - Y_{P_d}\uparrow .$$

Combining these two equations gives

$$Y_{C_{d+1}} \equiv_{d+1} Y_{P_d/v_{d+1}} - Y_{P_d}\uparrow - Y_{C_d}\uparrow . \tag{13}$$

The demonstration of Proposition 6.4 also showed us that

$$Y_{P_d} \equiv_d \sum_{(\tau)} \left(\left(c_{(\tau)} - \frac{c_{(\tau)}}{|B_\tau|} \right) e_{(\tau/d)} + \frac{c_{(\tau)}}{|B_\tau|} e_{(\tau+(d))} \right). \tag{14}$$

Applying Corollary 6.1 and Lemma 6.3 yields

$$\begin{aligned} Y_{P_d}\uparrow \equiv_{d+1} \sum_{(\tau)} \Bigg[& \left(c_{(\tau)} - \frac{c_{(\tau)}}{|B_\tau|} \right) e_{(\tau/d/d+1)} - \left(c_{(\tau)} - \frac{c_{(\tau)}}{|B_\tau|} \right) e_{(\tau/d,d+1)} \\ & + \frac{c_{(\tau)}}{|B_\tau|(|B_\tau|+1)} e_{(\tau+(d)/d+1)} - \frac{c_{(\tau)}}{|B_\tau|(|B_\tau|+1)} e_{(\tau+(d)+(d+1))} \Bigg] \end{aligned}$$

and

$$Y_{P_d/v_{d+1}} \equiv_{d+1} \sum_{(\tau)} \left(c_{(\tau)} - \frac{c_{(\tau)}}{|B_\tau|} \right) e_{(\tau/d/d+1)} + \frac{c_{(\tau)}}{|B_\tau|} e_{(\tau+(d)/d+1)}.$$

By the induction hypothesis,

$$\begin{aligned} Y_{C_d}\uparrow &\equiv_{d+1} \sum_{(\tau)} c_{(\tau)} e_{(\tau+(d))}\uparrow \\ &\equiv_{d+1} \sum_{(\tau)} \left(\frac{c_{(\tau)}}{|B_\tau|+1} e_{(\tau+(d)/d+1)} - \frac{c_{(\tau)}}{|B_\tau|+1} e_{(\tau+(d)+(d+1))} \right). \end{aligned}$$

Plugging these expressions for $Y_{P_d/v_{d+1}}$, $Y_{P_d}\uparrow$, and $Y_{C_d}\uparrow$ into Eq. (13), grouping the terms according to type, and simplifying gives

$$Y_{C_{d+1}} \equiv_{d+1} \sum_{(\tau)} \left(c_{(\tau)} - \frac{c_{(\tau)}}{|B_\tau|} \right) e_{(\tau/d,d+1)} + \frac{c_{(\tau)}}{|B_\tau|} e_{(\tau+(d)+(d+1))}.$$

This corresponds to the expression in Eq. (14) for Y_{P_d} in exactly the desired manner, and so we are done. □

From the previous proposition and the fact that $Y_{C_1} = 0$ we get an immediate corollaries.

Proposition 6.8 *For all $d \geq 1$, Y_{C_d} is (e)−positive.*

Corollary 6.9 *For all $d \geq 1$, X_{C_d} is e−positive.*

We are also able to use our recurrence to show the (e)-positivity of complete graphs with one edge removed.

Proposition 6.10 *For $d \geq 2$, if $e = v_{d-1}v_d$ then*

$$Y_{K_d \backslash e} \equiv_d \frac{d-2}{d-1} e_{([d])} + \frac{1}{d-1} e_{([d-1]/d)}.$$

Proof: Consider the complete graph K_d and apply deletion-contraction to the edge $e = v_{d-1}v_d$. Together with Corollary 6.1 this will give us

$$\begin{aligned} e_{[d]} &= Y_{K_d} \\ &= Y_{K_d \backslash e} - Y_{K_{d-1}}\uparrow \\ &= Y_{K_d \backslash e} - e_{[d-1]}\uparrow \\ &\equiv_d Y_{K_d \backslash e} - \frac{1}{d-1} e_{([d-1]/d)} + \frac{1}{d-1} e_{([d])}. \end{aligned}$$

Simplifying gives the result. □

This also immediately specializes.

Corollary 6.11 *For $d \geq 2$,*

$$X_{K_d \backslash e} = d(d-2)(d-2)! e_d + (d-2)! e_{(d-1,1)}.$$

7. The $(\mathbf{3}+\mathbf{1})$-free Conjecture

One of our original goals in setting up this inductive machinery was to make progress on the $(\mathbf{3}+\mathbf{1})$-free Conjecture of Stanley and Stembridge, which we now state. Let $\mathbf{a}+\mathbf{b}$ be the poset which is a disjoint union of an a-element chain and a b-element chain. The poset P is said to be *$(\mathbf{a}+\mathbf{b})$-free* if it contains no induced subposet isomorphic to $\mathbf{a}+\mathbf{b}$. Let $G(P)$ denote the incomparability graph of P whose vertices are the elements of P with an edge uv whenever u and v are incomparable in P. The $(\mathbf{3}+\mathbf{1})$-free Conjecture of Stanley and Stembridge [13] states:

Conjecture 7.1 If P is $(\mathbf{3}+\mathbf{1})$-free, then $X_{G(P)}$ is e-positive.

Gasharov [4] has demonstrated the weaker result that $X_{G(P)}$ is s-positive, where s refers to the Schur functions.

A subset of the $(\mathbf{3}+\mathbf{1})$-free graphs is the class of *indifference graphs*. They are characterized [12] as having vertices and edges

$$V = \{v_1, \ldots, v_d\} \quad \text{and} \quad E = \{v_i v_j : i, j \text{ belong to some } [k, l] \in \mathcal{C}\},$$

where $\mathcal{C}$ is a collection of intervals $[k, l] = \{k, k+1, \ldots, l\} \subseteq [d]$. We note that without loss of generality, we can assume no interval in the collection is properly contained in any other. These correspond to incomparability graphs of posets which are both $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free.

Indifference graphs have a nice inductive structure that should make it possible to apply our deletion-contraction techniques. Although we have not been able to do this for the full family, we are able to resolve a special case. For any composition of $n, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, let $\tilde{\alpha_i} = \sum_{j \le i} \alpha_j$. A K_α*-chain* is the indifference graph using the collection of intervals $\{[1, \tilde{\alpha_1}], [\tilde{\alpha_1}, \tilde{\alpha_2}], \ldots, [\tilde{\alpha_{k-1}}, \tilde{\alpha_k}]\}$. This is just a string of complete graphs, whose sizes are given by the parts of α, which are attached to one another sequentially at single vertices. We notice that the K_α-chain for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ can be obtained from the K_τ-chain for $\tau = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1})$ by attaching the graph K_{α_k} to its last vertex.

We will be able to handle this type of attachment for any graph G with vertices $\{v_1, v_2, \ldots, v_d\}$. Hence, we define $G + K_m$ to be the graph with

$$V(G + K_m) = V(G) \cup \{v_{d+1}, \ldots, v_{d+m-1}\}$$

and

$$E(G + K_m) = E(G) \cup \{e = v_i v_j : i, j \in [d, d + m - 1]\}.$$

Using deletion-contraction techniques, we are able to exhibit the relationship between the (e)-expansion of $G + K_m$ and the (e)-expansion of G. However, we will also need some more notation. For $\pi \in \Pi_d$, we let $\pi + i$ denote the partition given by π with the additional i elements $d+1, d+2, \ldots, d+i$ added to B_π. This is in contrast to $\pi + (i)$, which denotes the partition given by π with the element i inserted into B_π. We denote the falling factorial by

$$\langle m \rangle_i \stackrel{\text{def}}{=} m(m-1)\cdots(m-i+1)$$

and the rising factorial by

$$(m)_i \stackrel{\text{def}}{=} m(m+1)\cdots(m+i-1).$$

We begin studying the behavior of $Y_{G+K_m}\uparrow_d^{d+j}$ with two lemmas the first of which follows easily from Eq. (10).

Lemma 7.2 *For any graph G on d vertices and* $1 \le i, j, k \le d + 1$ *we have* $Y_G \uparrow_i^j \equiv_i Y_G \uparrow_i^k$.

Lemma 7.3 *If G is a graph on d vertices with*

$$Y_G \equiv_d \sum_{(\pi) \subseteq \Pi_d} c_{(\pi)} e_{(\pi)},$$

then

$$Y_{G+K_m}\uparrow_d^{d+m} \equiv_{d+m} \sum_{(\pi)} \sum_{i=0}^{m-1} \frac{c_{(\pi)} \langle m-1\rangle_i \left[e_{(\pi+i/d+i+1,\ldots,d+m)} - e_{(\pi+i+(d+m)/d+i+1,\ldots,d+m-1)}\right]}{(b)_{i+1}},$$

where $b = |B_\pi|$.

Proof: We prove the lemma by induction on m. The case $m = 1$ is merely a restatement of Corollary 6.1. So we may assume this lemma is true for $Y_{G+K_m}\uparrow_d^{d+m}$, and proceed to prove it for $Y_{G+K_{m+1}}\uparrow_d^{d+m+1}$.

From Lemma 7.2, it follows that for $1 \leq j \leq m$, we have

$$Y_{G+K_m}\uparrow_d^{d+j}\uparrow_d^{d+m+1} \equiv_{d+m+1} Y_{G+K_m}\uparrow_d^{d+m}\uparrow_d^{d+m+1}.$$

Now, from $G + K_{m+1}$ we may delete the edge set $\{v_d v_{d+j} : 1 \leq j \leq m\}$ and combine all the terms $Y_G\uparrow_d^{d+j}\uparrow_d^{d+m+1}$ for $1 \leq j \leq m$ to obtain

$$\begin{aligned} Y_{G+K_{m+1}}\uparrow_d^{d+m+1} &\equiv_{d+m+1} Y_{G\uplus K_m}\uparrow_d^{d+m+1} - mY_{G+K_m}\uparrow_d^{d+m}\uparrow_d^{d+m+1} \\ &\equiv_{d+m+1} Y_{G\uplus K_m}\uparrow_d^{d+m+1} - mY_{G+K_m}\uparrow_d^{d+m}\uparrow_{d+m}^{d+m+1}. \end{aligned}$$

From this point on, we need only concern ourselves with the clerical details, making sure that everything matches up properly. We can see from Lemma 5.1, Lemma 6.3 and the original hypothesis on Y_G, that

$$Y_{G\uplus K_m}\uparrow_d^{d+m+1} \equiv_{d+m+1} \sum_{(\pi)} \frac{c_{(\pi)}}{b} \left(e_{(\pi_1)} - e_{(\pi_2)}\right). \tag{15}$$

where

$$\begin{aligned} &\pi_1 = \pi/d+1,\ldots,d+m/d+m+1, \\ &\pi_2 = \pi+(d+m+1)/d+1,\ldots,d+m. \end{aligned}$$

Similarly, the induction hypothesis shows

$$mY_{G+K_m}\uparrow_d^{d+m}\uparrow_{d+m}^{d+m+1} \equiv_{d+m+1} \sum_{(\pi)} \sum_{i=0}^{m-1} \frac{c_{(\pi)} m \langle m-1\rangle_i}{(b)_{i+1}} \left(\frac{e_{(\pi_3)} - e_{(\pi_4)}}{m-i} - \frac{e_{(\pi_5)} - e_{(\pi_6)}}{b+i+1}\right) \tag{16}$$

where

$$\begin{aligned} &\pi_3 = \pi+i/d+i+1,\ldots,d+m/d+m+1, \\ &\pi_4 = \pi+i/d+i+1,\ldots,d+m+1, \\ &\pi_5 = \pi+i+(d+m)/d+i+1,\ldots,d+m-1/d+m+1, \\ &\pi_6 = \pi+i+(d+m)+(d+m+1)/d+i+1,\ldots,d+m-1. \end{aligned}$$

Simplifying the terms and combining both Eqs. (15) and (16) gives

$$\begin{aligned}
&Y_{G+K_{m+1}}\uparrow_d^{d+m+1}\\
&\quad\equiv_{d+m+1} \sum_{(\pi)} c_{(\pi)} \left(\frac{e_{(\pi_1)} - e_{(\pi_2)}}{b} - \sum_{i=0}^{m-1} \frac{\left(e_{(\pi_3)} - e_{(\pi_4)}\right) \langle m\rangle_i}{(b)_{i+1}} \right.\\
&\qquad\qquad \left. + \sum_{i=0}^{m-1} \frac{\left(e_{(\pi_5)} - e_{(\pi_6)}\right) \langle m\rangle_{i+1}}{(b)_{i+2}} \right).
\end{aligned}$$

Note that modulo $d + m + 1$ we have

$$\begin{aligned}
&(\pi_5) = (\pi + i + 1/d + i + 2, \ldots, d + m/d + m + 1) \quad \text{and}\\
&(\pi_6) = (\pi + i + 1 + (d + m + 1)/d + i + 2, \ldots, d + m).
\end{aligned}$$

So by shifting indices and simplifying, we obtain

$$\begin{aligned}
&Y_{G+K_{m+1}}\uparrow_d^{d+m+1}\\
&\quad\equiv_{d+m+1} \sum_{(\pi)} \sum_{i=0}^{m} \frac{c_{(\pi)} \langle m\rangle_i \left[e_{(\pi+i/d+i+1,\ldots,d+m+1)} - e_{(\pi+i+(d+m+1)/d+i+1,\ldots,d+m)} \right]}{(b)_{i+1}},
\end{aligned}$$

which completes the induction step and the proof. □

This lemma is useful because it will help us to find an explicit formula for $Y_{G+K_{m+1}}$ in terms of Y_G. Once this formula is in hand, it will be easy to verify that if G is (e)-positive, then so is $G + K_{m+1}$. To complete the induction step in establishing this formula, we will need the following observation which follows from Eq. (10).

Lemma 7.4 *For any graph G on d vertices, and $\sigma \in \mathcal{S}_d$,*

$$Y_G\uparrow_i^{d+1} \equiv_{d+1} Y_{\sigma(G)}\uparrow_{\sigma(i)}^{d+1}.$$

We now give the formula for $Y_{G+K_{m+1}}$ in terms of Y_G.

Lemma 7.5 *If $m \geq 1$, and*

$$Y_G \equiv_d \sum_{(\pi)\subseteq\Pi_d} c_{(\pi)} e_{(\pi)},$$

then

$$Y_{G+K_{m+1}} \equiv_{d+m} \sum_{(\pi)\subseteq\Pi_d} \sum_{i=0}^{m-1} \frac{c_{(\pi)}\langle m-1\rangle_i}{(b)_{i+1}} \left[(b-m+i)e_{(\hat{\pi})} + (i+1)e_{(\bar{\pi})}\right]$$

where $b = |B_\pi|$ *and*

$$\hat{\pi} = \pi + i/d+i+1,\ldots,d+m,$$
$$\overline{\pi} = \pi + i + (d+m)/d+i+1,\ldots,d+m-1.$$

Proof: We induct on m. If $m = 1$, then $Y_{G+K_2} = Y_{G\uplus K_1} - Y_{G\uparrow_d^{d+1}}$. This shows that

$$Y_{G+K_2} \equiv_{d+1} \sum_{(\pi)} \left(\frac{c_{(\pi)}(b-1)}{b} e_{(\pi/d+1)} + \frac{c_{(\pi)}}{b} e_{(\pi+(d+1))} \right),$$

which verifies the base case.

To begin the induction step, we repeatedly utilize the Deletion-Contraction Recurrence to delete the edges $v_{d+i}v_{d+m+1}$ for $0 \leq i \leq m$, and obtain

$$Y_{G+K_{m+2}} \equiv_{d+m+1} Y_{G+K_{m+1}\uplus v_{d+m+1}} - mY_{G+K_{m+1}\uparrow_{d+m}^{d+m+1}} - Y_{G+K_{m+1}\uparrow_{d}^{d+m+1}}. \tag{17}$$

Note that we are able to combine all the terms from $Y_{G+K_{m+1}\uparrow_{d+i}^{d+m+1}}$ for $1 \leq i \leq m$ using Lemma 7.4, since in these cases the necessary permutation exists.

We now expand each of the terms in Eq. (17). For the first, using Lemma 6.3,

$$Y_{G+K_{m+1}\uplus v_{d+m+1}} \equiv_{d+m+1} \sum_{(\pi)} \sum_{i=0}^{m-1} \frac{c_{(\pi)}\langle m-1\rangle_i}{(b)_{i+1}} \left[(b-m+i)e_{(\pi_1)} + (i+1)e_{(\pi_2)}\right],$$

where

$$\pi_1 = \pi + i/d+i+1,\ldots,d+m/d+m+1,$$
$$\pi_2 = \pi + i + (d+m)/d+i+1,\ldots,d+m-1/d+m+1.$$

For the second term, using Corollary 6.1, we have

$$mY_{G+K_{m+1}\uparrow_{d+m}^{d+m+1}}$$
$$\equiv_{d+m+1} \sum_{(\pi)} \sum_{i=0}^{m-1} \frac{c_{(\pi)}\langle m\rangle_{i+1}}{(b)_{i+1}} \left[\frac{b-m+i}{m-i} \left(e_{(\pi_1)} - e_{(\pi_3)}\right) + \frac{i+1}{b+i+1} \left(e_{(\pi_2)} - e_{(\pi_4)}\right) \right],$$

where

$$\pi_3 = \pi + i/d+i+1,\ldots,d+m+1,$$
$$\pi_4 = \pi + i + (d+m) + (d+m+1)/d+i+1,\ldots,d+m-1.$$

And finally, using Lemma 7.3,

$$Y_{G+K_{m+1}}\uparrow_{d}^{d+m+1} \equiv_{d+m+1} \sum_{(\pi)} \sum_{i=0}^{m} \frac{c_{(\pi)}\langle m\rangle_i}{(b)_{i+1}} \left(e_{(\pi_3)} - e_{(\pi_5)}\right)$$

where

$$\pi_5 = \pi + i + (d+m+1)/d+i+1, \ldots, d+m.$$

Grouping the terms appropriately and shifting indices where needed gives

$$Y_{G+K_{m+2}} \equiv_{d+m+1} \sum_{(\pi)} \sum_{i=0}^{m} \frac{c_{(\pi)}\langle m\rangle_i \left[(b-(m+1)+i)e_{(\pi_3)} + (i+1)e_{(\pi_5)}\right]}{(b)_{i+1}}.$$

This completes the induction step and the proof. □

Examining this lemma, we can see that in $Y_{G+K_{m+1}}$ we have the same sign on all the coefficients as we had in Y_G, with the possible exception of the terms where $b < m - i$. But it is easy to see that in this case we have

$$e_{(\pi+i/d+i+1,\ldots,d+m)} \equiv_{d+m} e_{(\pi+m-i-b-1+(d+m)/d+m-i-b,\ldots,d+m-1)}.$$

This means that in the expression for $Y_{G+K_{m+1}}$ as a sum over congruence classes modulo $d+m$, we can combine the coefficients on these terms. And so upon simplification, the coefficient on $e_{(\pi+i/d+i+1,\ldots,d+m)}$ will be:

$$\left(\frac{(b-m+i)\langle m-1\rangle_i}{(b)_{i+1}} + \frac{(m-i-b)\langle m-1\rangle_{m-i-b-1}}{(b)_{m-i-b}}\right) c_{(\pi)},$$

where $c_{(\pi)}$ is the coefficient on $e_{(\pi)}$ in Y_G.

Adding these fractions by finding a common denominator, we see that this is actually zero, which gives us the next result.

Theorem 7.6 *If Y_G is (e)-positive, then Y_{G+K_m} is also (e)-positive.*

Notice that Proposition 6.4 follows easily from Theorem 7.6 and induction, since for paths $P_{m+1} = P_m + K_2$. As a more general result we have the following corollary.

Corollary 7.7 *If G is a K_α-chain, then Y_G is (e)-positive. Hence, X_G is also e-positive.*

We can also describe another class of (e)-positive graphs. We define a *diamond* to be the indifference graph on the collection of intervals {[1, 3], [2, 4]}. So a diamond consists of two K_3's sharing a common edge. Then the following holds.

Theorem 7.8 *Let D be a diamond. If G is (e)-positive, then so is $G + D$.*

Proof: The proof of this result is analogous to the proof for the case of $G + K_m$, and so is omitted. □

8. Comments and open questions

We will end with some questions raised by this work. We hope they will stimulate future research.

(a) Obviously it would be desirable to find a way to use deletion-contraction to prove that indifference graphs are e-positive (or even demonstrate the full $(\mathbf{3}+\mathbf{1})$-Free Conjecture). The reason that it becomes difficult to deal with the case where the last two complete graphs overlap in more than one vertex is because one has to keep track of all ways the intersection could be distributed over the block sizes of an e_π. Not only is the bookkeeping complicated, but it becomes harder to find groups of coefficients that will sum to zero.

Another possible approach is to note that if G is an indifference graph, then for the edge $e = v_k v_d$ (where $[k, d]$ is the last interval) both $G\backslash e$ and G/e are indifference graphs. Furthermore $G\backslash e$ is obtained from G/e by attaching a K_{d-k} so that it intersects in all but one vertex with the final K_{d-k} of G/e. Unfortunately, the relationship between the coefficients in the (e)-expansion of $Y_{G\backslash e}$ and $Y_{G/e}\uparrow$ does not seem to be very simple.

(b) Notice that if T is a tree on d vertices, we have $\mathcal{X}_T(n) = n(n-1)^{d-1}$. Since X_G is a generalization of the chromatic polynomial, it might be reasonable to suppose that it also is constant on trees with d vertices. This is far from the case! In fact, it has been verified up to $d = 9$ [2] that, for non-isomorphic trees T_1, T_2 we have $X_{T_1} \neq X_{T_2}$. This leads to the following question posed by Stanley.

Question 8.1 ([12]) *Does X_T distinguish among non-isomorphic trees?*

We should note that the answe r to this question is definitely "yes" for Y_T. In fact more is true.

Proposition 8.2 *The function Y_G distinguishes among all graphs G with no loops or multiple edges.*

Proof: We know from Proposition 3.2 that $Y_G = \sum_P m_{\pi(P)}$ for the stable partitions P. Construct the graph H with vertex set $V(G) = \{v_1, v_2, \ldots, v_d\}$ and edge set $E(H) = \{v_i v_j \mid \text{there exists a } \pi(P) \text{ such that } i, j \text{ are in the same block of } \pi(P)\}$. Since $\pi(P)$ comes from a stable partition P of G, v_i and v_j are in the same block of some $\pi(P)$ if and only if there is no edge $v_i v_j$ in G. Hence the graph H constructed is the (edge) complement of G and so we can recover G from H. □

Of course we can have $Y_G \neq Y_H$ but $X_G = X_H$. So a first step towards answering Stanley's question might be to see if Y_T still distinguishes trees under congruence. It seems reasonable to expect to investigate this using our deletion-contraction techniques since trees are reconstructible from their leaf-deleted subgraphs [9]. We proceed in the following manner.

If $T_1 \not\cong T_2$ then by the reconstructibility of trees there must exist labelings of these trees so that v_d is a leaf of T_1, $\tilde{v}_d$ is a leaf of T_2 and $T_1 - v_d \not\cong T_2 - \tilde{v}_d$. By induction we will have $Y_{T_1 - v_d} \not\equiv_{d-1} Y_{T_2 - \tilde{v}_d}$. Furthermore, our recurrence gives

$$\begin{aligned} Y_{T_1} &= Y_{T_1 - v_d / v_d} - Y_{T_1 - v_d}\uparrow \\ Y_{T_2} &= Y_{T_2 - \tilde{v}_d / \tilde{v}_d} - Y_{T_2 - \tilde{v}_d}\uparrow . \end{aligned}$$

One now needs to investigate what sort of cancelation occurs to see if these two differences could be equal or not. Concentrating on a term of a particular type could well be the key.

(c) It would be very interesting to develop a wider theory of symmetric functions in noncommuting variables. The only relevant paper of which we are aware is Doubilet's [3] where he talks more generally about functions indexed by set partitions, but not the noncommutative case per se. His work is dedicated to finding the change of basis formulae between 5 bases (the three we have mentioned, the complete homogeneous basis, and the so-called forgotten basis which he introduced). However, there does not as yet seem to be any connection to representation theory. In particular, there is no known analog of the Schur functions in this setting.

Note added in Proof: Rosas and Sagan have recently come up with a definition of the Schur function in noncommuting variables and are investigating its properties.

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