Journal of Algebraic Combinatorics **13** (2001), 295–315

Orthogonal Matroids

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Received May 18, 1999; Revised June 16, 2000

Abstract. The notion of matroid has been generalized to Coxeter matroid by Gelfand and Serganova. To each pair (W, P) consisting of a finite irreducible Coxeter group W and parabolic subgroup P is associated a collection of objects called Coxeter matroids. The (ordinary) matroids are the special case where W is the symmetric group (the A_n case) and P is a maximal parabolic subgroup. This generalization of matroid introduces interesting combinatorial structures corresponding to each of the finite Coxeter groups. Borovik, Gelfand and White began an investigation of the B_n case, called symplectic matroids. This paper initiates the study of the D_n case, called orthogonal matroids. The main result (Theorem 2) gives three characterizations of orthogonal matroid: algebraic, geometric, and combinatorial. This result relies on a combinatorial description of the Bruhat order on D_n (Theorem 1). The representation of orthogonal matroids by way of totally isotropic subspaces of a classical orthogonal space (Theorem 5) justifies the terminology *orthogonal matroid*.

Keywords: orthogonal matroid, Coxeter matroid, Coxeter group, Bruhat order

1. Introduction

Matroids, introduced by Hassler Whitney in 1935, are now a fundamental tool in combinatorics with a wide range of applications ranging from the geometry of Grassmannians to combinatorial optimization. In 1987 Gelfand and Serganova [9, 10] generalized the matroid concept to the notion of Coxeter matroid. To each finite Coxeter group W and parabolic subgroup P is associated a family of objects called Coxeter matroids. Ordinary matroids correspond to the case where W is the symmetric group (the A_n case) and P is a maximal parabolic subgroup.

This generalization of matroid introduces interesting combinatorial structures corresponding to each of the finite Coxeter groups. Borovik, Gelfand and White [2] began an investigation of the B_n case, called symplectic matroids. The term "symplectic" comes from examples constructed from symplectic geometries. This paper initiates the study of the D_n case, called orthogonal matroid because of examples constructed from orthogonal geometries.

The first goal of this paper is to give three characterizations of orthogonal matroids: algebraic, geometric and combinatorial. This is done in Sections 3, 4 and 6 (Theorem 2) after preliminary results in Section 2 concerning the family D_n of Coxeter groups. The

*The work of Neil White on this paper was partially supported at UMIST under funding of EPSRC Visiting Fellowship GR/M24707.

algebraic description is in terms of left cosets of a parabolic subgroup P in D_n. The Bruhat order on D_n plays a central role in the definition. The geometric description is in terms of a polytope obtained as the convex hull of a subset of the orbit of a point in $\mathbb{R}^n$ under the action of D_n as a Euclidean reflection group. The roots of D_n play a central role in the definition. The combinatorial description is in terms of k-element subsets of a certain set and flags of such subsets. The Gale order plays a central role in the definition. Section 5 gives a precise description of the Bruhat order on both B_n and D_n in terms of the Gale order on the corresponding flags (Theorem 1). A fourth characterization, in terms of oriflammes, holds for an important special case (Theorem 3 of Section 6).

Section 7 concerns the relationship between symplectic and orthogonal matroids. Every orthogonal matroid is a symplectic matroid. Necessary and sufficient conditions are provided when a Lagrangian symplectic matroid is orthogonal (Theorem 4). More generally, the question remains open.

Section 8 concerns the representation of orthogonal matroids and, in particular, justifies the term orthogonal. Just as ordinary matroids arise from subspaces of projective spaces, symplectic and orthogonal matroids arise from totally isotropic subspaces of symplectic and orthogonal spaces, respectively (Theorem 5).

2. The Coxeter group D_n

We give three descriptions of the family D_n of Coxeter groups: (1) in terms of generators and relations; (2) as a permutation group; and (3) as a reflection group in Euclidean space.

Presentation in terms of generators and relations. A *Coxeter group* W is defined in terms of a finite set S of generators with the presentation

$$\langle s \in S \mid (ss')^{m_{ss'}} = 1 \rangle,$$

where $m_{ss'}$ is the order of ss', and $m_{ss} = 1$ (hence each generator is an involution). The cardinality of S is called the *rank* of W. The *diagram* of W is the graph where each generator is represented by a node, and nodes s and s' are joined by an edge labeled $m_{ss'}$ whenever $m_{ss'} \geq 3$. By convention, the label is omitted if $m_{ss'} = 3$. A Coxeter system is *irreducible* if its diagram is a connected graph. A reducible Coxeter group is the direct product of the Coxeter groups corresponding to the connected components of its diagram. The finite irreducible Coxeter groups have been completely classified and are usually denoted by $A_n (n \geq 1)$, $B_n (=C_n)(n \geq 2)$, $D_n (n \geq 4)$, E_6, E_7, E_8, F_4, G_2, H_3, H_4, and $I_2(m)$ $(m \geq 5,\ m \neq 6)$, the subscript denoting the rank. The diagrams of the families A_n, B/C_n and D_n appear in figure 1, these being the families of concern in this paper.

Permutation representation. Throughout the paper we will use the notation $[n] = \{1, 2, \ldots, n\}$ and $[n]^* = \{1^*, 2^*, \ldots, n^*\}$. As a permutation group, A_n is isomorphic to the symmetric group on the set $[n+1]$ with the standard generators being the adjacent transpositions

$$S = \{(1\,2), (2\,3), \ldots, ((n-1)\,n)\}.$$

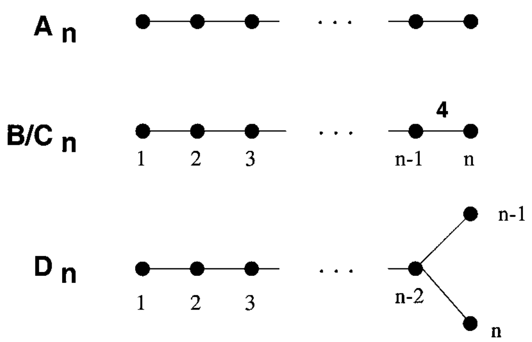

Figure 1. Diagrams of three Coxeter families.

Likewise B_n is isomorphic to the permutation group acting on $[n] \cup [n]^*$ generated by the involutions

$$S = \{(1\,2)(1^*\,2^*), (2\,3)(2^*\,3^*), \ldots, ((n-1)\,n)((n-1)^*\,n^*), (n\,n^*)\}.$$

We will use the convention $i^{**} = i$ for any $i \in [n]$. Call a subset $X \subset [n] \cup [n]^*$ *admissible* if $X \cap X^* = \emptyset$, i.e., if X does not contain both i and i^* for any i. The group B_n acts simply transitively on ordered, admissible n-tuples; hence

$$|B_n| = 2^n\, n!.$$

The group D_n is isomorphic to the permutation group acting on $[n] \cup [n]^*$ and generated by the involutions

$$S = \{(12)(1^*2^*), (23)(2^*3^*), \ldots, ((n-1)n)((n-1)^*n^*), ((n-1)n^*)((n-1)^*n)\}.$$

Note that D_n is a subgroup of B_n. More precisely, D_n consists of all the even permutations in B_n; hence

$$(B_n : D_n) = 2 \quad \text{and} \quad |D_n| = 2^{n-1}\, n!.$$

Reflection group. A *reflection* in a Coxeter group W is a conjugate of some involution in S. Let $T = T(W)$ denote the set of all reflections in W. Every finite Coxeter group W can be realized as a reflection group in some Euclidean space $\mathbb{E}$ of dimension equal to the rank of W. In this realization, each element of T corresponds to the orthogonal reflection through a hyperplane in $\mathbb{E}$ containing the origin.

It is not difficult to give an explicit representation of B_n and D_n as reflection groups. If $i \in [n]$, let e_i denote the ith standard coordinate vector. Moreover, let $e_{i^*} = -e_i$. Regard B_n, or its subgroup D_n, as a permutation group as given above. Then for $w \in B_n$, the representation of w as an orthogonal transformation is given by letting

$$w(e_i) = e_{w(i)} \tag{2.1}$$

for each $i \in [n]$ and expanding linearly.

As a reflection group, each finite Coxeter group W acts on its Coxeter complex. Let Σ denote the set of all reflecting hyperplanes of W, and let $E' = \mathbb{E} \setminus \bigcup_{H \in \Sigma} H$. The connected components of E' are called *chambers*. For any chamber Γ, its closure $\bar{\Gamma}$ is a simplicial cone in $\mathbb{E}$. These simplicial cones and all their faces form a simplicial fan called the *Coxeter complex* and denoted $\Delta := \Delta(W)$. It is known that W acts simply transitively on the set of chambers of $\Delta(W)$.

A *flag* of an n-dimensional polytope is a nested sequence $F_0 \subset F_1 \subset \cdots \subset F_{n-1}$ of faces. A polytope is *regular* if its symmetry group is flag transitive. Each of the irreducible Coxeter groups listed above, except D_n, E_6, E_7, and E_8, is the symmetry group of a regular convex polytope. In particular A_n is the symmetry group of the $(n-1)$-simplex, the permutation representation above being the action on the set of n vertices, each vertex labeled with an element of $[n]$. The group B_n is the symmetry group of the n-cube or its dual, the cross polytope (generalized octahedron). For this reason, the group B_n is referred to as the *hyperoctahedral group*. The permutation representation is the action on the set of $2n$ vertices of the cross polytope, each vertex labeled with an element of $[n] \cup [n]^*$, the vertex i^* being the vertex antipodal to the vertex i. Dually, the action is on the set of $2n$ facets of the n-cube, the facet i^* being the one opposite the facet i. In the cases where Coxeter group W is the symmetry group of a regular polytope, the intersection of the Coxeter complex $\Delta(W)$ with a sphere centered at the origin is essentially the barycentric subdivision of the polytope.

The Coxeter group of type D_n also acts on the n-cube Q_n, although not quite flag transitively; D_n acts transitively on the set of k-dimensional faces of Q_n for all k except $k = 0$. However, there are two orbits in its action on the set of vertices of Q_n, and hence two orbits in its action on the set of flags of Q_n.

3. D_n Matroids

Three definitions of D_n matroid (orthogonal matroid) are now given: (1) algebraic, (2) geometric, and (3) combinatorial. That these three definitions are equivalent is the subject of Sections 4, 5 and 6. Three such definitions are also given of A_n (ordinary) matroids and B_n (symplectic) matroids.

Algebraic description. We begin with a definition of the Bruhat order on a Coxeter group W; for equivalent definitions see e.g., [7, 12]. We will use the notation $\succeq$ for the Bruhat order. For $w \in W$ a factorization $w = s_1 s_2 \cdots s_k$ into the product of generators in S is called *reduced* if it is shortest possible. Let $l(w)$ denote the length k of a reduced factorization of w.

Definition 1 Define $u \succeq v$ if there exists a sequence $v = u_0, u_1, \ldots, u_m = u$ such that $u_i = t_i u_{i-1}$ for some reflection $t_i \in T(W)$, and $l(u_i) > l(u_{i-1})$ for $i = 1, 2 \ldots, m$.

Every subset $J \subset S$ gives rise to a (standard) *parabolic subgroup* P_J generated by J. The Bruhat order can be extended to an ordering on the left coset space W/P for any parabolic subgroup P of W.

Definition 2 Define Bruhat order on W/P by $\bar{u} \succeq \bar{v}$ if there exists a $u \in \bar{u}$ and $v \in \bar{v}$ such that $u \succeq v$.

Associated with each $w \in W$ is a shifted version of the Bruhat order on W/P, which will be called the w-Bruhat order and denoted $\succeq_w$.

Definition 3 Define $\bar{u} \succeq_w \bar{v}$ in the *w-Bruhat order* on W/P if $w^{-1}\bar{u} \succeq w^{-1}\bar{v}$.

Definition 4 The set $L \subseteq W/P$ is a *Coxeter matroid* (for W and P) if, for each $w \in W$, there is a $\bar{u}_0 \in L$ such that $\bar{u} \preceq_w \bar{u}_0$ for all $\bar{u} \in L$.

The condition in Definition 4 is referred to as the *Bruhat maximality condition*. A Coxeter diagram with a subset of the nodes circled will be referred to as a marked diagram. The *marked diagram* G of W/P is the diagram of W with exactly those nodes circled that do not correspond to generators of P. Likewise, if L is a Coxeter matroid for W and P, then G is referred to as the *marked diagram* of L.

Geometric description. Consider the representation of a Coxeter group W as a reflection group in Euclidean space $\mathbb{E}$ as discussed in Section 2. A *root* of W is a vector orthogonal to some hyperplane of reflection (a hyperplane in the Coxeter complex). For D_n, with respect to the same coordinate system used in Eq. (2.1), the roots are precisely the vectors

$$R = \{e_i \pm e_j \mid i, j \in [n], i \neq j\},$$

while the roots of B_n are

$$R \cup \{e_i \mid : i \in [n]\}.$$

For our purposes the norm of the root vector is not relevant.

In the Coxeter complex of W choose a *fundamental chamber* that is bounded by the hyperplanes of reflection corresponding to the generators in S. With the Coxeter complex of D_n as described in Section 2, a fundamental chamber is the convex cone spanned by the vectors

$$\begin{aligned} &e_1,\ e_1 + e_2,\ \ldots,\ e_1 + e_2 + \cdots + e_{n-2},\\ &e_1 + e_2 + \cdots + e_{n-1} + e_n,\\ &e_1 + e_2 + \cdots + e_{n-1} - e_n. \end{aligned} \tag{3.1}$$

Let x be any nonzero point in the closure of this fundamental chamber. Denote the orbit of x by

$$O_x = \{w(x) \mid w \in W\}.$$

If $L \subseteq O_x$, then the convex hull of L, denoted $\Delta(L)$, is a polytope. The following formulation was originally stated by Gelfand and Serganova [10]; also see [12].

Definition 5 The set $L \subseteq O_x$ is a Coxeter matroid if every edge in $\Delta(L)$ is parallel to a root of W.

The *marked diagram* G of a point x is the diagram of W with exactly those nodes circled that correspond to hyperplanes of reflection not containing point x. Note that x and y have the same diagram if and only if they have the same stabilizer in W. If $L \subseteq O_x$ is a Coxeter matroid, then G is referred to as the *marked diagram* of L. The polytope $\Delta(L)$ is independent of the choice of x in the following sense. The proof appears in [5].

Lemma 1 *If x and y have the same diagram and $L_x \subseteq O_x$ and $L_y \subseteq O_y$ are corresponding subsets of the orbits* (*i.e.*, *determined by the same subset of W*), *then $\Delta(L_x)$ and $\Delta(L_y)$ are combinatorially equivalent and corresponding edges of the two polytopes are parallel.*

Because of Lemma 1, there is no loss of generality in taking, for each diagram, one particular representative point x in the fundamental chamber and the corresponding orbit O_x. In particular, we can take x in the set Z_0 of points $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ where each of the following quantities equals either 0 or 1: $x_i - x_{i+1}$, $i = 1, 2 \ldots, n-2$, and $x_{n-1} - |x_n|$, and $|x_n|$. The set $\mathbb{Z}_0$ consists essentially of all possible barycenters of subsets of the vectors in (3.1) that span the fundamental chamber. (Except, however, the single vector $e_1 + \cdots + e_{n-1}$ is used instead of the last two vectors in (3.1) when both of those vectors are present. Thus, if $n = 3$ for example, we get $(2, 1, 0) \in Z_0$ instead of $(3, 2, 0)$.)

Combinatorial description. Our combinatorial description of symplectic and orthogonal matroids is analogous to the definition of an ordinary matroid in terms of its collection of bases. Whereas the algebraic and geometric descriptions hold for any finite irreducible Coxeter group, the definitions in this section are specific to the A_n, B_n and D_n cases. For a generalization see [13].

An essential notion in these definitions is Gale ordering. Given a partial ordering $\preceq$ on a finite set X, the corresponding *Gale ordering* on the collection of k-element subsets of X is defined as follows: $A \leq B$ if, for some ordering of the elements of A and B,

$$A = (a_1, a_2, \ldots, a_k)$$
$$B = (b_1, b_2, \ldots, b_k),$$

we have $a_i \preceq b_i$ for all i. Equivalently, we need a bijection $\kappa : B \to A$ so that $\kappa(b) \leq b$ for all $b \in B$. In later proofs when constructing such a bijection, we will refer to b as *dominating* $\kappa(b)$. The following lemma is straightforward.

Lemma 2 *Let*

$$A = (a_1, a_2, \ldots, a_k)$$
$$B = (b_1, b_2, \ldots, b_k),$$

and assume the elements have been ordered so that $a_i \prec a_{i+1}$ and $b_i \prec b_{i+1}$ for all $i \leq k - 1$. Then $A \leq B$ in the Gale order if and only if $a_i \prec b_i$ for all i.

Define a *flag F of type* $(k_1, k_2, \dots, k_m)$, $0 < k_1 < k_2 < \dots < k_m$, as a nested sequence of subsets of X

$$F = (A_1 \subset A_2 \subset \dots \subset A_m)$$

such that $|A_i| = k_i$. A flag of type k is just a single k-element set. Extend the Gale ordering on sets to the Gale ordering on flags as follows. If

$$F_A = (A_1 \subset A_2 \subset \dots \subset A_m)$$
$$F_B = (B_1 \subset B_2 \subset \dots \subset B_m)$$

are two flags, then $F_A \le F_B$ if $A_i \le B_i$ for all i.

A_n matroids. Consider flags $F = (A_1 \subset A_2 \subset \dots \subset A_m)$ where $A_i \subseteq [n]$ for each i. Let

$$\mathcal{A}_{(k_1,k_2,\dots,k_m)}$$

denote the set of all flags of type $(k_1, k_2, \dots, k_m)$.

Definition 6 A collection $L \subseteq \mathcal{A}_{(k_1,k_2,\dots,k_m)}$ is an A_n matroid if, for any linear ordering of $[n]$, L has a unique maximal member in the corresponding Gale order.

If the flag consists of more than one subset, the A_n matroid is often referred to as an ordinary *flag matroid*. In the case of single sets (flags of type k), it is a standard result in matroid theory [14] that an A_n matroid is simply an ordinary matroid of rank k.

B_n matroids. Now consider flags $F = (A_1 \subset A_2 \subset \dots \subset A_m)$ where $A_i \subset [n] \cup [n]^*$ for each i. Call such a flag *admissible* if $A_m \cap A_m^* = \emptyset$, i.e., if A_m does not contain both i and i^* for any i. Let

$$\mathcal{A}_{(k_1,k_2,\dots,k_m)}$$

denote the set of all admissible flags of type $(k_1, k_2, \dots, k_m)$ where $0 < k_1 < k_2 < \dots < k_m \le n$.

Define a partial order on $[n] \cup [n]^*$ by

$$1 \prec 2 \prec \dots \prec n-1 \prec n \prec n^* \prec (n-1)^* \prec \dots \prec 2^* \prec 1^*. \tag{3.2}$$

A partial order on $[n] \cup [n]^*$ is called *B_n-admissible* if it is a shifted ordering $\prec_w$ for some $w \in B_n$. By a *shifted ordering* we mean:

$$a \preceq_w b \quad \text{if and only if} \quad w^{-1}(a) \preceq w^{-1}(b).$$

Note that an ordering $\preceq$ on $[n] \cup [n]^*$ is admissible if and only if (1) $\preceq$ is a linear ordering and (2) from $i \prec j$ it follows that $j^* \prec i^*$ for any distinct elements $i, j \in [n] \cup [n]^*$. For example, $2 \prec 4^* \prec 1^* \prec 3 \prec 3^* \prec 1 \prec 4 \prec 2^*$ is an admissible ordering.

Definition 7 A collection $L \subseteq \mathcal{A}_{(k_1,k_2,\ldots,k_m)}$ is a B_n matroid (symplectic flag matroid) if, for any admissible ordering of $[n] \cup [n]^*$, L has a unique maximal member in the corresponding Gale ordering.

The *marked diagram* G of $\mathcal{A}_{(k_1,k_2,\ldots,k_m)}$ is the diagram for B_n with the nodes k_i, $i = 1, 2, \ldots m$, circled. Likewise, if $L \subset \mathcal{A}_{(k_1,k_2,\ldots,k_m)}$ is a B_n matroid, then G is referred to as the *marked diagram* of L.

D_n matroids. Again consider flags $F = (A_1 \subset A_2 \subset \cdots \subset A_m)$ where $A_i \subset [n] \cup [n]^*$ for each i. Let $\mathcal{A}_{(k_1,k_2,\ldots,k_m)}$ denote the same set of admissible flags of type $(k_1, k_2, \ldots, k_m)$ as in the B_n case, except when $k_m = n$. If $k_m = n$ and $k_{m-1} \le n - 2$, then let

$$\mathcal{A}^+_{(k_1,k_2,\ldots,k_{m-1},n)}$$

denote the set of admissible flags of type $(k_1, k_2, \ldots, k_{m-1}, n)$ with A_m containing an even number of starred elements, and let

$$\mathcal{A}^-_{(k_1,k_2,\ldots,k_{m-1},n)}$$

denote the set of admissible flags of type $(k_1, k_2, \ldots, k_{m-1}, n)$ with A_m containing an odd number of starred elements. Note that we do not permit both $k_{m-1} = n - 1$ and $k_m = n$. In the case $k_m = n$ it should be understood, without explicitly stating it, that $\mathcal{A}$ means either $\mathcal{A}^+$ or $\mathcal{A}^-$.

Define a partial order on $[n] \cup [n]^*$ by

$$1 \prec 2 \prec \ldots n - 1 \prec \begin{matrix} n \\ n^* \end{matrix} \prec (n-1)^* \prec \cdots \prec 2^* \prec 1^*. \tag{3.3}$$

Note that the elements n and n^* are incomparable in this ordering. A partial order on $[n] \cup [n]^*$ is *D_n-admissible* if it is a shifted order $\prec_w$ for some $w \in D_n$:

$$a \preceq_w b \quad \text{if and only if} \quad w^{-1}(a) \preceq w^{-1}(b).$$

Note that an ordering $\preceq$ is D_n-admissible if and only if it is of the form

$$a_1 \prec a_2 \prec \cdots a_{n-1} \prec \begin{matrix} a^n \\ a_n^* \end{matrix} \prec a_{n-1}^* \prec \ldots \prec a_2^* \prec a_1^*,$$

where $\{a_1, a_2, \ldots, a_n\}$ is admissible and we again use the convention $i^{**} = i$.

Definition 8 A collection $L \subseteq \mathcal{A}_{(k_1,k_2,\ldots,k_m)}$ is a D_n matroid (orthogonal matroid) if, for each admissible ordering, L has a unique maximal member in the corresponding Gale order.

The elements of L will be called *bases* of the matroid. If $m > 1$, then the matroid is sometimes referred to as an *orthogonal flag matroid*. The marked diagram G of $\mathcal{A}_{(k_1,k_2,\ldots,k_m)}$ is obtained from the diagram of D_n by considering three cases:

Case 1. If $k_m \leq n-2$, then circle the nodes $k_1, k_2, \ldots, k_m$.
Case 2. If $k_{m-1} \leq n-2$ (or $m = 1$) and $k_m = n$, then circle the nodes $k_1, \ldots, k_{m-1}$ and the node $n-1$ or n depending on whether the collection of flags is $\mathcal{A}^-_{(k_1,k_2,\ldots,k_m)}$ or $\mathcal{A}^+_{(k_1,k_2,\ldots,k_m)}$, respectively.
Case 3. If $k_{m-1} \leq n-2$ (or $m = 1$) and $k_m = n-1$, then circle nodes $k_1, \ldots, k_{m-1}$ and both nodes $n-1$ and n.

Note that all possibilities for marked diagrams are realized. If $L \subseteq \mathcal{A}_{(k_1,k_2,\ldots,k_m)}$ is a D_n matroid, then G is referred to as its *marked diagram*.

4. Bijections

The definitions of D_n matroid in the previous section are cryptomorphic in the sense of matroid theory; that is, they define the same object in terms of its various aspects. Likewise for B_n matroids. For ordinary matroids there are additional definitions in terms of independent sets, flats, cycles, the closure operator, etc. In this section we make the crytomorphisms explicit for orthogonal matroids.

The definitions given for D_n matroid in the previous section are

(1) in terms of the set W/P of cosets (Definition 4),
(2) in terms of the set O_x of points in Euclidean space (Definition 5), and
(3) in terms of a collection $\mathcal{A}_{(k_1,k_2,\ldots,k_m)}$ of admissible flags (Definition 8).

Explicit bijections are now established between W/P, O_x and $\mathcal{A}_{(k_1,k_2,\ldots,k_m)}$, each with the same marked diagram:

$$f : D_n/P \to O_x$$
$$g : D_n/P \to \mathcal{A}_{(k_1,\ldots,k_m)}$$
$$h : \mathcal{A}_{(k_1,\ldots,k_m)} \to O_x.$$

To define f, start with the collection D_n/P of cosets with marked diagram G. Fix a point $x \in \mathbb{R}^n\backslash\{0\}$ in the fundamental chamber with marked diagram G. In fact, we can take x to be a point in the set Z_0 defined in Section 3. In other words, the stabilizer of x in W is exactly P. For $w \in D_n$, the point $w(x)$ depends only on the coset wP. This gives a bijection

$$f\ : D_n/P \to O_x$$
$$\bar{w} \mapsto w(x).$$

To describe the inverse of f, let $y \in O_x$ and let $w \in D_n$ be such that $w(x) = y$. If P is the parabolic subgroup of D_n generated by exactly those reflections that stabilize x, then $f^{-1}(y) = wP$.

To define g, again consider the collection D_n/P of cosets with marked diagram G. Let $\mathcal{A}_{(k_1,\ldots,k_m)}$ be the collection of admissible sets with the same marked diagram G (as described by the three cases in Section 3). If $F = (A_1 \subset A_2 \subset \cdots \subset A_m)$ is a flag and $A_i = \{a_1, \ldots, a_{k_i}\}$, then the flag will often be denoted

$$F = a_1 a_2 \ldots a_{k_1} \bigm| \ldots a_{k_2} \bigm| \ldots \quad \ldots \bigm| \ldots a_{k_m}. \tag{4.1}$$

For example the flag $\{1, 2\} \subset \{1, 2, 3^*, 5\} \subset \{1, 2, 3^*, 5, 7^*\}$ will be denoted simply $12 \mid 3^*5 \mid 7^*$. Let F_0 be the flag

$$F_0 = 1\,2 \ldots | \ldots \quad \ldots | \ldots k_m$$

of type $(k_1, \ldots, k_m)$, $k_m < n$, or, in the case $k_m = n$,

$$F_0 = 1\,2 \ldots | \ldots \quad \ldots | \ldots n \quad \text{or}$$
$$F_0 = 1\,2 \ldots | \ldots \quad \ldots | \ldots n^*,$$

depending, respectively, on whether the node $n - 1$ or n is circled. Let F be an arbitrary flag given in the form (4.1). The action of D_n as a permutation group on $[n] \cup [n]^*$ as described in Section 2 extends to an action on $\mathcal{A}_{(k_1, k_2, \ldots, k_m)}$:

$$w(F) = w(a_1)w(a_2) \ldots w\big(a_{k_1}\big) \bigm| \ldots w\big(a_{k_2}\big) \bigm| \ldots \quad \ldots \bigm| \ldots w\big(a_{k_m}\big),$$

where $w \in D_n$. For this action of D_n on $\mathcal{A}_{(k_1,\ldots,k_m)}$ the stabilizer of F_0 is P. So, for $w \in D_n$, the flag $w(F)$ depends only on the coset wP. Thus a bijection is induced:

$$g \;:\; D_n/P \to \mathcal{A}_{(k_1,\ldots,k_m)}$$
$$\bar{w} \mapsto w(F_0).$$

The map g is surjective because, if $k_m < n$, then there is one orbit, $\mathcal{A}_{(k_1,\ldots,k_m)}$, in the action of D_n on the set of admissible flags of type $(k_1, \ldots, k_m)$. If $k_m = n$, then there are two orbits, $\mathcal{A}^+_{(k_1,\ldots,k_m)}$ and $\mathcal{A}^-_{(k_1,\ldots,k_m)}$, the one consisting of those admissible flags such that A_m contains an even number of starred elements, and the other those admissible flags such that A_m contains an odd number of starred elements.

To describe the inverse of g, let F be an arbitrary flag in $\mathcal{A}_{(k_1,\ldots,k_m)}$ and let $w \in D_n$ be such that $w(F_0) = F$. If P is the parabolic subgroup of D_n that stabilizes F_0, then $g^{-1}(F) = wP$.

The third bijection is $h = f \circ g^{-1}$. However, it is useful to provide a direct construction, as follows. Let F be a flag of type $(k_1, \ldots, k_m)$ where $k_i \in [n]$:

$$F = (A_1 \subset A_2 \subset \cdots \subset A_m) = a_1 a_2 \ldots a_{k_1} \bigm| \ldots a_{k_2} \bigm| \ldots \quad \ldots \bigm| \ldots a_{k_m}.$$

Recall that $e_{i^*} = -e_i$ for any $i \in [n]$ and define the map $h : \mathcal{A}_{(k_1,\ldots,k_m)} \to \mathbb{R}^n$ by

$$h(F) = \sum_{i=1}^{k_m} \alpha_i e_{a_i},$$

where α_i is the number of sets in the flag $(A_1 \subset A_2 \subset \cdots \subset A_m)$ that contain a_i. In particular let

$$x = h(1\,2\ldots k_1 \mid \ldots k_2 \mid \ldots \quad \ldots \mid \ldots k_m) \in Z_0,$$

where we allow $k_m = n$ or $k_m = n^*$ when we are in case $\mathcal{A} = \mathcal{A}^+$ or $\mathcal{A} = \mathcal{A}^-$, respectively. Now we have our map

$$h : \mathcal{A}_{(k_1,\ldots,k_m)} \to O_x.$$

There is an alternative way to describe the map h, in terms of the barycentric subdivision $\beta(Q)$ of the n-cube Q centered at the origin with edges parallel to the axes. Let the faces of Q be labeled by $[n] \cup [n]^*$, where i and i^* are antipodal faces. If $F \in \mathcal{A}_k$, then $h(F)$ is a vertex of $\beta(Q)$ (appropriately scaled). For a flag $F = (A_1 \subset A_2 \subset \cdots \subset A_m) \in \mathcal{A}_{(k_1,\ldots,k_m)}$, the image $h(F)$ is the barycenter of the simplex of $\beta(Q)$ determined by the vertices $h(A_1), h(A_2), \ldots, h(A_m)$.

To describe the inverse of h, note that each vertex v in $\beta(Q)$ represents a face f_v of Q. If $A \subset [n] \cup [n]^*$ is the set of k facets of Q whose intersection is f_v, then label v by A. Point x is the barycenter of a simplex of $\beta(Q)$ whose vertices are labeled, say $[k_1], [k_2], \ldots, [k_m]$, where $k_1 < k_2 < \cdots < k_m$ (again, in the case $\mathcal{A} = \mathcal{A}^-$, replace $[k_m] = [n] = \{1, 2, \ldots, n\}$ by $\{1, 2, \ldots, -n\}$). Likewise each point $y \in O_x$ is the barycenter of some simplex of $\beta(Q)$ whose vertices are labeled $A_1, A_2, \ldots, A_m$ where $A_1 \subset A_2 \subset \cdots \subset A_m$ and $|A_i| = k_i$ for each i. Then

$$h^{-1}(y) = F = (A_1 \subset A_2 \subset \cdots \subset A_m).$$

Proposition *With the maps as defined above*: $h \circ g = f$.

Proof: Let $\bar{w} \in D_n/P$ and use formula (2.1):

$$\begin{aligned}(h \circ g)(\bar{w}) &= h\,(w(1)w(2)\ldots w(k_1) \mid \ldots w(k_2) \mid \ldots \quad \ldots \mid \ldots w(k_m))\\ &= \sum_{i=1}^{k_m} \alpha_i e_{w(i)} = w\left(\sum_{i=1}^{k_m} \alpha_i e_i\right) = w(x) = f(\bar{w}).\end{aligned}$$

□

5. The Bruhat order on D_n

Let D_n/P and $\mathcal{A}_{(k_1,\ldots,k_m)}$ have the same marked diagram. In this section a combinatorial description of the Bruhat order on D_n/P is given in terms of $\mathcal{A}_{(k_1,\ldots,k_m)}$. This is also done in the B_n case.

Consider two flags of the same type:

$$F_A = (A_1 \subset A_2 \subset \cdots \subset A_m)$$
$$F_B = (B_1 \subset B_2 \subset \cdots \subset B_m)$$

Alternatively,

$$A = a_1 a_2 \dots a_{k_1} \Big| \dots a_{k_2} \Big| \dots \quad \dots \Big| \dots a_{k_m}$$
$$B = b_1 b_2 \dots b_{k_1} \Big| \dots b_{k_2} \Big| \dots \quad \dots \Big| \dots b_{k_m}.$$

Assume, without loss of generality, that in A and B the elements between consecutive vertical bars are arranged in descending order with respect to (3.2). This is possible because, by definition, elements n and n^* do not appear together in an admissible set.

Now we define a partial order on $\mathcal{A}_{(k_1,\dots,k_m)}$ called the *weak* D_n *Gale order*. Consider two distinct flags A and B (denoted as above) where, for each i, either (1) $b_i = a_i$ or (2) $b_i = a_i^*$. In case (2) also assume that

(a) a_i is unstarred and $a_i \neq n$;
(b) a_j (and thus also b_j) is less than a_i in numerical value for all $j > i$; and
(c) all elements greater than a_i in numerical value appear to the left of a_i in A.

Clearly $B > A$ in the Gale order for D_n. Call two flags with the above properties *close*. For example, if $n = 4$ then $B = 4 \,|\, 3^*1$ and $A = 4 \,|\, 31$ are close, but $B = 43^*1 = 3^*41$ and $A = 431$ are not close and $B = 3^* \,|\, 1$ and $A = 3 \,|\, 1$ are not close. For any pair of flags that are close, it is easy to check that one covers the other in the Gale order. Consider the Hasse diagram of the D_n Gale order on the set $\mathcal{A}_{(k_1,\dots,k_m)}$ of flags. Remove from the diagram all covering relations that are close. Call the resulting partial order on $\mathcal{A}_{(k_1,\dots,k_m)}$ the *weak* D_n *Gale order*.

The following notation is used in the next lemma, which provides a formula for the length $l(u)$ of any element $u \in D_n$. Let $F_u \in \mathcal{A}_{(1,\dots,n-1)}$. Since all nodes in the corresponding diagram are circled, the parabolic subgroup in this case is trivial, so we are justified in using the notation F_u, where $u \in D_n$. Let F_u' be the flag in $\mathcal{A}_{(1,\dots,n)}$ obtained from F_u by adjoining the missing element at the end so that the number of starred elements is even. For a flag A in $\mathcal{A}_{(1,\dots,n)}$ define a *descent* as a pair (a_i, a_j) of elements such that $j > i$ and $a_i \succ a_j$. Let $d(A)$ denote the number of descents in flag A. Note that, as a permutation of $[n] \cup [n]^*$, a reflection $t \in D_n$ is an involution of the form

$$(i\ j)(i^*\ j^*) \quad \text{where } i, j \in [n] \cup [n]^*. \tag{5.1}$$

A generating reflection is of the form $(i(i+1))(i^*(i+1)^*)$, $i = 1, 2, \dots, n-1$ or $((n-1)\, n^*)((n-1)^*\, n)$.

Lemma 3 *Let F_u be the flag corresponding to an element $u \in D_n$. Then*

$$l(u) = d(F_u') + \sum_{s^* \in [n]^* \cap F_u'} (n - s).$$

Proof: The proof is by induction. For a flag in $F = F_u \in \mathcal{A}_{(1,\dots,n-1)}$, denote the parameter $d(F') + \sum_{s^* \in [n]^* \cap F'} (n - s)$ by $p(F)$. Note that applying any generating reflection s to F

changes $p(F)$ by at most 1. Since $p(1 \mid 2 \mid \ldots \mid n) = 0$, necessarily $l(u) \geq p(F)$. But we can always arrange it so that $p(sF) = p(F) - 1$, so $l(u) = p(F)$. □

Theorem 1

(1) *The correspondence* $g : D_n/P \to \mathcal{A}_{(k_1,\ldots,k_m)}$ *is a poset isomorphism between* D_n/P *with respect to the Bruhat order and* $\mathcal{A}_{(k_1,\ldots,k_m)}$ *with respect to the weak* D_n *Gale order. In other words, for* D_n*, Bruhat order is weaker than Gale order.*
(2) *On the other hand, for* B_n*, Bruhat order on* B_n/P *is isomorphic to Gale order on* $\mathcal{A}_{(k_1,\ldots,k_m)}$*.*

Proof: In this proof the relation $\geq$ refers to the D_n weak Gale order. Let F_A and F_B be two flags in $\mathcal{A}_{(k_1,\ldots,k_m)}$ and $\bar{v}$ and $\bar{u}$ the corresponding cosets in D_n/P. It must be shown that $F_B \geq F_A$ if and only if $\bar{u} \succeq \bar{v}$. According to Definitions 1 and 2 in Section 3, in D_n/P we have $\bar{u} \succeq \bar{v}$ in the Bruhat order if and only if there exists a sequence $\bar{v} = \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_m = \bar{u}$ such that $\bar{u}_i = t_i\bar{u}_{i-1}$ for some reflection $t_i \in T(D_n)$, and $l(u_i) > l(u_{i-1})$ for $i = 1, 2 \ldots, m$. Let $\bar{u} = t\bar{v}$ where t is a reflection, and let flags F_v and F_u be the corresponding flags. Note that, according to Lemma 3, $\bar{u} \succ \bar{v}$ if and only if $F_u > F_v$.

Below it will be shown that, in $\mathcal{A}_{(k_1,\ldots,k_m)}$, we have $F_B \geq F_A$ in the weak D_n order if and only if there exists a sequence of flags $F_A = F_0, F_1, \ldots, F_m = F_B$ such that $F_i = t_i(F_{i-1})$ for some involution t_i of the form (5.1), and $F_i > F_{i-1}$ for $i = 1, 2 \ldots, m$. This will prove the theorem. Thus it is now sufficient to show that, for any two flags $F_B > F_A$, there is an involution t of the form (5.1) such that either $F_B \geq F_C = t(F_A) > F_A$ or $F_B > F_C = t(F_B) \geq F_A$.

To simplify notation assume that $B > A$ are two flags. Let j be the first index such that $a_j \neq b_j$, and such that, furthermore, if $b_j = a_j^*$ then assume that either

(i) there is an a_i, $i > j$, that is greater than a_j in numerical value or
(ii) there is an element c greater than a_j in numerical value such that neither c nor c^* lies to the left of a_j in A.

Since A and B are not close, such a j must exist. To simplify notation let $a = a_j$, $b = b_j$. If $a \succ b$, then it is not possible that $B > A$. Also $b = n^*$, $a = n$ is not possible; otherwise there would be an earlier pair a_i, $b_i = a_i^*$, $i < j$, that would qualify as the first pair such $a_j \neq b_j$. Thus $b \succ a$. Let $[a, b] = \{x \mid a \preceq x \preceq b\}$. One of the following cases must hold:

(1) There is a $c \in [a, b]$ such that either c lies to the right of a in A or neither c nor c^* appears in A.
(2) There is a $c \in [a, b]$ such that c lies to the right of b in B or neither c nor c^* appears in B.
(3) There is a $c^* \in [a, b]^*$ such that c^* lies to the right of a in A.

In fact, if neither (1) nor (2) holds, then clearly b can play the role of c in (3) unless $b = a^*$. But if $b = a^*$ and neither (1) nor (2) holds, then the non-closeness of A and B is violated.

Case 1. Assume that such a c exists. If such c exists to the right of a in A, let $c = a_k$ be the first such element as we move to the right of a in A. (If neither c nor c^* appear in A for all such c, let $k = \infty$.) The elements $a_i \in A$ between a and a_k do not lie in $[a, b]$.

There are now two subcases. Assume that $a_i = b_i$ for all $i < j$. Let C be obtained from A by applying the involution $(ac)(a^*c^*)$. Clearly $C > A$, since c must occur in a separate block of A from a, since $c > a$ and we assumed elements in a block are ordered in decreasing order. It remains to show that $B \geq C$. We must show that $B_i \geq C_i$ for all $j < i < k$. We have $B_i > A_i$ and we may assume that all elements to the left of b in B_i are used to dominate the elements to the left of a in A. The element b in B_i must be used to dominate an element of A that is less than or equal to a. But then, without loss of generality, we may assume that b is used to dominate a. Using the same correspondence we have $B_i \geq C_i$.

Now assume the other subcase, that $b_i = a_i^*$ for some $i < j$. Arguing as above, for $B_i > A_i$ it may be the case that b is used to dominate a_i. By the choice of j satisfying properties (i) and (ii) above, b is less than a_i in numerical value, and hence it must be the case that $b \succ b_i = a_i^*$. But then, without loss of generality, we can take b_i to dominate a_i and b to dominate a. Then proceed as in the paragraph above.

Case 2 is proved in an analogous fashion to Case 1, finding a C such that $B > C \geq A$.

Case 3. Assume that neither case (1) nor case (2) holds. We have already seen that $b \neq a^*$. The situation is now divided into two possibilities. Either

(1) a and b are both unstarred or a is unstarred and $b = n^*$; or
(2) a and b are both starred or b is starred and $a = n$.

The two cases are exhaustive. To see this let $a \neq n$ be unstarred and $b \neq n^*$ starred. Either a is greater than b in numerical value or b is greater than a in numerical value. Assume the former; the argument is the same in either case. Then $a \in [a, b]$ and $a^* \in [a, b]$. One of the following must be true: neither a nor a^* appear in B (which is case 2) or one of a or a^* appears to the right of b in B (also case 2). (Note that $b = n^*,\ a = n$ is not possible.)

We consider the case where both a and b are unstarred or a is unstarred and $b = n^*$; the argument in the second case is analogous. We assume that there is a $c^* \in [a, b]^*$ such that c^* lies to the right of a in A. Let $c^* = a_k$ be the last such starred element as we move to the right of a in A. In other words, all other elements in $[a, b]^*$ that lie to the right of a in A lie to the left of c^*. Let C be obtained from A by applying the involution $(ac)(a^*c^*)$. Clearly $C > A$; it remains to show that $B \geq C$. In particular, we must show that $B_i \geq C_i$ for all i such that $k_i > j$. We have $B_i > A_i$ and we can assume, by the same reasoning as in case (1), that all elements to the left of b in B are used to dominate the elements to the left of a in A. Since no elements in $[a, b]$ appear to the right of a in A, there is no loss of generality in assuming that b is used to dominate a. Then, unless $k_i \geq k$, the same correspondence between the elements of A_i and B_i shows that $B_i \geq C_i$.

If $k_i \geq k$, then consider the set X of starred elements of A_i (and n if $b = n^*$) that lie to the right of a and the set Y of starred elements of B_i that lie to the right of b. It is not necessary to consider elements to the left of a or b because, even if for some such pair we have $b_i = a_i^*$, the element b_i cannot be used to dominate any element of X in the Gale order $B_i > A_i$ since all elements in X are less than b_i in numerical value (and hence greater in order). So in the Gale order $B_i > A_i$ the elements of X must be dominated by elements of Y. Indeed, $X \cap ([a, b]^*) \setminus \{b^*\} \supseteq Y \cap ([a, b]^*) \setminus \{a^*\}$. There may be elements of $([a, b]^*) \setminus \{a^*\}$ not in B_i, by virtue of appearing to the right of position k_i. But in that case, even larger elements of Y must be used to dominate $X \cap ([a, b]^*) \setminus \{b^*\}$. By letting each element be used to dominate itself, where possible, we see that there is no loss of generality in assuming that $x \geq a^*$, $x \in B_i$ dominates $b^* \in A_i$. But now almost the same correspondence shows that $B_i \geq C_i$. Merely make the changes that $x \in B_i$ dominates $a^* \in C_i$ and $c^* \in B_i$ (or some larger element) dominates $b^* \in C_i$, whereas b now dominates c. This completes the proof of statement (1) of Theorem 1.

A similar though considerably simpler proof of statement (2) in Theorem 1 can be given. In fact, a general proof for all Coxeter groups with a linear diagram was given in [13]. This would include the B_n case, but not the D_n case. □

6. Cryptomorphisms

We have seen that for D_n, the bijection $g : D_n/P \to \mathcal{A}_{(k_1,\dots,k_m)}$ is not a poset isomorphism between Bruhat order and Gale order, but rather that Bruhat order is weaker than Gale order. It is therefore somewhat surprising that the Bruhat maximality condition is still equivalent to the Gale maximality condition. We now prove the equivalence of our three definitions of D_n matroid, the algebraic, geometric, and combinatorial descriptions.

Theorem 2 *Let $L \subseteq D_n/P$ be a collection of cosets, let $\Delta = \Delta(f(L))$ be the corresponding polytope, and let $\mathcal{F} = g(L) \subseteq \mathcal{A}_{(k_1,\dots,k_m)}$ be the corresponding collection of flags. Then the following are equivalent.*
(1) *L satisfies the Bruhat maximality condition,*
(2) *$\Delta(L)$ satisfies the root condition,*
(3) *$\mathcal{F}$ satisfies the Gale maximality condition.*

Proof: This theorem follows from Theorem 3 in [13]. However, that paper considers a much more general setting, and the proof, including that of the prerequisite Theorem 1 of [13], is quite long and involved. The situation simplifies considerably in the current setting. The equivalence of statements (1) and (2) is a special case of the Gelfand-Serganova Theorem and is proved for any finite irreducible Coxeter group in [12, Theorem 5.2.] Now we show the equivalence of (1) and (3).

Assume that $L \subseteq D_n/P$ satisfies the maximality condition with respect to Bruhat order. This means that, for any $w \in D_n$, there is a maximum $\bar{u}_0 \in L$ such that $w^{-1}\bar{u}_0 \succeq w^{-1}\bar{u}$ for all $\bar{u} \in L$. But by statement (1) of Theorem 1 this implies that $w^{-1}(\bar{u}_0(F_0)) = g(w^{-1}\bar{u}_0) \geq g(w^{-1}\bar{u}) = w^{-1}(\bar{u}(F_0))$ in the weak D_n Gale order for all $\bar{u} \in L$, where F_0 is the flag defined in Section 4. So $\mathcal{F}$ satisfies the Gale maximality condition.

Conversely, suppose that $\mathcal{F}$ satisfies the Gale maximality condition with respect to admissible D_n-orderings. Since every admissible B_n-ordering is a refinement of an admissible

D_n-ordering, it is clear that $\mathcal{F}$ satisfies the Gale maximality condition with respect to admissible B_n-orderings. By statement (2) of Theorem 1, the corresponding collection of cosets of B_n/P satisfies the Bruhat maximality condition, and hence the corresponding polytope $\Delta(L)$ satisfies the root condition for B_n by the above-mentioned equivalence of (1) and (2) for finite irreducible Coxeter groups. Note that if we consider the same set of flags $\mathcal{F}$ as both a D_n matroid and a B_n matroid, $\Delta(L)$ is the same polytope in both cases. If $\Delta(L)$ also satisfies the root condition for D_n, we are done. Therefore we assume, by way of contradiction, that $\Delta(L)$ does not satisfy the root condition for D_n, and hence that there is an edge of $\Delta(L)$ which is parallel to e_i for some i. Let this edge be $\delta_A\delta_B$ for some pair of flags $A, B \in \mathcal{F}$, $h(A) = \delta_A$ and $h(B) = \delta_B$, where h is the bijection from Section 4. Then A and B differ only in element i, and hence for some k, $a_k = i$, $b_k = i^*$. Now let $f = \sum_{j=1}^n \gamma_j x_j$ be a linear functional which takes its maximum value on the polytope $\Delta(L)$ only on the two vertices $\delta_A\delta_B$ and, of course, the edge between them. Clearly $\gamma_i = 0$. Without loss of generality, we may assume that f is chosen so that $\gamma_j \neq 0$ for all $j \neq i$. Now choose an admissible D_n-ordering on $[n] \cup [n]^*$ according to the values of γ_j: if we write $\gamma_{j^*} = -\gamma_j$, then when $\gamma_j > \gamma_k$ for $j, k \in [n] \cup [n]^*$, set $j > k$, and when $\gamma_j = \gamma_k$, break the tie arbitrarily, as long as admissibility is attained. Also i and i^* remain incomparable. In the Gale order induced by this admissible order, A and B are unrelated. Suppose there exists some flag $X > A$ in the Gale order. Clearly $f(\delta_X) \geq f(\delta_A)$, contradicting the fact that δ_A and δ_B are the unique vertices of $\Delta(L)$ on which f is maximized. Thus A, and likewise B, are both maximal in the D_n Gale order, contrary to assumption. □

We now consider a fourth equivalent definition of orthogonal matroid in the case that the marked diagram has both nodes $n-1$ and n circled. In this case an orthogonal matroid $L \subseteq \mathcal{A}_{(k_1,\ldots,k_m)}$ has $k_m = n-1$, and the largest member A_m of each flag is an admissible set of cardinality $n-1$. It is easily seen that the collection of all such $(n-1)$-sets for all members of L itself constitutes an orthogonal matroid of rank $n-1$. (This is likewise true for smaller ranks, as well as for all ranks for B_n and A_n matroids.) However, the present case is the only one among all of these in which the parabolic subgroup P is not *maximal* since, by the way the diagram is defined, the two generators corresponding to $n-1$ and n are both deleted to get the generators of P. Thus the idea presents itself that such an orthogonal matroid of rank $n-1$ should be equivalent in some way to a pair of orthogonal matroids of opposite parity, corresponding to the two marked diagrams with either $n-1$ or n circled. This is indeed the case. Let

$$F_A = (A_1 \subset A_2 \subset \cdots \subset A_m) \in \mathcal{A}_{(k_1,k_2,\ldots,k_m)}$$

be a flag with $k_m = n-1$, and let A_m^+ and A_m^- denote the unique extensions of A_m to admissible sets of cardinality n having an even and an odd number of starred elements, respectively. Let us denote

$$\Theta_A = (A_1 \subset A_2 \subset \cdots \subset A_{m-1} \subset \{A_m^+, A_m^-\}),$$

where the notation is intended to convey that A_{m-1} is a subset of both A_m^+ and A_m^-, whereas the latter two are not to be regarded as occurring in any particular order since they are unrelated by containment. Θ_A is sometimes referred to as an *oriflamme*. Given a D_n admissible order, we will write $\Theta_A \leq \Theta_B$ if $A_i \leq B_i$ in D_n Gale order, for each $i = 1, \ldots, m-1$, and if, furthermore, $\{A_m^+, A_m^-\} \leq \{B_m^+, B_m^-\}$, which we define to mean that either $A_m^+ \leq B_m^+$ and $A_m^- \leq B_m^-$, or $A_m^+ \leq B_m^-$ and $A_m^- \leq B_m^+$. We will refer to this ordering as modified Gale order.

Theorem 3 *For any two flags F_A, F_B of the same type $(k_1, k_2, \ldots, k_m)$ with $k_m = n-1$, with corresponding oriflammes Θ_A, Θ_B, we have $F_A \geq F_B$ if and only if $\Theta_A \geq \Theta_B$. Hence a collection of flags is an orthogonal matroid if and only if the corresponding collection of oriflammes satisfies the maximum condition for modified Gale order.*

Proof: It suffices to consider the case $m = 1$, so that $A = A_1$ and $B = B_1$ are admissible sets of cardinality $n-1$. We may also write $\Theta_A = \{A \cup \{x\}, A \cup \{x^*\}\}$, $\Theta_B = \{B \cup \{y\}, B \cup \{y^*\}\}$, where x, x^* are the unique pair neither of which is in A, and similarly y, y^* for B. Note that if $x = y$ or y^* then the theorem is trivial, so we assume henceforth that this is not the case.

Suppose $\Theta_A \geq \Theta_B$. If $x > y, y^*$ and $y, y^* > x^*$, then since $A \cup \{x^*\} \geq B \cup \{y\}$ or $A \cup \{x^*\} \geq B \cup \{y^*\}$, we see immediately that $A \geq B$. Similar arguments cover the remaining possible cases of the admissible order restricted to x, x^*, y, y^*.

To prove the converse, assume that $A \geq B$. We must prove that $\{A \cup \{x\}, A \cup \{x^*\}\} \geq \{B \cup \{y\}, B \cup \{y^*\}\}$. The assumption that $A \geq B$ means that there is a bijection κ from A to B, so that $a \geq \kappa(a)$ for all $a \in A$. Note that we can assume without loss of generality that κ maps any element b of $A \cap B$ to itself, for if $\kappa(a) = b$ and $\kappa(b) = b'$, then we can reassign $\kappa(a) = b'$ and $\kappa(b) = b$. Thus we have $\kappa(a) \in B \setminus A$ if and only if $a \in A \setminus B$.

Let us assume that our admissible ordering restricted to x, x^*, y, y^* gives $x > y, y^*$ and $y, y^* > x^*$. The argument for the remaining cases is similar. (In particular, when x and x^* are between y and y^*, we just need to reverse the roles of A and B and reverse the ordering to transform the argument to the above case.) Notice that $A \cup \{x\} \geq B \cup \{y\}$ and $A \cup \{x\} \geq B \cup \{y^*\}$ as well. Thus we only need to prove either $A \cup \{x^*\} \geq B \cup \{y\}$ or $A \cup \{x^*\} \geq B \cup \{y^*\}$. Since $x \neq y, y^*$, we have two cases: either $x \in B$ or $x^* \in B$.

Case 1: $x \in B$. Write $x = b_1$, and since $b_1 \notin A$, we have $b_1 = \kappa(a_1)$ for some $a_1 \in A \setminus B$. Since $a_1 > b_1 \geq y, y^*$, and $a_1 \notin B$ we have $a_1^* \in B \setminus A$. We denote $b_2 = a_1 < x^*$ and have $b_2 = \kappa(a_2)$ for $a_2 \in A \setminus B$. Now if $a_2 \geq y$ or $a_2 \geq y^*$, then we are done, for we can define $\kappa'(a_2) = y$ and $\kappa'(x^*) = b_2$ (or similarly with y^*), with the rest of κ' agreeing with κ, and κ' establishes the desired (modified) Gale dominance. Hence we may now assume that $a_2 < y, y^*$. Since $a_2 \in A \setminus B$ and $a_2 \neq y, y^*$, we have $a_2^* \in B \setminus A$. Denoting $b_3 = a_2^* = \kappa(a_3)$ for $a_3 \in A \setminus B$ and $a_3 > a_2^* > y, y^*$, so $a_3^* \in B \setminus A$, and we write $b_4 = a_3^* = \kappa(a_4)$ for some $a_4 \in A \setminus B$.

We continue in this fashion until either two a_i's coincide (and hence so do the corresponding b_i's), or until some $a_i \geq y$ or $a_i \geq y^*$ for some even i, one of which must occur eventually since A is finite. In the case of a coincidence, since $a_i > y, y^*$ for i odd but not for i even, our coincidence is of the form $a_i = a_j$ for i, j of the same parity. Assume that i is the minimal index in such a coincidence. But then $b_i = b_j$ since κ is a bijection, which if $i \geq 1$ means $a_{i-1}^* = a_{j-1}^*$, and hence $a_{i-1} = a_{j-1}$. By minimality of i, we must have had $b_1 = b_k$ for some odd $k \geq 3$. But $b_1 = x$ and $b_k = a^*_{k-1}$, showing that $x^* \in A$, a contradiction. Thus we could not have two a_i's coincide, so we must instead have $a_i \geq y$ (or y^*) for some even i. Note that for even $j \geq 2$, we have $a_j = b^*_{j+1} \geq a^*_{j+1} = b_{j+2}$. We now define $\kappa'(a_i) = y$ (or y^*), $\kappa'(a_j) = b_{j+2}$ for all even j with $2 \leq j \leq i-2$, $\kappa'(x^*) = b_2$, and otherwise κ' agrees with κ. This gives the desired result for Case 1.

Case 2: $x^* \in B$. Starting with $b_1 = x^*$, we construct a_i's and b_i's exactly as above, except that now it is the even-numbered a_i which are always larger than y, y^* and we want to find one odd-numbered a_i which is greater than either y or y^*. The rest of the proof is similar to Case 1, with $b_1 = x^*$ being used to contradict any coincidence among the a_i's. □

7. Relation between symplectic and orthogonal matroids

In this section we view both symplectic matroids and orthogonal matroids in terms of their combinatorial description. Both are defined in terms of admissible k-element subsets of $[n] \cup [n]^*$. The number k is called the *rank* of the symplectic or orthogonal matroid.

Corollary 1 *Every orthogonal (flag) matroid is a symplectic (flag) matroid.*

Proof: This follows directly from Theorem 2 since every admissible set of flags for D_n is admissible for B_n and every root of D_n is a root of B_n. □

In general, the converse is false. For example, $\{12, 12^*\}$ is a rank 2 symplectic (B_2) matroid, but is not an orthogonal (D_2) matroid. We are not able, in general, to give a simple combinatorial characterization of when a symplectic matroid is orthogonal (the geometric characterization is obvious). Below, however, is a characterization for the special case of a rank n symplectic matroid $L \subseteq \mathcal{A}_n$, called a Lagrangian matroid in [2] or a symmetric matroid in [6].

Theorem 4 *A B_n matroid L of rank n is a D_n matroid if and only if L lies either entirely in $\mathcal{A}_n^+$ or entirely in $\mathcal{A}_n^-$.*

Proof: Assume that L is a symplectic matroid. In one direction the result follows directly from the definition of orthogonal matroid.

For the other direction, assume that L lies either entirely in $\mathcal{A}_n^+$ or entirely in $\mathcal{A}_n^-$. Consider any admissible orthogonal ordering $\preceq$ of $[n] \cup [n]^*$. Without loss of generality, let $\preceq$ be the ordering in (3.3). Use the notation $\preceq_s$ for either one of the two admissible symplectic orderings that are linear extensions of $\preceq$. With respect to $\preceq_s$ there is a Gale maximum $A = (a_1, \ldots, a_n)$, where the a_i can be taken in descending order with respect to $\preceq_s$. Thus for any $B = (b_1, \ldots, b_k)$, also in descending order, we have $b_i \preceq_s a_i$ for all i. However, the same inequality $b_i \preceq a_i$ also holds for the orthogonal ordering unless n and n^* appear in the same position in A and B, resp. But that can happen only if $A \cap [n-1] = B \cap [n-1]$. This would imply $A \in \mathcal{A}_n^+$ and $B \in \mathcal{A}_n^-$ or the other way around, a contradiction. □

8. Representable orthogonal matroids

Some symplectic matroids and orthogonal matoids arise naturally from symplectic and orthogonal geometries, respectively, in much the same way that ordinary matroids arise from projective geometry. The representation of symplectic matroids was discussed in [2]; the representation of orthogonal matroids is discussed in this section. However, it is convenient to consider the symplectic and the orthogonal case simultaneously; this leads to a simplified treatment of the symplectic case as well as a proof in the orthogonal case. We will consider only the representation of symplectic and orthogonal matroids of type k, for some $k \leq n$. Flag symplectic and flag orthogonal matroids can similarly be represented using flags of totally isotropic subspaces.

Both a *symplectic space* and an *orthogonal space* consist of a pair (V, f) where V is a vector space over a field of characteristic $\neq 2$ with basis

$$E = \{e_1, \dots, e_n, e_{1^*}, \dots, e_{n^*}\},$$

and f is a bilinear form hereafter denoted just $(\cdot, \cdot)$. The bilinear form is antisymmetric for a symplectic space and symmetric in an orthogonal space. In both cases

$$\begin{aligned} (e_i, e_{i^*}) &= 1 \quad \text{for all } i \in [n] \\ (e_i, e_j) &= 0 \quad \text{for all } i \neq j^*, \ \text{where } i, j \in [n] \cup [n]^*. \end{aligned}$$

A subspace U of V is *totally isotropic* if $(u, v) = 0$ for all $u, v \in U$.

Let U be a totally isotropic subspace of dimension k of either a symplectic or an orthogonal space V. Since $U \perp U$, and $\dim U^\perp = 2n - \dim U$, we see that $k \leq n$. Now choose a basis $\{u_1, u_2, \dots, u_k\}$ of U, and expand each of these vectors in terms of the basis E: $u_i = \sum_{j=1}^n a_{i,j} e_j + \sum_{j=1}^n b_{i,j} e_{j^*}$. Thus we have represented the totally isotropic subspace U as the row-space of a $k \times 2n$ matrix $(A \mid B)$, $A = (a_{i,j})$, $B = (b_{i,j})$, with the columns indexed by $[n] \cup [n]^*$, specifically, the columns of A by $[n]$ and those of B by $[n]^*$.

Given a totally isotropic subspace U of dimension k, let $C = (A \mid B)$ be a $k \times 2n$ matrix defined above. If X is any k-element subset of $[n] \cup [n]^*$, let C_X denote the $k \times k$ minor formed by taking the j-th column of C for all $j \in X$. Define a collection L_U of k-element subsets of $[n] \cup [n]^*$ by declaring $X \in L_U$ if X is an admissible k-element set and $\det(C_X) \neq 0$. Note that L_U is independent of the choice of the basis of U.

Theorem 5 *If U is a totally isotropic subspace of a symplectic or orthogonal space, then L_U is the collection of bases of a symplectic or orthogonal matroid, respectively.*

Proof: The fact that the row space is totally isotropic implies

$$\sum_j a_{ij} b_{lj^*} (e_j, e_{j^*}) + a_{ij^*} b_{lj} (e_{j^*}, e_j) = 0$$

for all i, l. In terms of the matrices A and B this is equivalent to

$$A_i \bullet B_l \pm A_l \bullet B_i = 0, \tag{8.1}$$

where A_i and B_i denote the respective row vectors and $\bullet$ denotes the the usual dot product. The sign is $+$ in the orthogonal case and $-$ in the symplectic case. In the orthogonal case, taking $i = l$, the equality (8.1) above implies

$$A_i \bullet B_i = 0. \tag{8.2}$$

Let $\preceq$ be any admissible ordering of $[n] \cup [n]^*$. Order the columns of C in descending order with respect to $\preceq$. (The order of n and n^* is arbitrary in the orthogonal case.) The re-ordering may be done by first interchanging pairs of columns indexed by j and j^* for some j. In order to maintain (8.1) in the symplectic case (where we still consider A to be the first n columns of C), one of the two interchanged columns must be multiplied by -1. Note that this does not change L_U. Second, do like column permutations on both A and B. Finally, reverse the order of the columns of B. Then (8.1) and (8.2) remain valid, provided we now interpret $X \bullet Y$ to mean the dot product of X with the reverse of Y.

In light of the preceding paragraph, we may, without loss of generality, assume $\preceq$ is the ordering (3.2) in the symplectic case and the ordering (3.3) in the orthogonal case. This will keep our notation simpler. We must show that L_U contains a maximum member with respect to the induced Gale ordering. Using the usual row operations, put C in echelon form so that each row has a leading 1, the leading 1 to the right of the leading one in the preceeding row, and zeros filling each column containing a leading 1. Let X_0 be the subset of $[n] \cup [n]^*$ corresponding to the columns with leading ones. It is now sufficient to show that (1) X_0 is admissible, and (2) $X \leq X_0$ for any X such that the determinant of the $k \times k$ minor C_X of C corresponding to X is non-zero.

Concerning (1), assume that X_0 is not admissible. Then both j and j^* appear in X_0 for some $j \in [n]$. Let i and l be the rows for which there is a leading 1 at positions j and j^*, respectively. Then $A_i \bullet B_l \pm A_l \bullet B_i = 1$, contradicting equality (8.1).

Concerning (2), if it is not the case that $X \leq X_0$, then, for some j, the first j columns of C_X have at least $k - j + 1$ rows of zeros. But such a matrix C_X has determinant 0. The exception to this argument is in the orthogonal case when X and X_0 contain the incomparable elements, n and n^*, in the same position when the elements of X and X_0 are arranged in descending order. However, for this to happen, there must be a row of C, say the j-th, such that $A_j = (0, \ldots, 0, 0)$ or $A_j = (0, \ldots, 0, 1)$. If $A_j = (0, \ldots, 0, 1)$, then by equality (8.2) we have $b_{jn^*} = A_j \bullet B_j = 0$. In either case the first j columns of C_X or C_{X_0} again have at least $k - j + 1$ rows of zeros, implying that $\det(C_X) = 0$ or $\det(C_{X_0}) = 0$. □

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