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Simplicial Properties of the Set of Planar Binary Trees

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Abstract. Planar binary trees appear as the the main ingredient of a new homology theory related to dialgebras, cf.(J.-L. Loday, *C.R. Acad. Sci. Paris* **321** (1995), 141–146.) Here I investigate the simplicial properties of the set of these trees, which are independent of the dialgebra context though they are reflected in the dialgebra homology.

The set of planar binary trees is endowed with a natural (almost) simplicial structure which gives rise to a chain complex. The main new idea consists in decomposing the set of trees into classes, by exploiting the orientation of their leaves. (This trick has subsequently found an application in quantum electrodynamics, c.f. (C. Brouder, "On the Trees of Quantum Fields," Eur. Phys. J. C12, 535–549 (2000).) This decomposition yields a chain bicomplex whose total chain complex is that of binary trees. The main theorem of the paper concerns a further decomposition of this bicomplex. Each vertical complex is the direct sum of subcomplexes which are in bijection with the planar binary trees. This decomposition is used in the computation of dialgebra homology as a derived functor, cf. (A. Frabetti, "Dialgebra (co) Homology with Coefficients," Springer L.N.M., to appear).

Keywords: planar binary trees, almost-simplicial sets

Introduction

The planar binary trees have been widely studied for their combinatorial properties, which relate them to permutations, partitions of closed strings and other finite sets. In fact, the cardinality of the set Y_n of planar binary trees with $n + 1$ leaves and one root is the Catalan number $c_n = \frac{2n!}{n!(n+1)!}$, which is well known to have many combinatorial interpretations [6].

In 1994, in the paper [10] written by J.-L. Loday, these trees appear as the main ingredient in the homology of a new kind of algebras, called *dialgebras*, equipped with two binary associative operations. Instead of the single copy $A^{\otimes n}$, which forms the module of Hochschild n-chains of an associative algebra A, Loday finds out that the module of n-chains of a dialgebra D is made of c_n copies of $D^{\otimes n}$. The crucial observation is that labelling each copy of $D^{\otimes n}$ by an n-tree leads to a very natural and simple definition of the face maps: the i-th face of an n-tree is obtained by deleting its i-th leaf. Hence the set of rooted planar binary trees acquires an important role in the simplicial context of dialgebra homology. The study of this homology leads to the investigation of the simplicial structure of the set of trees, which is completely independent of the dialgebra context and constitutes the content of this paper.

The set of trees can be equipped with degeneracy operators s_j which satisfy all the classical simplicial relations except that $s_i s_i \neq s_{i+1} s_i$. For such a set, which is called *almost-simplicial*, some of the properties of simplicial sets still hold, for instance the Eilenberg-Zilber Theorem, cf. [8].

The main idea of the paper consists in decomposing the set of trees into classes, by exploiting the orientation of their leaves. This trick is purely combinatorial (set-theoretical), and it is explained in Section 1. Then I show that this decomposition is compatible with the almost-simplicial structure and yields a chain bicomplex whose total chain complex is that of binary trees. Consequently, in the application to dialgebras, we obtain a canonical spectral sequence converging to the dialgebra homology.

The main theorem of the paper concerns a further decomposition of this bicomplex. I show that each vertical complex is in fact the direct sum of subcomplexes, which I call *towers*. It turns out that these towers are in bijection with the planar binary trees. The vertical complex in degree p is the sum of the towers indexed by the planar binary trees of order p. Again, this decomposition is purely combinatorial. An illustrative picture of the situation is placed at the beginning of Section 2, where I define the vertical towers, by means of a new kind of degeneracy operator, and prove the decomposition theorem (2.12).

The main application of this decomposition is the interpretation of dialgebra homology as a Tor functor given in [5].

The basic idea of discerning the orientation of the leaves of planar binary trees finds another application in [2], where C. Brouder employs planar binary trees to describe the solution of the Schwinger equations coupling the full fermion and photon propagators of quantum electrodynamics. The left (resp. right) orientation of the leaves correspond to the photon (resp. fermion) component of the solutions.

I would like to thank J.-L. Loday and T. Pirashvili for the useful conversations about the planar binary trees and the simplicial tricks, and finally the Referees for their accurate suggestions, which brought remarkable ameliorations to the exposition of the results. I am very grateful to the “Institut de Recherche Mathémetique Avancée de Strasbourg (France)” for supporting me during the preparation of the paper.

Notation. For any set X and any field k, I denote by $k[X]$ the vector space over k generated by the elements of X.

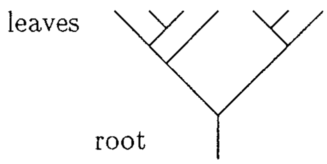
1. Double simplicial structure on the set of binary trees

In this section I recall the notion of almost-simplicial structure (cf. [8]) and prove that the family of planar binary trees is an almost-simplicial set (announced in [4]). I also show that the associated chain complex is acyclic.

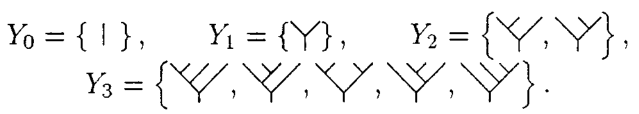
Then I introduce some classes of planar binary trees, whose cardinality is computed in Appendix A. The faces and degeneracies previously defined are compatible with the decomposition of the set of planar binary trees into the classes. As a result, there exists a chain bicomplex whose total chain complex is that of planar binary trees. An important application, given in [5], concerns the dialgebra homology defined in [10]: the bicomplex of trees induces a non trivial spectral sequence converging to the dialgebra homology.

Faces and degeneracies on the sets of planar binary trees

1.1. Planar binary trees. By a *planar binary tree* I mean an open planar graph with 3-valent internal vertices. Among the external vertices, fix a preferred one and call it the *root*. Usually I draw the root at the bottom of the tree. The remaining external vertices, called *leaves*, are drawn at the top of the tree:



For any natural number n, let Y_n be the set of planar binary trees with $n+1$ leaves, which are labelled as $0, 1, \ldots, n$ from left to right. Given a tree y with $n+1$ leaves, call *order of* y the natural number $|y| := n$. Notice that the order of a tree is the number of internal vertices. Therefore, for $n = 0$, I consider the unique tree with one leaf, the root and no internal vertices. Here is a picture of the sets Y_n for $n = 0, 1, 2, 3$:



The cardinality of the set Y_n is given by the *Catalan number* (see [9], [1], [3] and [6])

$$c_n = \frac{2n!}{n!(n+1)!}.$$

Hence the sets $Y_0, Y_1, Y_2, \ldots$ have cardinality 1, 1, 2, 5, 14, 42, 132 and so on.

In the sequel I abbreviate “planar binary tree” into “tree” and “planar binary tree with $n+1$ leaves” into “n-tree”.

1.2. Pseudo and almost-simplicial sets. Recall that a *pre-simplicial set* E is a collection of sets E_n, one for each $n \geq 0$, equipped with face maps $d_i : E_n \to E_{n-1}$, for any $i = 0, \ldots, n$, satisfying the relations

$$(d) \qquad d_i d_j = d_{j-1} d_i, \qquad i < j.$$

Given a field k, consider the k-linear span $k[E_n]$ of the elements of the set E_n. The faces give rise to the boundary operator $d : k[E_n] \to k[E_{n-1}]$, $d = \sum_{i=0}^{n} (-1)^i d_i$ which satisfies $d \circ d = 0$. Therefore any pre-simplicial set $\{E_n, d_i\}$ always gives rise to a chain complex $(k[E_*], d)$.

Also recall that a *simplicial* set is equipped with degeneracy maps $s_j : E_n \to E_{n+1}$, for any $j = 0, \dots, n$, which satisfy the relations

$$(ds) \qquad d_i s_j = \begin{cases} s_{j-1} d_i, & i < j, \\ id, & i = j, j+1, \\ s_j d_{i-1}, & i > j+1, \end{cases}$$

$$(s) \qquad s_i s_j = s_{j+1} s_i, \qquad i \leq j.$$

By definition, a *pseudo-simplicial* set is a family of sets endowed with faces and degeneracies satisfying relations (d) and (ds) but not necessarily relations (s) (see [11] and [18]).

Define an *almost-simplicial* set to be a pseudo-simplicial set whose degeneracies satisfy relations (s) *except* for $i = j$, which means that $s_i s_j = s_{j+1} s_i$ for $i < j$ and in general (but not necessarily) $s_i s_i \neq s_{i+1} s_i$.

Clearly all simplicial or almost-simplicial sets are pseudo-simplicial,

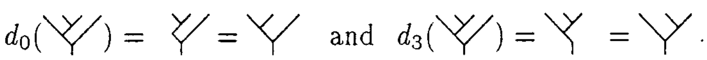
$$\{\text{simplicial sets}\} \subset \{\text{almost-simplicial sets}\} \subset \{\text{pseudo-simplicial sets}\} \subset \{\text{pre-simplicial sets}\}.$$

Let us consider now the set of binary trees described in (1.1). Trees can be obtained one from another by repeating two basic operations: deleting and adding leaves. The operation of deleting leaves allows us to define face maps $Y_n \to Y_{n-1}$ and thus to consider the associated chain complex $k[Y_*]$ for any given field k. The operation of adding leaves allows us to define degeneracy maps $Y_n \to Y_{n+1}$.

1.3. Face maps on trees. For any $n \geq 0$, and any $i = 0, \dots, n$, the ith *face* is the map

$$d_i : Y_n \to Y_{n-1}$$

which associates to an n-tree y the $(n-1)$-tree $d_i(y)$ obtained by *deleting* the ith leaf from y. For example:



1.4. Lemma. *The face maps d_i satisfy the above relations (d). Hence, given a field k, the sequence*

$$k[Y_0] \longleftarrow k[Y_1] \longleftarrow k[Y_2] \longleftarrow \cdots \longleftarrow k[Y_n] \longleftarrow \dots$$

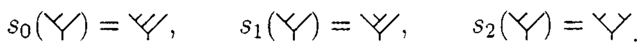
is a chain complex, with boundary operator $d : k[Y_n] \to k[Y_{n-1}]$ given by $d = \sum_{i=0}^{n} (-1)^i d_i$.

Proof: In fact, since the leaf number j of the tree y is the leaf number $j-1$ of the tree $d_i(y)$, the maps $d_i d_j$ and $d_{j-1} d_i$ produce the same modification: they delete the leaves number i and j. □

1.5. Degeneracies on trees. For any $n \geq 0$, and any $j = 0, \ldots, n$, the jth *degeneracy* is the map

$$s_j : Y_n \to Y_{n+1}$$

which *bifurcates* the jth leaf of an n-tree, i.e. which replaces the jth leaf | by the branch. For example:



1.6. Lemma. *The degeneracy maps satisfy the above relations* (ds). *They also satisfy* (s) *for* $i < j$, *hence the set of binary trees* $\{Y_n, d_i, s_j\}$ *is almost-simplicial.*

Proof: (ds) The operations $d_i s_j$ on a tree y first adds a leaf replacing the leaf number j by the branch, and then deletes the leaf number i. So, when $i < j$ or $i > j+1$, we obtain the same tree if we invert the operations on the suitable leaves. When $i = j$ or $j+1$, the operator d_i evidently brings the tree $s_j(y)$ (with branch labelled by j, $j+1$) back to the original tree.

(s) The operation $s_i s_j$ on a tree y first bifurcates the leaf number j and then bifurcates the leaf number i. So it is clear that if $i < j$ the same tree can be obtained performing the two bifurcations in the inverted order, observing that the jth leaf of y is the leaf number $j+1$ of $s_i(y)$. □

1.7. Remark. The set of binary trees $\{Y_n, d_i, s_j\}$ is *not* simplicial. In fact, for $i = j$ the operator $s_i s_i$ replace the ith leaf with the branch, operator $s_{i+1} s_i$ produces the branch, hence they do not coincide.

1.8. Theorem. *For any field* k, *the chain complex of binary trees is acyclic, that is*

$$H_n(k[Y_*], d) = \begin{cases} k, & \text{for } n = 0, \\ 0, & \text{for } n > 0. \end{cases}$$

Proof: It is straightforward to check that the map

$$h : Y_n \longrightarrow Y_{n+1}, \qquad h(y) :=$$

satisfies $d_0 h = id$ and $d_i h = h d_{i-1}$ for any $i > 0$, that is, h is an extra-degeneracy (i.e. $h = s_{-1}$ satisfies relations (s)) for the almost-simplicial set of binary trees. It follows that $dh + hd = id$, hence the induced map $h : k[Y_n] \to k[Y_{n+1}]$ is a homotopy between the maps id and 0. □

Classes of planar binary trees and the bicomplex of trees

1.9. Classes of trees. For any pair of natural numbers p, q, let $Y_{p,q}$ be the set of $(p+q+1)$-trees with p leaves oriented like\(excluding the 0-th leaf), and q leaves oriented like/

(excluding the last one). The *class* of an n-tree is specified by the component $Y_{p,q} \subset Y_n$, with $n = p + q + 1$, to which the tree belongs. For example:

 $\in Y_{2,1} \subset Y_4$ and  $\in Y_{1,3} \subset Y_5$.

For any $p, q \geq 0$, the set $Y_{p,0}$ (resp. $Y_{0,q}$) contains only one tree (resp.), called a *comb*.

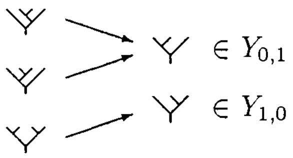
The sets $Y_{p,q}$ are obviously disjoint, for different pairs (p, q), and their disjoint union covers the set Y_{p+q+1}. Hence we have

$$Y_n = \bigsqcup_{p+q+1=n} Y_{p,q}, \quad n \geq 1. \tag{1}$$

For example, $Y_1 = Y_{0,0}$, $Y_2 = Y_{1,0} \sqcup Y_{0,1}$ and $Y_3 = Y_{2,0} \sqcup Y_{1,1} \sqcup Y_{0,2}$. Notice that the number of classes in the set Y_n is precisely $n = \text{ card } \{(p, q) \mid 0 \leq p, q \leq n-1, \ p+q+1 = n\}$.

The orientation of the leaves of an n-tree, given by the numbers p and q of \- and /-leaves, permits us to define a double complex of binary trees, by considering maps which do not change one of the two numbers p, q.

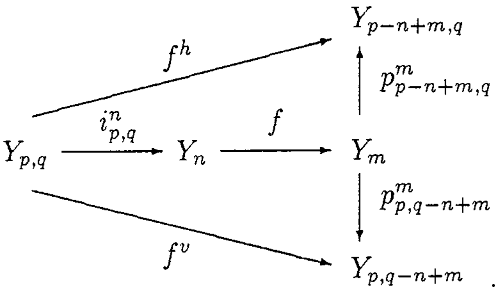
However, in general, a map defined on a class of oriented trees cannot be specified to preserve *globally* one orientation, since it usually changes both values p and q acting on different trees. This happens, in particular, to the face $d_i : Y_n \to Y_{n-1}$, for a fixed $i \in \{0, \ldots, n\}$: when restricted to each component $Y_{p,q} \subset Y_n$, it takes value in one of the two components $Y_{p-1,q}$, $Y_{p,q-1}$ of Y_{n-1}, depending on the tree y of $Y_{p,q}$. Consider, for example, the face d_0 restricted to the component $Y_{1,1} = \{\ ,\ ,\ \} \subset Y_3$. Then $d_0 : Y_{1,1} \to Y_{1,0} \sqcup Y_{0,1}$ takes value in $Y_{0,1}$ on , and in $Y_{1,0}$ on and .

$d_0 : Y_{1,1} \ni$ 

This motivates the following definition.

1.10. Oriented maps. Let $f : k[Y_n] \to k[Y_m]$ be a linear map. Let $k[Y_n] = \oplus_{p+q+1=n} k[Y_{p,q}]$ and $k[Y_m] = \oplus_{r+s+1=m} k[Y_{r,s}]$ be the decompositions into oriented classes induced by (1). Denote by $i^n_{p,q} : k[Y_{p,q}] \hookrightarrow k[Y_n]$ the inclusion for $p + q + 1 = n$, and by $p^m_{r,s} : k[Y_m] \to k[Y_{r,s}]$ the projection for $r + s + 1 = m$. Define the *oriented maps of* f to be the *horizontal* and *vertical* maps f^h and f^v obtained by the composition of the inclusion $i^n_{p,q}$, then f, then the projection $p^m_{r,s}$ for $r = p$ (hence $s = q - n + m$) and respectively

$s = q$ (hence $r = p - n + m$):



Hence, the oriented maps of f take the following values:

- the horizontal $f^h : k[Y_{p,q}] \to k[Y_{p-(n-m),q}]$ has value

$$f^h(y) := \begin{cases} f(y), & \text{if } f(y) \in k[Y_{p-(n-m),q}], \\ 0, & \text{otherwise,} \end{cases}$$

- the vertical $f^v : k[Y_{p,q}] \to k[Y_{p,q-(n-m)}]$ has value

$$f^v(y) := \begin{cases} f(y), & \text{if } f(y) \in k[Y_{p,q-(n-m)}], \\ 0, & \text{otherwise.} \end{cases}$$

In particular, we can consider the oriented maps defined by the faces d_i and the degeneracies s_j.

1.11. Bicomplex of trees. For any natural numbers p, q, take $k[Y_{p,q}]$ as the module of (p, q)-chains, and define horizontal and vertical boundary operators $d^h : k[Y_{p,q}] \to k[Y_{p-1,q}]$, $d^v : k[Y_{p,q}] \to k[Y_{p,q-1}]$ respectively as

$$d^h := \sum_{i=0}^{n} (-1)^i d_i^h \quad \text{and} \quad d^v := \sum_{i=0}^{n} (-1)^i d_i^v, \qquad n = p + q + 1.$$

1.12. Lemma. *The oriented boundaries defined above satisfy $d^h d^h = 0$ and $d^v d^v = 0$, hence $(k[Y_{p,*}], d^v)$ and $(k[Y_{*,q}], d^h)$ are chain complexes for any $p, q \geq 0$.*

Proof: It suffices to show that the oriented faces d_i^h and d_i^v still satisfy the simplicial relations (d) of (1.2). Let us show, for instance, that $d_i^v d_j^v = d_{j-1}^v d_i^v$ for any $i < j$. It suffices to prove that $d_i d_j$ is a vertical map (i.e. $d_i d_j : k[Y_{p,q}] \to k[Y_{p,q-2}]$) if and only if $d_{j-1} d_i$ is vertical. A face d_i deletes a/-leaf if the ith-leaf itself is oriented like /, i.e., ⁄, or if it is a\-leaf such that ⋌.Then it is easy to see that both $d_i d_j$ and $d_{j-1} d_i$ delete two/-leaves only on the four combinations of these two possibilities for the leaves i and j. □

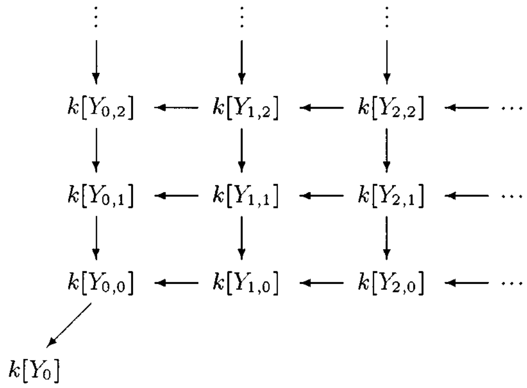


Figure 1. Bicomplex of rooted planar binary trees.

1.13. Remark. By assumption, in a pre-simplicial module the faces are all non-zero maps. Therefore, even if the horizontal (resp. vertical) faces satisfy relations (d), the horizontal families $k[Y_{*,q}]$ (resp. vertical families $k[Y_{p,*}]$) are not considered to be pre-simplicial modules.

1.14. Proposition. *The triple* $(k[Y_{*,*}], d^v, d^h)$ *forms a chain bicomplex, described in Fig. 1, whose total complex is the shifted complex of binary trees* $(k[Y_{*+1}], d)$.

Proof: On any tree y, the map d_i acts either as d_i^h (because $d_i^v(y) = 0$), or as d_i^v (when $d_i^h(y) = 0$). Thus, for any $i = 0, \ldots, n$, we have an obvious identity $d_i = d_i^h + d_i^v$. Consequently, the boundary operator $d : k[Y_{p,q}] \to k[Y_{p-1,q}] \oplus k[Y_{p,q-1}]$ is the sum $d = d^h + d^v$. Then we have $dd = d^h d^h + d^h d^v + d^v d^h + d^v d^v = 0$. From (1.12) it follows that $d^h d^v + d^v d^h = 0$. This shows at the same time that $(k[Y_{*,*}], d^h, d^v)$ is a bicomplex, and that $k[Y_{*+1}] = \mathrm{Tot}(k[Y_{*,*}])$. □

1.15. Remark. The bicomplex of trees gives rise to a spectral sequence

$$E^2_{p,q} = H_p H_q(k[Y_{*,*}]) \Longrightarrow H_{p+q}(k[Y_{*+1}])$$

which is zero everywhere, since the complex of trees is acyclic and the E^1-plane, in a similar way, can be shown to be zero. However the peculiar structure of trees becomes interesting when the vector spaces $k[Y_n]$ appear as tensor components of some chain-modules, as for the chain complex of dialgebras (see [4, 5] and [10]). In this case, the bicomplex of trees permits us to find a spectral sequence which converges to the homology of the given complex.

2. Decomposition of the bicomplex of trees into towers

In this section I show a technical result which helps drastically in the computation of dialgebra homology as a derived functor (see [5]). The main theorem says that any vertical complex $k[Y_{p,*}]$ is a direct sum of subcomplexes whose homology can be computed for some dialgebras.

At the same time, being related to intrinsic properties of the trees, this result clarifies the simplicial structure of the bicomplex. Each subcomplex, called the *vertical tower* and denoted by $T_*(y)$, is constructed on a single tree, called the *base tree*, whose vertical faces are all zero, by applying all possible vertical increasing maps of degree 1, i.e. by adding $/$-leaves in all possible distinct ways. It turns out, due to the particular shape of planar binary trees, that such towers are all disjoint from each other and that they cover the whole bicomplex. This structure yields a decomposition of the bicomplex of trees which has the following remarkable structure:

- The base trees arising in the vertical chain complex $k[Y_{p,*}]$, for fixed $p \geq 0$, are in bijection with p-trees (see Lemma (2.11)), i.e. they are counted exactly by $c_p = \text{ card } Y_p$.
- The vertical tower $T_*(y)$, associated to a p-tree y, is a multi-complex with dimension $d = 2p + 1$ (see Proposition (2.13)).
- The vertical tower $T_*(y)$, associated to a p-tree y, is a subcomplex of $k[Y_{p,*}]$ shifted by the number of$/$-leaves of y (excluding its last leaf). This means that if y belongs to the class $Y_{p',q'}$ of Y_p, then $T_m(y) \subset k[Y_{p,q'+m}]$ for any $m \geq 0$ (see again Lemma (2.11)). A geometrical meaning of the number q' is given in the Appendix B.

Figure 2 is a summarizing picture of the vertical towers at small dimension. The details of the definitions and proofs are given in the remaining part of this section.

New kinds of degeneracies: Grafting operators

In order to construct a vertical complex on a given tree, I need to introduce a second kind of increasing maps $Y_n \to Y_{n+1}$, besides the usual degeneracies s_j.

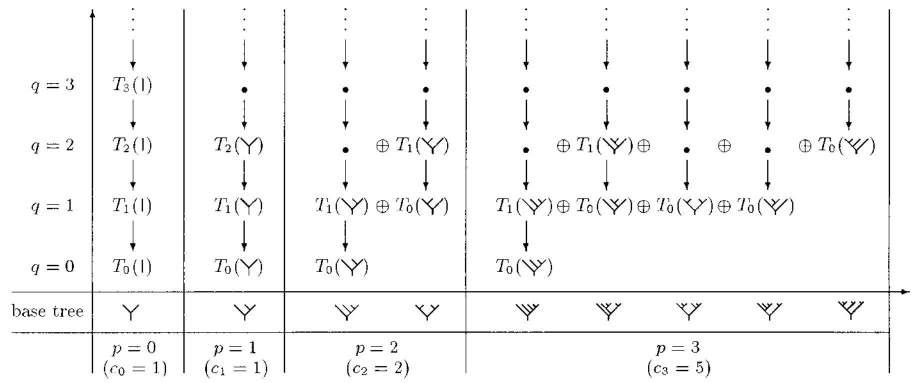
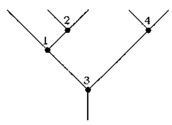


Figure 2. Decomposition of the bicomplex of trees into vertical towers.

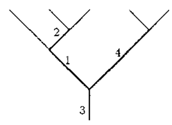
The operation of adding a leaf to a tree consists, more precisely, of grafting a new leaf into a given edge of the tree. The degeneracy operators defined in (1.5), in fact, graft a new leaf into the edge which starts from any existing leaf. Thus, to define the remaining increasing operators, I need a rule to label the internal edges of a tree.

2.1 Labels of internal vertices and internal edges. Any binary tree with $n+1$ leaves and one root has precisely n internal vertices. Let us choose the following rule to label them. An internal vertex is labelled by i if it closes a descending path which starts between the leaves number $i-1$ and i.

An internal edge of the tree is the branch delimited by two adjacent vertices, including the root. I label by i the edge whose 'upper' extreme is a vertex labelled by i. (If we extend this rule to the external edges, each leaf has the same label as the edge which starts from it.) For instance:



and



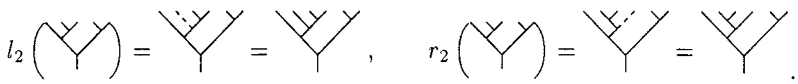
Labels of internal vertices. Labels of internal edges.

In conclusion, any n-tree has $n+1$ external edges (the leaves), labelled from 0 to n, and n internal edges (including the one which ends with the root), labelled from 1 to n.

2.2. Grafting operators. For any $n \geq 1$, and for any $i = 1, \ldots, n$, the ith *left* and *right grafting operator* are the maps

$$l_i, r_i : Y_n \to Y_{n+1},$$

which graft a new leaf into the ith internal edge of a tree, respectively from the left and from the right. In the example above:



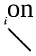
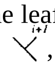
Notice that the operation of grafting a new leaf into an *external* edge produces the same result whether it is performed from the left or from the right: it consists in *bifurcating* the leaf. Thus the grafting operators on external edges coincide with the degeneracies.

I wish to determine whether increasing maps are horizontal or vertical. In the next lemma it is shown that the orientation of grafting operators does not depend on the index i nor on the tree on which the map is acting. Instead, the orientation of the degeneracy s_i changes with the index $i = 0, \ldots, n$ depending on the particular tree on which it is acting.

2.3. Lemma. *Let p, q be natural numbers, and $n = p + q + 1$.*

1. *The left grafters l_i are horizontal maps, i.e. $l_i : Y_{p,q} \to Y_{p+1,q}$ for any $i = 1, \ldots, n$. Similarly, the right grafters r_i are vertical maps, i.e. $r_i : Y_{p,q} \to Y_{p,q+1}$ for any $i = 1, \ldots, n$.*
2. *For any (p, q)-tree y, and for any index $i \in \{0, \ldots, n\}$, the degeneracy s_i is horizontal on y, i.e. $s_i^v(y) = 0$, if and only if the ith leaf of y is oriented like/. Similarly, s_i is vertical on y, i.e. $s_i^h(y) = 0$, if and only if the ith leaf of y is oriented like\.*

Proof:

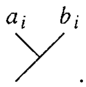
1. The statement is obvious, since by definition any left grafter l_i acts by adding a\-leaf and any right grafter r_i acts by adding a/-leaf.
2. The map s_i acts on the leaf/as , thus s_i adds a\-leaf (it is horizontal). Similarly, s_i acts on the leaf  as , thus s_i adds a/-leaf (it is vertical). □

Since I wish to deal with vertical complexes $k[Y_{p,*}]$, throughout the remaining part of this section I fix a $p \geq 0$, and observe (p, q)-trees for different values of $q \geq 0$.

The next lemma says whether an increasing map is distinct from any other or produces the same tree as some other map.

2.4. Labels of oriented leaves. Let y be a (p, q)-tree, and $n = p + q + 1$. Define a map $a^y : \{1, \ldots, p\} \to \{1, \ldots, n\}$ by assigning to the integer i the label $a^y(i) = a_i^y$ of the ith \-leaf of y, counting leaves from left to right and excluding the 0th leaf.

Any\-leaf (except the first one) is grafted into a/-leaf (including the last one). Thus there is a map $b^y : \{1, \ldots, p\} \to \{1, \ldots, n\}$ which assigns to the integer i the label $b^y(i) = b_i^y$ of the/-leaf into which the ith\-leaf is grafted, i.e.



.

Call $A(y) := \{a_1^y, \ldots, a_p^y\} \subset \{1, \ldots, n\}$ the image of a. Since the p\-leaves of y are distinct by assumption, the map a_y is a bijection between the set $\{1, \ldots, p\}$ and the set $A(y)$. Thus we can also define a map $b : A(y) \to \{1, \ldots, n\}$ by $b(a_i^y) = b_i^y$. Call $B(y) := \{b_1^y, \ldots, b_p^y\} \subset \{1, \ldots, n\}$ the image of b. Some properties of the maps a and b are given in Appendix B.

Finally, let $c^y : \{1, \ldots, q + 1\} \to \{1, \ldots, n\}$ be the map which counts all/-leaves (including the last one), and let $C(y)$ be its image. Clearly $C(y) = \{1, \ldots, n\} \backslash A(y)$ and $B(y) \subset C(y)$.

2.5. Lemma. *Let y be a (p, q)-tree, and $n = p + q + 1$.*

1. *The degeneracy maps are all distinct from each other, i.e. for any $i, j \in \{0, \ldots, n\}$, if $i \neq j$ then $s_i(y) \neq s_j(y)$. (In particular this holds for any index in the set $A(y)$.)*
2. *Any right grafting map into an internal edge labelled as a/-leaf (i.e. whose label is the same as a/-leaf) produces the same tree as some degeneracy map or a right grafting*

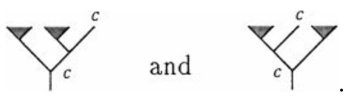
map into an edge labelled as a$\backslash$-leaf. In other words, for any index $c \in C(y)$, there exists an $a \in A(y)$ such that $r_c(y) = s_a(y)$ or $r_c(y) = r_a(y)$.

3. *All right grafting maps into internal edges labelled as a$\backslash$-leaf are distinct from each other and from any degeneracy map. That is, for any $a \in A(y)$, $r_a(y) \neq s_{a'}(y)$ and $r_a(y) \neq r_{a'}(y)$ for any $a' \neq a \in A(y)$.*

Thus, for any (p, q)-tree y, there are precisely $p + 1$ distinct vertical non-zero degeneracies acting on y, namely $s_0, s_{a_1}, \ldots, s_{a_p}$, and p distinct vertical grafting maps, namely $r_{a_1}, \ldots, r_{a_p}$.

Proof:

1. The assertion is obvious.
2. Suppose that an internal edge is labelled as a/-leaf, by c. Then there are two Possible shapes of the branch around the cth leaf:

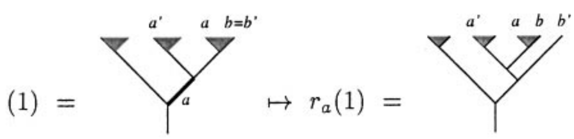


.

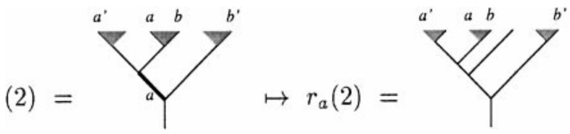
In the first case, we have $c = b(a) \in B(y)$ for some $a \in A(y)$. Choose the biggest one. Then, if there is no a' between a and $b = b(a)$, we have $r_c(y) = s_a(y)$. Otherwise, if there is some $a' \in A(y)$ such that $a < a' < b = c$, by Proposition (B.1) there must be some $b' = b(a') \in B(y)$ such that $a < a' < b' < b = c$. Choose b' to be the biggest such that $b' < b$, and choose a'' to be the smallest such that $b(a'') = b'$. Then we have $r_c(y) = r_{a''}(y)$.

In the second case, we have $c \in C(y)\backslash B(y)$, and the cth-leaf is grafted into a$\backslash$-leaf labelled, say, by a, so $a < c < b = b(a)$. Then, if there are no a' between a and c, we have $r_c(y) = s_a(y)$. Otherwise, chose a' as in the previous case, we then have $r_c(y) = r_{a'}(y)$.

3. The position of an internal edge which is labelled as a$\backslash$-leaf is very peculiar. Suppose that it is labelled by a. Then there must be an index $a' < a$ (possibly $a' = 0$) such that the internal edge a starts at the intersection between the$\backslash$-leaf a' and the/-leaf $b = b(a)$. By Proposition (B.1) it must be $b \leq b' = b(a')$. Thus there are only two possible shapes of the branch around the a^{th} leaf, for $b = b'$ and for $b < b'$, and r_a acts as follows:
for $b = b'$:

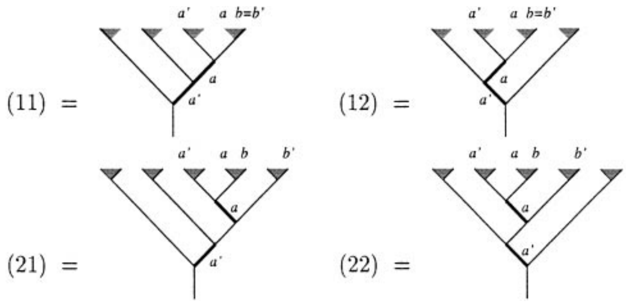


and for $b < b'$:

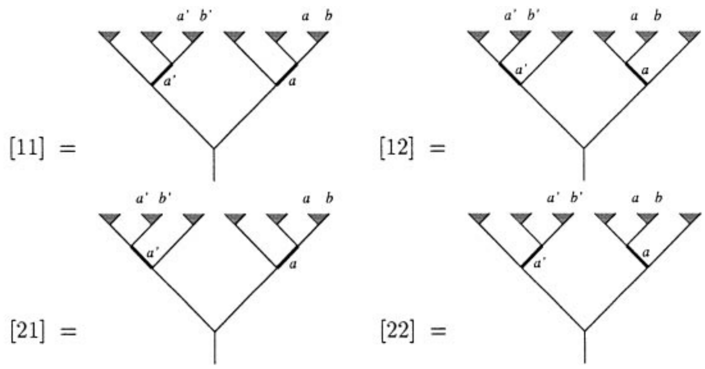


It is then clear that $r_a(y)$ can never be obtained by bifurcating a leaf: the branch  obstructs it. So $r_a(y) \neq s_{a''}(y)$ for any $a'' \neq a$.

Now consider the right grafter into another\-leaf, say $a' \neq a$. The internal edge labelled by a' is again placed in a peculiar position, such as the one labelled by a. Again by Proposition (B.1), there are exactely 8 mutual positions of two internal edges labelled by a and a'. Suppose $a' < a$. If $b \leq b'$:



If $b > b'$:



One can check that on these 8 trees we always have $r_a \neq r_{a'}$, so finally r_a is always different from $r_{a'}$. □

Since any map r_c coincides with some degeneracy, for $c \in C(y)$, I give the commutation relations between r_a and the faces d_i only for $a \in A(y)$.

2.6. Lemma. *The right grafting operators satisfy the relations*

$$(dr) \quad d_i r_a = \begin{cases} r_{a-1} d_i, & \textit{for } 0 \le i < a, \\ r_a d_i, & \textit{for } a \le i \le b(a), \\ id, & \textit{for } i = b(a)+1, \\ r_a d_{i-1}, & \textit{for } i > b(a)+1, \end{cases}$$

Proof: For any $a \in A(y)$, the operator r_a can act on the two basic trees (1) and (2) of lemma (2.5). Relations (dr) can be checked on (1) with the help of the following observations.

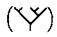
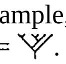
- If $i < a$, the leaf number a of (1) is labelled by $a-1$ in $d_i(1)$, hence also the internal edge labelled by a in (1) becomes $a-1$ in $d_i(1)$.
- If $a \le i \le b$, the edge labelled by a in (1) remains labelled by a in $d_i(1)$.
- If $i = b+1$, the face d_{b+1} deletes precisely the leaf which has just been added by r_a.
- If $i > b+1$, the edge labelled by a in (1) remains labelled by a in $d_i(1)$, but the leaf number i deleted in $r_a(1)$ by the face d_i was labelled by $i-1$ in (1).

The same observations hold for the tree (2). □

Decomposition of the vertical complexes into towers

2.7. Vertical towers. Let y be a (p, q)-tree. Define the *vertical tower* over y to be the graded set $T_*[y]$, where $T_0[y] := \{y\}$, and

$$T_m[y] := \{s_0(y'), \quad s_{a_i}(y'), \quad r_{a_i}(y') \mid i = 1, \ldots, p, \quad y' \in T_{m-1}[y]\} \subset Y_{p,q+m}, \quad m > 0.$$

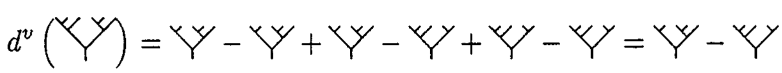
For example, the tree $y =$  $\in Y_{1,2}$ has $a_1 = 2$, s_0  $=$ , s_2  $=$  and  $=$ .

Thus

$$T_0[y] = \{$$  $$\}, \qquad T_1[y] = \{$$ , ,  $$\},$$ (2)

and so on. To simplify the notation, I use the same symbol $T_m[y]$ to denote the subset of trees and the k-module spanned by these trees.

2.8. Base trees. In general, a vertical tower is not a vertical complex. For instance, consider the tree  $\in T_1[y]$ in the example above. Since all the faces $d_0, d_1, \ldots, d_5$ act vertically on it, its vertical boundary $d^v = d_0 - d_1 + d_2 - d_3 + d_4 - d_5$ yields the combination of trees



which does not belong to $T_0[y] = k[$  $]$.

For any $p \geq 0$, I define a $(p, *)$-*base tree* to be any (p, q)-tree y such that $d_i^v(y) = 0$ for all $i = 0, \dots, n$. A "geometric" description of a base tree is the following: A tree y has $d_i^v(y) = 0$, for all i, if and only if every/-leaf belongs to a branch $\curlyvee$. This idea is used in the proof of Lemma (2.11).

2.9. Lemma. *For any (p, q)-tree y, if $d_i^v(y) = 0$ for any $i = 0, \dots, p+q+1$ then the vertical tower $T_*[y]$ is closed for the vertical faces d_i^v. Hence the vertical tower constructed on a base tree is a vertical complex.*

Proof: Assume that $d_i^v(y) = 0$ for all $i = 0, \dots, p+q+1$. I show that if y' belongs to $T_m[y]$ for some $m > 0$, then for any index $k \in \{0, \dots p+q+m+1\}$ such that $d_k^v(y') \neq 0$, the tree $d_k^v(y')$ belongs to $T_{m-1}[y]$. We proceed by induction on m.

- First assume that $y' \in T_1[y]$. Then by definition of vertical tower we know that $y' = s_0(y)$ or there exists an index $i \in \{1, \dots, p\}$ such that y' is equal either to $s_{a_i}(y)$ or to $r_{a_i}(y)$. Now consider a $k \in \{0, \dots, p+q+2\}$ such that $d_k^v(y') \neq 0$, then either

$$d_k^v(y') = d_k^v s_{a_i}(y) = \begin{cases} s_{a_i-1} d_k^v(y) = 0, & \text{if } k < a_i \\ y, & \text{if } k = a_i, a_i + 1 \\ s_{a_i} d_{k-1}^v(y) = 0, & \text{if } j \geq k+1 \end{cases}$$

  for a_i possibly equal also to 0, or

$$d_k^v(y') = d_k^v r_{a_i}(y) = \begin{cases} r_{a_i-1} d_k^v(y) = 0, & \text{if } k < a_i \\ r_{a_i} d_k^v(y) = 0, & \text{if } a_i \leq k \leq b_i \\ y, & \text{if } k = b_i + 1 \\ r_{a_i} d_{k-1}^v(y) = 0, & \text{if } k > b_i + 1 \end{cases}$$

  In conclusion we have that $d_k^v(y') = 0$ or $d_k^v(y') = y$ belongs to $T_0[y]$.
- Assume now that for any tree $y'' \in T_{m-1}[y]$, we have $d_k^v(y'') \in T_{m-2}[y]$ for any $k = 0, \dots p+q+m-1$ such that $d_k^v(y'') \neq 0$. I show that the same holds for any tree $y' \in T_m[y]$. In fact y' must be equal either to $s_0(\bar{y})$, $s_{a_i}(\bar{y})$ or to $r_{a_i}(\bar{y})$, for an index $i \in \{1, \dots, p\}$, with $\bar{y} \in T_{m-1}[y]$. Thus, in the first case (for a_i also equal to 0)

$$d_k^v(y') = d_k^v s_{a_i}(\bar{y}) = \begin{cases} s_{a_i-1} d_k^v(\bar{y}), & \text{if } k < a_i \\ \bar{y}, & \text{if } k = a_i, a_i + 1 \\ s_{a_i} d_{k-1}^v(\bar{y}), & \text{if } k > a_i + 1 \end{cases}$$

  belongs to $T_{m-1}[y]$, because for inductive hypothesis $d_k^v(\bar{y}) \in T_{m-2}[y]$, and in the second case

$$d_k^v(y') = d_k^v r_{a_i}(\bar{y}) = \begin{cases} r_{a_i-1} d_k^v(\bar{y}), & \text{if } k < a_i \\ r_{a_i} d_k^v(\bar{y}), & \text{if } a_i \leq k \leq b_i \\ \bar{y}, & \text{if } k = b_i + 1 \\ r_{a_i} d_{k-1}^v(\bar{y}), & \text{if } k > b_i + 1 \end{cases}$$

  belongs to $T_{m-1}[y]$ for the same reason. □

2.10. Corollary. *The vertical towers on two distinct base trees y and z are disjoint, that is,*

$$T_*[y] \bigcap T_*[z] = 0.$$

Proof: Let $y' \in T_n[y] \cap T_m[z]$. I show that $y' = 0$ by induction on n.

For $n = 0$, the base tree y cannot itself belong to $T_m[z]$, for any $m \geq 0$, because being in the image of at least one vertical map s_0, s_a, r_a coming from $T_{m-1}[z]$ it should have a non-zero corresponding vertical face, in contradiction with the assumption that it is a base tree. Hence $T_0[y] \cap T_*[z] = 0$.

Suppose we know that $T_q[y] \cap T_*[z] = 0$ for all $q < n$, and let y' belong to $T_n[y]$. If y' would belong to $T_*[z]$, by lemma (2.9) all its vertical faces would belong to $T_*[z]$, while y' must be in the image of one vertical map s_0, s_a, r_a coming from $T_{n-1}[y]$, and at least the corresponding vertical face takes values in $T_{n-1}[y]$. Hence y' can not belong to $T_*[z]$. □

2.11. Lemma-Notation. *There is a bijective correspondence between the set Y_p and the set of $(p, *)$-base trees. Therefore I denote by $T_*(y)$ the tower $T_*[\tilde{y}]$ on the $(p, *)$-base tree $\tilde{y}$ corresponding to the p-tree y. Moreover, the number of/-leaves of a p-tree y is equal to the number of /-leaves of its associated base tree $\tilde{y}$.*

Proof: Let

$$\varphi : Y_p \to \left\{ y \in \bigsqcup_{0 \leq s \leq p-1} Y_{p,s} \mid d_i^v(y) = 0 \ \forall i = 0, 1, \ldots, p+s+1 \right\}$$

be the map which sends a tree y into the tree $\varphi(y)$ obtained by bifurcating all the/-leaves. More precisely, suppose that the p-tree y lies in the component $Y_{r,s}$ of Y_p, i.e. suppose that y has r internal\-leaves and s internal/-leaves, with $r + s + 1 = p$. Let $c = c^y$: $\{1, 2, \ldots, s+1\} \to C(y)$ be the map which labels the/-leaves, as in (2.4). Then φ is defined by

$$\varphi(y) := s_{c_1} s_{c_2} \cdots s_{c_{s+1}}(y).$$

(i) Let us show that the map φ is well defined.

- If y is a p-tree, then the tree $\varphi(y)$ has exactly p internal leaves oriented like\. In fact, suppose that the p-tree y lies in the component $Y_{r,s}$ of Y_p. Then the tree $\varphi(y)$ has the original r internal\-leaves, and the new $s+1$ \-leaves appearing after the bifurcation of the $s+1$ total/-leaves: the total number is $r + s + 1 = p$.
- If y is a p-tree, then the tree $\varphi(y)$ can have at most $p - 1$ internal/-leaves. In fact, the/-leaves of $\varphi(y)$ are exactly the s original/-leaves of the p-tree y belonging to the component $Y_{r,s}$, and clearly $0 \leq s \leq p - 1$. Hence $\varphi(y)$ belongs to the union of the sets $Y_{p,s}$ for $0 \leq s \leq p - 1$.
- Let us show that if $\varphi(y)$ belongs to the set $Y_{p,s}$, then $d_i^v(\varphi(y)) = 0$ for any $i = 0, 1, \ldots, p+s+1$. If the index i labels a/-leaf of $\varphi(y)$, it comes by construction

from a bifurcated/-leaf of y, thus d_i^v produces a tree with the same number of/-leaves, and a\-leaf less. When the index i labels a\-leaf of $\varphi(y)$, the face d_i clearly deletes a\-leaf unless the $i+1st$ leaf is a/-leaf which is grafted into the ith leaf, and this is impossible in the tree $\varphi(y)$, because by construction any/-leaf is preceded by a\-leaf which is grafted into the/-leaf, and not the opposite.

(ii) To prove that the map φ is a bijection, I show that the map

$$\psi : \bigsqcup_{s \geq 0} Y_{p,s} \to Y_p$$

which deletes all the/-leaves, including the last one, is inverse to φ when restricted to the subset of trees with $d_i^v(y) = 0$ for all $i = 0, \dots, p+s+1$. The composition $\psi \circ \varphi$ is clearly the identity map on Y_p. On the other side, let y be a (p, s)-tree, for some $s \geq 0$. By construction, the tree $\varphi\psi(y)$ is obtained by deleting all the/-leaves from y, and then replacing all the new/-leaves with bifurcations. Thus y and $\varphi\psi(y)$ can only differ for some/-leaf, say labelled by k, such that the leaf labelled by $k-1$ is *not* a\-leaf grafting into it. Any such leaf produces a vertical non-zero face d_k^v. Since the domain of ψ is restricted to the trees with only zero vertical faces, the trees y and $\varphi\psi(y)$ must coincide. □

2.12. Theorem. *For any $p \geq 0$, the vertical complex $(k[Y_{p,*}], d^v)$ is the direct sum of the vertical towers based on p-trees, each shifted by the number of/-leaves of its base tree, that is,*

$$k[Y_{p,*}] = \bigoplus_{y \in Y_p} T_{*+q_y}(y),$$

where q_y is the number of /-leaves of y.

Proof:

(i) By lemma (2.9) we know that the towers constructed on $(p, *)$-base trees (and hence, by lemma (2.11) on their associated p-trees) are sub-complexes of the vertical complexes $(k[Y_{p,*}], d^v)$.

(ii) Corollary (2.10) tells us that the vertical towers on distinct base trees are disjoint subsets of the vertical complexes. Hence the sum is direct.

(iii) I show now that the sum covers the whole vertical complex $k[Y_{p,*}]$, i.e. that for any $y \in Y_{p,q}$, there exists a tree $y' \in Y_{p'}$ such that $y \in T_m(y')$ for some $m \geq 0$. Put $y' := \psi(y) \in Y_p$ and let $\tilde{y'} \in Y_{p,q_{y'}}$ be its $(p, *)$-base tree. Then y differs from $\tilde{y'}$ for some/-leaves which are not labelled by any b_i, with $i = 1, \dots, p$. In fact, by definition, any tree in degree m is obtained by adding a/-leaf to a tree in degree $m-1$, by means of the maps s_a or r_a. Thus y belongs to $T_m[\tilde{y'}] = T_m(y')$, with $m = q - q_{y'}$. □

2.13. Proposition. *The tower $T_*(y)$, associated to a p-tree y, is the total complex of a multi complex with dimension $d = 2p + 1$. Hence the number of its direct summands, at any degree $m \geq 0$, is given by the binomial coefficient*

$$\binom{d+m-1}{m} = \frac{(2p+m)!}{m!\,(2p)!}.$$

Proof: Apply definition (2.7) and remark, after (2.5), that $2p + 1$ is precisely the number of distinct maps which can act on a tree with p\-leaves by adding a/-leaf. □

Drawings of vertical towers.

2.14. Vertical tower for $p = 0$. The vertical complex $k[Y_{0,*}]$ coincides with the tower $T_*(|)$ with base Y, and it is pre-simplicial since all the faces are non-zero (Figure 3).

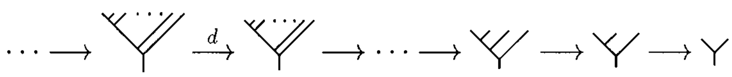


Figure 3. Vertical tower T_* (|) with base Y.

2.15. Vertical tower for $p = 1$. The vertical complex $k[Y_{1,*}]$ coincides with the tower T_*(Y) with base Y. This complex is the total of a multi-complex with dimension $d = 3$ (Figure 4).

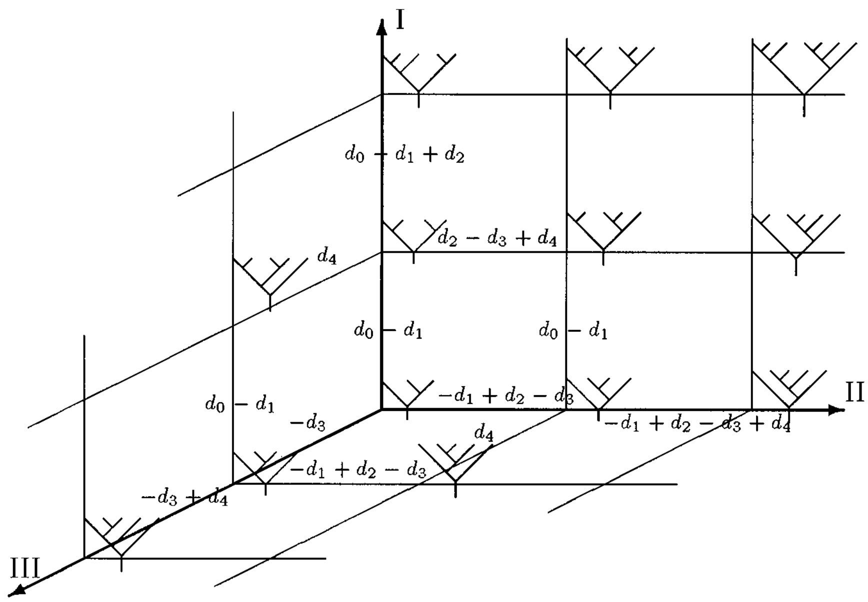


Figure 4. Vertical tower T_*(Y) with base Y.

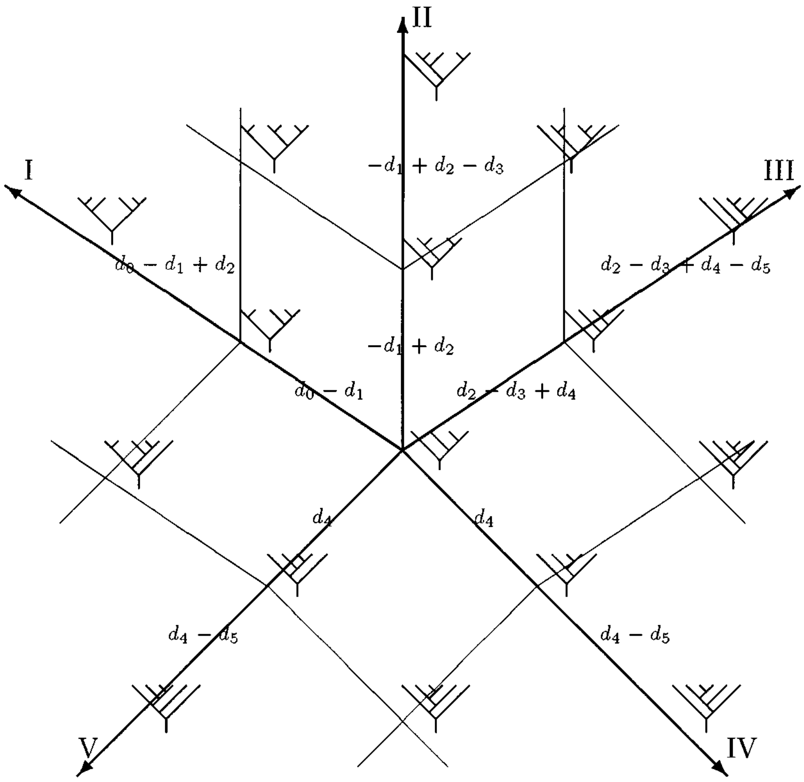


Figure 5. Vertical tower T_* ( ) with base .

2.16. Vertical towers for $p = 2$. The set Y_2 contains two trees, and , associated respectively to the base trees and . Hence $k[Y_{2,*}] = T_*$ ( )$\oplus$ T_{*+1} ( ), where the two towers are multi-complexes with dimension $d = 5$ (Figures 5 and 6).

2.17. Vertical complex for $p \geq 3$. The set Y_3 contains five trees, , , , and , which correspond, respectively, to the five following base trees:

Hence $k[Y_{3,*}]$ is the direct sum of five vertical towers, based on these five trees, which are multi-complexes with dimension $d = 7$.

In a similar way one can proceed for $p > 3$. Each vertical complex $k[Y_{p,*}]$ is the direct sum of c_p vertical towers (where c_p is the number of p-trees), and each vertical tower is a multi-complex with dimension $d = 2p + 1$.

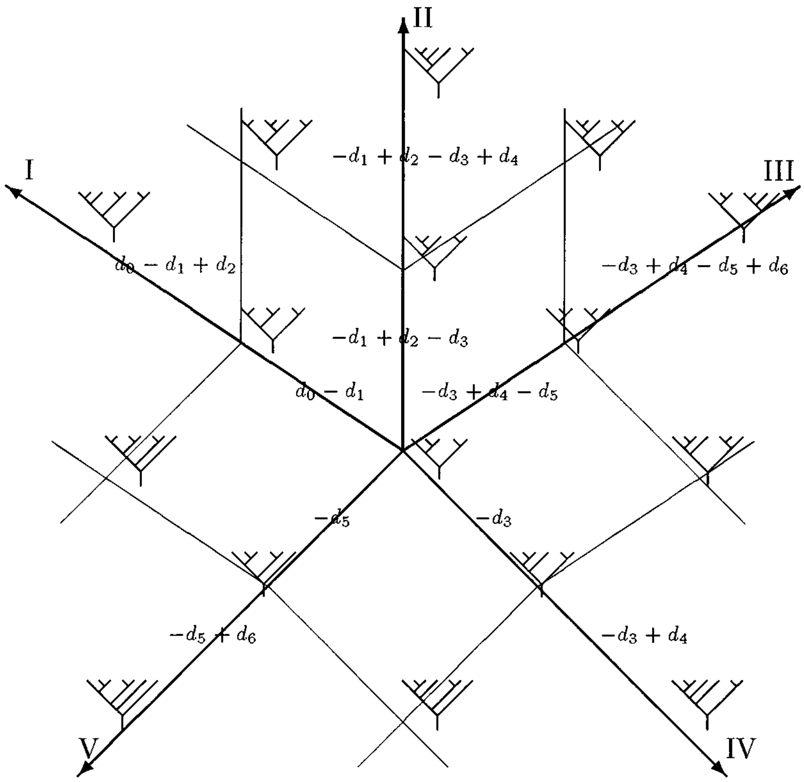


Figure 6. Vertical tower T_* ( ) with base .

A. Appendix. Cardinality of the classes of planar binary trees

In this appendix I compute the cardinality of the classes of planar binary trees.

For any couple of natural numbers p, q, let $Y_{p,q}$ be the set of $(p+q+1)$-trees with p leaves oriented like\(excluded the 0-th leaf), and q leaves oriented like/(excluded the last one), as in (1.9).

A.1. Proposition. *Let $c_{p,q}$ be the cardinality of the set $Y_{p,q}$. Then*

$$c_{p,q} = c_{q,p} = \frac{(p+q)!}{p!\,q!} \frac{(p+q+1)!}{(p+1)!\,(q+1)!}.$$

For small p, 9, the numbers $c_{p,q}$ are explicitely given in Figure 7.

q	⋮	⋮	⋮	⋮				
3	1	10	50	175	...			
2	1	6	20	50	105	...		
1	1	3	6	10	15	21	...	
0	1	1	1	1	1	1	1	...
$c_{p,q}$	0	1	2	3	4	5	6	p

Figure 7. The cardinality of the classes of rooted planar binary trees.

A.2. Lemma. *The cardinality $c_{p,q}$ of the set $Y_{p,q}$ is $c_{0,0} = 1$ when $p = q = 0$, $c_{p,0} = 1$ for any $p > 0$, $c_{0,q} = 1$ for any $q > 0$ and finally, for any $p, q \geq 1$, it satisfies the relation*

$$c_{p,q} = c_{p-1,q} + c_{p,q-1} + \sum_{\substack{p_1+p_2=p-1 \\ q_1+q_2=q-1}} c_{p_1,q_1} \cdot c_{p_2,q_2}.$$

Proof: When $p = q = 0$, there exists only one $(0,0)$-tree, namely . Thus $c_{0,0} = 1$. Similarly, when $p > 0$ and $q = 0$, there exists only one $(p,0)$-tree, namely the comb tree . The same for $p = 0$ and $q > 0$. Thus $c_{p,0} = 1$ for any $p > 0$ and $c_{0,q} = 1$ for any $q > 0$.

When $p, q \geq 1$, any (p,q)-tree y can have one of the following three shapes:

- $y =$, where, for $i = 1, 2$, y_i is a (p_i, q_i)-tree such that $p_1 + p_2 = p - 1$ and $q_1 + q_2 = q - 1$;
- $y =$, where y_1 is a (p_1, q_1)-tree with $p_1 = p$ and $q_1 = q - 1$;
- $y =$, where y_2 is a (p_2, q_2)-tree with $p_2 = p - 1$ and $q_2 = q$.

Thus, for any $p, q \geq 1$, $c_{p,q}$ is the sum of the cardinality of these three disjoint sets. □

Proof of (A.1): We have to count the number $c_{p,q}$ of (p,q)-trees, for $p, q \geq 0$. Consider the values $c_{p,q}$ as coefficients of Taylor's expansion of a function of two variables x and y, around the point $(0,0)$, and put

$$f(x,y) := 2xy \sum_{p,q \geq 0} c_{p,q}\, x^p y^q.$$

It is straightforward to show that the relations of lemma (A.2) lead us to the quadratic equation $f^2(x,y) + 2(x + y - 1) f(x,y) + 4xy = 0$ in the indeterminate $f(x,y)$. The solution of this equation is the function $f(x,y) = -(x + y - 1) \pm [(x + y - 1)^2 - 4xy]^{\frac{1}{2}}$.

By direct computations, choosing the sign "$-$" before the root, we obtain the values

$$f(0,0)=0, \qquad \frac{1}{n!}\frac{\partial^n f(0,0)}{\partial x^n}=0, \qquad \frac{1}{m!}\frac{\partial^m f(0,0)}{\partial y^m}=0.$$

In fact

$$\frac{\partial^n f(x,y)}{\partial x^n}=2n!\; y\,[1+g_{n,0}(x,y)]\;\Delta(x,y)^{-\frac{1}{2}-(n-1)},$$

where $g_{n,0}(x,y)$ is a polynomial with $g_{n,0}(0,0)=0$, $\Delta(x,y):=[(x+y-1)^2-4xy]^{\frac{1}{2}}$ is such that $\Delta(0,0)=1$, and similarly for $\frac{\partial^m f(x,y)}{\partial y^m}$. Therefore the function $f(x,y)$ has itself Taylor's expansion

$$f(x,y)=\sum_{n,m\geq 1}\frac{1}{n!\,m!}\frac{\partial^{n+m} f(0,0)}{\partial x^n\partial y^m}\;x^n y^m$$

and the coefficients $c_{p,q}$ satisfy

$$2c_{n-1,m-1}=\frac{1}{n!\,m!}\frac{\partial^{n+m} f(0,0)}{\partial x^n\partial y^m}.$$

Again by direct computation we obtain

$$\frac{\partial^{n+m} f(x,y)}{\partial x^n\partial y^m}=2\frac{(n+m-2)!\,(n+m-1)!}{(n-1)!\,(m-1)!}\;[1+g_{n,m}(x,y)]\;\Delta(x,y)^{-\frac{1}{2}-(n+m-1)},$$

where $g_{n,m}(0,0)=0$ and $\Delta(0,0)=1$. Hence we get the final formula

$$c_{p,q}=\frac{1}{2}\frac{1}{(p+1)!\,(q+1)!}\frac{\partial^{(p+1)+(q+1)} f(0,0)}{\partial x^{p+1}\partial y^{q+1}}=\frac{(p+q)!\,(p+q+1)!}{p!\,q!\,(p+1)!\,(q+1)!}. \qquad \square$$

A.3. Remark. The Catalan number c_n can be given in terms of binomial coefficients, $c_n=\frac{1}{n+1}\binom{2n}{n}$. Hence the discrete convolution formula for binomial coefficients, namely

$$\binom{i+j}{k}=\sum_{h=0}^{k}\binom{i}{h}\binom{j}{k-h},$$

evaluated at $i=n-1$, $j=n+1$ and $k=n$, yields exactly the identity

$$c_n=\sum_{0\leq p\leq n-1}\frac{1}{n+1}\binom{n-1}{p}\binom{n}{p}=\sum_{p+q=n-1}\frac{(p+q)!}{p!\,q!}\frac{(p+q+1)!}{(p+1)!\,(q+1)!}$$
$$=\sum_{p+q=n-1} c_{p,q}.$$

B. Appendix. An invariant of the towers

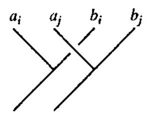
In this appendix I show that the classes of trees are in bijection with certain classes of set maps. From this construction it is then clear that the number of/-leaves of a tree y characterizes the shape of all trees belonging to the vertical tower $T_*(y)$ associated to y.

B.1. Proposition. *For any $p, q \geq 0$, there is a bijective correspondence between (p, q)-trees and pairs of set maps $a, b : \{1, \dots, p\} \to \{1, \dots, n\}$, with $n = p + q + 1$, satisfying the following conditions:*

1. *if $i < j$ then $a(i) < a(j)$, hence the map a is monotone strictly increasing*;
2. *$a(i) < b(i)$ for any i, in particular the maps a and b have disjoint image*;
3. *if $i < j$ and $a(j) < b(i)$, then $b(i) \geq b(j)$ (equivalently, if $i < j$ and $b(i) < b(j)$ then $b(i) < a(j)$).*

Proof:

(i) Let us show that for any (p, q)-tree y, the set maps $a, b : \{1, \dots, p\} \to \{1, \dots, n\}$ defined in (2.4), with $n = p + q + 1$, which label the oriented leaves of y, satisfy conditions 1, 2, 3. The first two conditions are evident: 1 means that the p\-leaves are distinct, and 2 means that any\-leaf is distinct from the/-leaf into which is grafted. Condition 3 is due to the facts that any\-leaf cannot coincide with any/-leaf, so $b_i \neq a_j$, and that for $i < j$ and $b_i < b_j$, the relation $b_i > a_j$ would correspond to the following impossible picture:



(ii) Let $a, b : \{1, \dots, p\} \to \{1, \dots, n\}$ be two maps satisfying conditions 1, 2, 3 above, with $n = p + q + 1$. Then we can construct a tree y with the following algorithm.

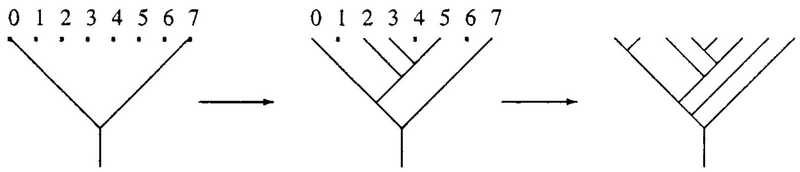
- Draw $p + q + 2$ points, and label them from 0 to $p + q + 1$. Draw an edge\from the 0-th leaf, an edge/from the last leaf and the root.
- From any leaf labelled by $a(i)$, draw an edge\and graft it into an edge/ drawn from the leaf labelled by $b(i)$. Extend all the edges until they reach an edge of opposite orientation.
- From any remaining leaf, draw an/-edge, and reach an\-edge.

None of these operations has any freedom of choice, so the tree thus obtained is uniquely determined, and it is clearly described by the given maps a, b. □

Here is an example of the algorithm above. Let $n = p + q + 1$ be 7, and $p = 2$. Choose two maps according to conditions 1, 2, 3 of (B.1), for instance,

$$a(1) = 2, \qquad a(2) = 3, \qquad b(1) = 5, \qquad b(2) = 5.$$

Now follow the three steps in the drawing.



B.2. Blocks. The map b is not necessarily monotone. However we can say that it is "block" monotone, since it satisfies

4. For any triple of indices $i < j < k$ such that $b(i) < b(j)$, we have $b(i) < b(k)$.

This condition says that whenever the map b satisfies $b(i) < b(j)$, for $i < j$, the inequality sign "$<$" separates two *blocks* in the image of b, given, respectively, by indices preceding and following the inequality sign. This follows easily from the above conditions 1, 2, 3. By 3, the inequality $b(i) < b(j)$ implies that $b(i) < a(j)$. Condition 1 says that $a(j) < a(k)$ and condition 2 says that $a(k) < b(k)$. Thus, combining the three inequalities, we obtain $b(i) < a(j) < a(k) < b(k)$.

Remark that the *number of blocks* of the (p, q)-tree associated to the maps a and b can vary between 1 and p, for $p > 0$, and is assumed to be 1 for $p = 0$.

B.3. Proposition. *All the trees belonging to a vertical tower $T_*(y)$ have the same number $q_y + 1$ of blocks, where q_y is the number of/-leaves of the tree y.*

Hence the number q_y has a geometrical meaning which is invariant in the vertical tower $T_*(y)$, being related to the number of blocks of leaves of any tree in the tower.

Proof: If a p-tree y has q_y\-leaves, by (2.11) we know that its associated base tree $\tilde{y}$ is a (p, q_y)-tree. The tower $T_*(y)$ is based on this tree, and by construction the tree $\tilde{y}$ is the one with minimal number of/-leaves in the tower. Grafting new/-leaves into any\-leaf does not affect the ordering of the indices b_i, and hence of the number of blocks. Thus we only need to show that $\tilde{y}$ itself has $q_y + 1$ blocks.

Since $\tilde{y}$ is in the image of the map φ defined in (2.11), by construction each/-leaf has a\-leaf grafted onto when φ is applied. Hence each/-leaf is labelled by a certain b_i ($i = 1, \ldots, p$) and the map b is strictly increasing, that is, each/-leaf is its own block. $\square$

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