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Vertex Operators for Standard Bases of the Symmetric Functions

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Abstract. We present formulas for operators which add a row or a column to the partition indexing the power, monomial, forgotten, Schur, homogeneous and elementary symmetric functions. As an application of these operators we show that the operator that adds a column to the Schur functions can be used to calculate a formula for the number of pairs of standard tableaux the same shape and height less than or equal to a fixed k.

Keywords: symmetric functions, vertex operators

1. Notation

Using the notation of [3], we will consider the power $\{p_\lambda[X]\}_\lambda$, Schur $\{s_\lambda[X]\}_\lambda$, monomial $\{m_\lambda[X]\}_\lambda$, homogeneous $\{h_\lambda[X]\}_\lambda$, elementary $\{e_\lambda[X]\}_\lambda$ and forgotten $\{f_\lambda[X]\}_\lambda$ bases for the symmetric functions. We will often appeal to [3] for proofs any symmetric function identities.

These bases are all indexed by partitions, non-increasing sequences of non- negative integers. The ith entry of the partition will be denoted by λ_i. The length of a partition λ is the largest i such that λ_i is non-zero and will be denoted by $l(\lambda)$. The size of the partition will be denoted by $|\lambda|$ and is equal to the sum over all the entries of λ. The symbol $n_i(\lambda)$ will be used to represent the number of parts of size i in the partition λ. The conjugate partition will be denoted by λ' and is the partition such that $\lambda'_i =$ the number of j such that λ_j is greater than or equal to i.

There is a standard inner product on symmetric functions $\langle p_\lambda, p_\mu\rangle = z_\lambda \delta_{\lambda\mu}$ where $\delta_{xy} = 1$ if $x = y$ and 0 otherwise and $z_\lambda = \prod_{i\geq 1} i^{n_i(\lambda)} n_i(\lambda)!$.

We will use a few non-standard operations on partitions that will require some notation. The first is adding a column (or a sequence of columns) to a partition. Let $a^k \mid \lambda$ denote the partition $(\lambda_1 + a, \lambda_2 + a, \ldots, \lambda_k + a)$. We will assume that this partition is undefined when $l(\lambda) > k$.

Use the notation $\lambda - (\mu)$ to denote the partition formed by removing the parts that are equal to μ from the partition λ. This of course assumes that there is a sequence $I = \{i_1, i_2, \ldots i_{l(\mu)}\} \subset \{1, 2, \ldots l(\lambda)\}$ such that $\mu = (\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{l(\mu)}})$. If this sequence does not exist then $\lambda - (\mu)$ is again undefined.

The last operation will be inserting parts into a partition and will be represented by $\lambda+(\mu)$. This will be the partition formed by ordering the sequence $(\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}, \mu_1, \mu_2, \ldots, \mu_{l(\mu)})$ into a partition.

Occasionally within sums we will have an expression such as $m_{\lambda-(\mu)}$ or $h_{1^k|\lambda}$ when it is not necessarily the case that μ is a subpartition of λ or that λ has height less than or equal k. We will consider expressions like these to be zero. This simplifies many summation formulas which otherwise would have to have three or four conditions to their arguments, but it is important to verify at each step that transformations to equations are in fact legal.

We will say that two bases for the symmetric functions $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$ are dual if they have the property that $\langle a_\lambda, b_\mu\rangle = \delta_{\lambda\mu}$. By definition, the power symmetric functions are dual to the basis $\{p_\lambda/z_\lambda\}_\lambda$. The monomial and homogeneous symmetric functions are dual. The forgotten and the elementary symmetric functions are dual. The Schur symmetric functions are self dual ($\langle s_\lambda, s_\mu\rangle = \delta_{\lambda\mu}$).

There exists an involution, ω, on symmetric functions that relates the elementary and homogeneous bases by $\omega h_\mu = e_\mu$ and the monomial and forgotten bases by $\omega m_\mu = f_\mu$. It also has the property that $\omega s_\lambda = s_{\lambda'}$.

Denote the operation of 'skewing' by a symmetric function f by $f^\perp$. It is defined as being the operation dual to multiplication by the symmetric function f in the sense that $\langle f^\perp P, Q\rangle = \langle P, fQ\rangle$. Its action on an arbitrary symmetric function P may be calculated by the formula $f^\perp P = \sum_\lambda \langle P, f a_\lambda\rangle b_\lambda$ for any dual bases $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$.

Using results and notation in [3] (p. 92–93 example (I.5.25)), by setting $\Delta f = \sum_\mu (a_\mu^\perp f) \otimes b_\mu$ where $\{a_\mu\}_\mu$ and $\{b_\mu\}_\mu$ are any dual bases. It follows that if $\Delta f = \sum_i c_i \otimes d_i$ then $f^\perp(PQ) = \sum_i c_i^\perp(P) d_i^\perp(Q)$. Using this result and considering f to be multiplication by some symmetric function we have,

$$h_k^\perp f = \sum_i \left(h_i^\perp f\right) h_{k-i}^\perp \tag{1}$$

$$e_k^\perp f = \sum_i \left(e_i^\perp f\right) e_{k-i}^\perp \tag{2}$$

$$p_k^\perp f = \left(p_k^\perp f\right) + f p_k^\perp \tag{3}$$

By the phrase 'vertex operators' we are referring to linear symmetric function operators that add a row or a column to the partitions indexing a particular family of symmetric functions. Formulas of this type for symmetric functions are sometimes called Rodrigues formulas. In this article we look at those symmetric function operators which lie in the linear span of $\{f_i g_i^\perp\}_i$ where f_i and g_i are symmetric functions to find expressions for vertex operators for each basis.

By Corollary 3 in Section 4 we know that it is always possible to produce such a vertex operator. Yet for some applications a more refined formula is necessary, and it seems that only for a very small class of operators will this formula reduce to something elegant.

The vertex operators for the elementary and forgotten symmetric function basis are related to the operators for the homogeneous and monomial (resp.) symmetric functions by conjugating by the operator ω.

The existence of such operators for the Schur ([3] p. 96–97, [5] p. 69), and (row only) Hall-Littlewood ([3] p. 237–238, [2]) symmetric functions are known. For the multiplicative bases, it is clear that there exists operators that add a row to the symmetric functions of this form since $e_k e_\lambda = e_{\lambda+(k)}$, $h_k h_\lambda = h_{\lambda+(k)}$ and $p_k p_\lambda = p_{\lambda+(k)}$, but adding a column is not an obvious operation.

In general, formulas for adding a row or a column can be useful in proving a combinatorial interpretation for a symmetric function or deriving new formulas or properties. The author's interest in this particular question comes from trying to find vertex operators for the Macdonald symmetric functions. The Macdonald vertex operator must specialize to the vertex operators for other symmetric functions and so understanding these operators is an important first step.

2. The power vertex operator

This is the warm up case for the other 5 bases. The commutation relation between $p_k^\perp$ and p_j (given by Eq. (3)) has a nice expression: $p_k^\perp p_j = p_j p_k^\perp + k\delta_{kj}$. This can be used to show the slightly more general relation $p_\lambda^\perp p_k = p_k p_\lambda^\perp + kn_k(\lambda) p_{\lambda-(k)}^\perp$ (where it is assumed that $p_{\lambda-(k)}^\perp = 0$ if λ does not contain a part of size k).

The vertex operator is given by the following theorem

Theorem 1 *For $a \geq 0$ and $k \geq 0$ define the following linear operator*

$$CP_{a^k} = \sum_{\lambda : l(\lambda) \leq k} p_a^{k-l(\lambda)} \prod_{i=1}^{l(\lambda)} (p_{\lambda_i+a} - p_{\lambda_i} p_a) p_\lambda^\perp / z_\lambda$$

where the sum is over all partitions λ with less than or equal to k parts (if $k = 0$ then $CP_{a^0} = 1$). CP_{a^k} has the property that $CP_{a^k} p_\mu = p_{a^k|\mu}$ for all μ such that $l(\mu) < k$.

Proof: The proof is by induction on the number of parts of μ. Clearly this operator has the property that $CP_{a^k} 1 = p_a^k$ since $p_\lambda^\perp$ kills 1 for $|\lambda| > 0$. From the commutation relation of $p_\lambda^\perp$ and p_k we derive that

$$CP_{a^k} p_j = (p_{j+a} - p_j p_a) CP_{a^{k-1}} + p_j CP_{a^k} \tag{4}$$

The proof by induction follows from this relation. □

The formula CP_{a^k} was chosen so that it has two properties: it adds a column to the power symmetric functions, and it has a relatively simple expression when written in this notation. The action of this operator on p_λ when $l(\lambda) > k$ is not specified by these conditions, but it is determined.

If one wishes to give an expression for an operator that has the same action on p_λ for $l(\lambda) \leq k$ and the action on p_λ for $l(\lambda) > k$ is something else (say for instance 0), this is possible by adding in terms of the form $p_\mu p_\lambda^\perp$ where $l(\lambda) > k$ to CP_{a^k}.

3. Homogeneous and elementary vertex operators

We require the following commutation relation between the homogeneous and monomial symmetric functions.

Lemma 1 *For* $k \geq 0$,

$$h_k^\perp m_\lambda = \sum_{i \geq 0} m_{\lambda-(i)} h_{k-i}^\perp$$
$$m_\lambda^\perp h_k = \sum_{i \geq 0} h_{k-i} m_{\lambda-(i)}^\perp$$

where we will assume the convention $m_{\lambda-(i)} = 0$ *whenever* $\lambda - (i)$ *is undefined.*

Proof: Note that $\langle h_i^\perp m_\lambda, h_\mu \rangle = \langle m_\lambda, h_i h_\mu \rangle = \delta_{\mu,\lambda-(i)}$. Therefore

$$h_i^\perp(m_\lambda) = m_{\lambda-(i)} \tag{5}$$

and $h_i^\perp(m_\lambda) = 0$ if λ does not have a part of size i. The first identity follows from Eq. (1).

The second identity is a restatement of the first since

$$\left\langle m_\lambda^\perp h_k P, Q \right\rangle = \left\langle P, h_k^\perp m_\lambda Q \right\rangle = \sum_{i \geq 0} \left\langle P, m_{\lambda-(i)} h_{k-i}^\perp Q \right\rangle \tag{6}$$

$$= \sum_{i \geq 0} \left\langle h_{k-i} m_{\lambda-(i)}^\perp P, Q \right\rangle \tag{7}$$

□

Define CH_{1^k} to be the operator $CH_{1^k} = \sum_{\lambda : \ell(\lambda) \leq k} (-1)^{|\lambda|} e_{1^k|\lambda} m_\lambda^\perp$, and the operator CE_{1^k} to be $CE_{1^k} = \sum_{\lambda : \ell(\lambda) \leq k} (-1)^{|\lambda|} h_{1^k|\lambda} f_\lambda^\perp$.

The vertex operator property that we prove for the homogeneous and elementary symmetric functions is

Theorem 2 *If* $l(\lambda) \leq k$, *then* $CH_{1^k} h_\lambda = h_{1^k|\lambda}$ *and* $CE_{1^k} e_\lambda = e_{1^k|\lambda}$.

Proof: The proof is a matter of showing that for $k > 1$ operator CH_{1^k} and h_n (considered as an operator that consists of multiplication by h_n) has the commutation relation $CH_{1^k} h_n = h_{n+1} CH_{1^{k-1}}$ and $CH_{1^1}(h_n) = h_{n+1}$.

$$CH_{1^k} h_n = \sum_{\lambda:\ell(\lambda) \leq k} (-1)^{|\lambda|} e_{1^k|\lambda} m_\lambda^\perp h_n \tag{8}$$

The sum here is over λ with the number of parts less than or equal to k. Apply Lemma 1 and rearrange the terms in the sum.

$$CH_{1^k} h_n = \sum_{\lambda:\ell(\lambda) \leq k} (-1)^{|\lambda|} e_{1^k|\lambda} \sum_{i \geq 0} h_{n-i} m_{\lambda-(i)}^\perp \tag{9}$$

$$= \sum_{\lambda:\ell(\lambda)\le k} \sum_{i\ge 0} (-1)^{|\lambda|-i}(-1)^i e_{1^k|\lambda} h_{n-i} m^{\perp}_{\lambda-(i)} \tag{10}$$

$$= \sum_{\lambda:\ell(\lambda)\le k} \sum_{i\ge 0} (-1)^{|\lambda|-i}(-1)^i h_{n-i} e_{i+1} e_{1^{k-1}|(\lambda-(i))} m^{\perp}_{\lambda-(i)} \tag{11}$$

In the last equation, there is an assumption that $e_{1^{k-1}|(\lambda-(i))} = 0$ if $\lambda - (i)$ is undefined. As long as $k > 1$, making the substitution $\lambda \to \lambda + (i)$ yields the equation:

$$= \left(\sum_{i=1}^{n} (-1)^i h_{n-i} e_{i+1} \right) \left(\sum_{\lambda:\ell(\lambda)\le k-1} (-1)^{|\lambda|} e_{1^{k-1}|\lambda} m^{\perp}_{\lambda} \right) \tag{12}$$

This is equal to $= h_{n+1} CH_{1^{k-1}}$ using the well known relation $\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0$ for $n \ge 0$. If $k = 1$ then

$$CH_{1^1}(h_n) = \sum_{i=1}^{n} (-1)^i h_{n-i} e_{i+1} = h_{n+1} \tag{13}$$

Notice also that CH_{1^k} acting on 1, yields h_1^k since only one term is not 0.

The corresponding result for the CE_{1^k} operator follows by noting that $CE_{1^k} = \omega CH_{1^k} \omega$. □

The action of CH_{1^k} on h_λ when $l(\lambda) > k$ is not known. The sum in the formula for CH_{1^k} is only over partitions λ such that $l(\lambda) \le k$ and by adding terms of the same form but with $l(\lambda) > k$ it is possible to modify the formula so that the action on the h_λ when $l(\lambda) > k$ is 0, but the formula will not be as simple.

It would be interesting to know the action of these vertex operators on other bases besides the one that it adds a row and column to. For instance, actions of e_k, h_k, and p_k are known on the Schur basis, but what is the action of an operator that adds a column to the homogeneous, elementary, or power basis when it acts on the Schur basis?

Note the following two formulas that relate CH_{1^k} and CE_{1^k}.

$$CH_{1^k} = \sum_{\lambda:l(\lambda)\le k} (-1)^{|\lambda|} CE_{1^k}(e_\lambda) m^{\perp}_{\lambda} \tag{14}$$

$$CE_{1^k} = \sum_{\lambda:l(\lambda)\le k} (-1)^{|\lambda|} CH_{1^k}(h_\lambda) f^{\perp}_{\lambda} \tag{15}$$

This is the first instance when a pair of operators share a relation like this, and it will occur with pairs of the other operators that appear in this article. These relations fall under the category of ‘eerie coincidences’ (since they are very unexpected and can probably be explained on some higher dimensional plane).

4. Monomial and forgotten vertex operators

The vertex operators for the monomial and forgotten symmetric functions require a few identities.

Lemma 2 *Let* $r_\mu = (-1)^{|\mu|-l(\mu)} \frac{l(\mu)!}{n_1(\mu)!n_2(\mu)!\ldots}$ *then for* $k \geq 0$, $e_k = \sum_{\mu \vdash k} r_\mu h_\mu$

Proof: [3] example I.2.20, p. 33 □

Lemma 3 *For* μ *a partition with* $|\mu| > 0$, $\sum_{j\geq 0}(-1)^j r_{\mu-(j)} = 0$ *where it is assumed that if* $\mu - (j)$ *does not exist then* $r_{\mu-(j)} = 0$.

Proof: $\sum_{j=0}^{k}(-1)^j e_{k-j} h_j = 0$. Now expand e_{k-j} in terms of the homogeneous basis using the last lemma and equate coefficients of h_μ on both sides of the equation. □

Lemma 4 *For* $k \geq 0$, $e_k^\perp m_\lambda = \sum_\mu r_\mu m_{\lambda-(\mu)} e_{k-|\mu|}^\perp$ *where* $m_{\lambda-(\mu)} = 0$ *if* $\lambda - (\mu)$ *is undefined.*

Proof: By Eq. (2) $e_k^\perp m_\lambda = \sum_{i\geq 0} e_i^\perp(m_\lambda) e_{k-i}^\perp$. The expansion of the $e_i^\perp$ in terms of $h_\mu^\perp$ is given in the last lemma and so we have that

$$e_k^\perp m_\lambda = \sum_{i\geq 0}\left(\sum_{\mu\vdash i} r_\mu h_\mu^\perp(m_\lambda)\right) e_{k-i}^\perp = \sum_{i\geq 0}\left(\sum_{\mu\vdash i} r_\mu m_{\lambda-(\mu)}\right) e_{k-i}^\perp \tag{16}$$

by Eq. (5), and this is equivalent to the statement of the lemma. □

Lemma 5 *For* $a > 0$, $m_{(a)} m_\lambda = \sum_{i\geq 0}(1 + n_{a+i}(\lambda)) m_{\lambda-(i)+(a+i)}$ *where it is assumed that* $m_{\lambda-(i)+(a+i)} = 0$ *if* $\lambda - (i)$ *is undefined.*

Proof: For partitions μ of $|\lambda| + a$, one has that the coefficient of m_μ in $m_{(a)} m_\lambda$ is equal to $h_\mu^\perp(m_{(a)} m_\lambda)$.

We note that for all $n \geq 0$ that $h_n^\perp m_{(a)} = m_{(a)} h_n^\perp + h_{n-a}^\perp$. Apply this to the expression for the coefficient of m_μ

$$h_\mu^\perp(m_{(a)} m_\lambda) = \sum_{j=1}^{l(\mu)} h_{\mu-(\mu_j)+(\mu_j-a)}^\perp(m_\lambda) \tag{17}$$

This implies that for the coefficient to be non-zero that μ must be equal to λ with a part (say of size i) pulled away and a part of size $a + i$ added in. The coefficient will be the number of times that $a + i$ appears in the partition μ (one more time than it appears in the partition λ). □

The first vertex operator that is presented here for the monomial symmetric functions adds a row but it also multiplies by a coefficient, but this operator provides an easy method for obtaining an operator that does not have this coefficient.

Proposition 1 *For $a > 0$, let $RM_a^{(1)} = \sum_{i\geq 0}(-1)^i m_{(a+i)} e_i^{\perp}$ then*

$$RM_a^{(1)} m_\lambda = (1 + n_a(\lambda)) m_{\lambda+(a)}$$

Proof:

$$RM_a^{(1)} m_\lambda = \sum_{i\geq 0}(-1)^i m_{(a+i)} e_i^{\perp} m_\lambda \tag{18}$$

Apply Lemma 4 to get

$$= \sum_{i\geq 0}(-1)^i m_{(a+i)} \sum_{\mu\vdash i} r_\mu m_{\lambda-(\mu)} \tag{19}$$

The sum over i and μ may be combined to form one sum over all partitions μ.

$$= \sum_{\mu}(-1)^{|\mu|} r_\mu m_{(a+|\mu|)} m_{\lambda-(\mu)} \tag{20}$$

Now multiplying by a monomial symmetric function with one part has an expansion given in Lemma 5.

$$= \sum_{\mu}\sum_{j\geq 0}(-1)^{|\mu|} r_\mu (1 + n_{a+|\mu|+j}(\lambda - (\mu))) m_{\lambda-(\mu)-(j)+(j+a+|\mu|)} \tag{21}$$

The terms indexed by the same monomial symmetric function may be grouped together by letting $\nu = \mu + (j)$.

$$= \sum_{\nu}\sum_{j\geq 0}(-1)^{|\nu|-j} r_{\nu-(j)} (1 + n_{a+|\nu|}(\lambda - ((\nu) - (j)))) m_{\lambda-(\nu)+(a+|\nu|)} \tag{22}$$

$$= \sum_{\nu}(1 + n_{a+|\nu|}(\lambda)) m_{\lambda-(\nu)+(a+|\nu|)} \sum_{j\geq 0}(-1)^{|\nu|-j} r_{\nu-(j)} \tag{23}$$

But $\sum_{j\geq 0}(-1)^{|\nu|-j} r_{\nu-(j)} = 0$ if $|\nu| > 0$ by Lemma 3. There is one term left.

$$= (1 + n_a(\lambda)) m_{\lambda+(a)} \tag{24}$$

□

An expression for an operator that adds a row without a coefficient can be written in terms of this operator.

Theorem 3 *For $a > 0$ define $RM_a = \sum_{k\geq 0}(-1)^k \frac{(RM_a^{(1)})^{k+1}}{(k+1)!}(h_a^k)^{\perp}$ then $RM_a m_\lambda = m_{\lambda+(a)}$.*

Proof: Apply the previous proposition to this formula and reduce using the following steps.

$$RM_a m_\lambda = \sum_{i\geq 0}(-1)^i \frac{\left(RM_a^{(1)}\right)^{i+1}}{(i+1)!}\left(h_a^i\right)^\perp m_\lambda \tag{25}$$

$$= \sum_{i\geq 0}(-1)^i \frac{(n_a(\lambda)+1)\dots(n_a(\lambda)+i+1)}{(i+1)!} m_{\lambda-(a^i)+(a^{i+1})} \tag{26}$$

$$= \sum_{i=0}^{n_a(\lambda)}(-1)^i \binom{n_a(\lambda)+1}{i+1} m_{\lambda+(a)} \tag{27}$$

$$= m_{\lambda+(a)} \tag{28}$$

The last equality is true because for $n > 0$, $\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i} = 1$. □

Notice that the action of the RM_a operators on the monomial basis implies that $RM_a RM_b = RM_b RM_a$. This property is difficult to derive just from the definition of the operator.

This expression for the operator RM_a is a little unsatisfying since the computation of $(RM_a^{(1)})^i$ can be simplified. The following operator shows how RM_a can be reduced to closer resemble CH_{1^k}. To add more than one row at a time to a monomial symmetric function, the formula resembles the vertex operator that adds a column to the homogeneous basis.

Proposition 2 *For $a > 0$ and $k \geq 0$, we have that*

$$RM_a^{(k)} = \frac{\left(RM_a^{(1)}\right)^k}{k!} = \sum_\lambda (-1)^{|\lambda|} m_{a^k|\lambda} e_\lambda^\perp$$

with the understanding that $m_{a^k|\lambda} = 0$ if $a^k|\lambda$ is undefined. It follows that $RM_a^{(k)} m_\lambda = \binom{n_a(\lambda)+k}{k} m_{\lambda+(a^k)}$.

Proof: By induction on k, we will show that $RM_a^{(1)} RM_a^{(k)} = (k+1) RM_a^{(k+1)}$. It follows that $(RM_a^{(1)})^k = k! RM_a^{(k)}$. Since $(RM_a^{(1)})^k m_\lambda = (n_a(\lambda)+1)(n_a(\lambda)+2)\cdots(n_a(\lambda)+k) m_{\lambda+(a^k)}$, then $RM_a^{(k)} m_\lambda = \frac{(n_a(\lambda)+1)(n_a(\lambda)+2)\cdots(n_a(\lambda)+k)}{k!} m_{\lambda+(a^k)}$.

$$RM_a RM_a^{(k)} = \sum_{j\geq 0}(-1)^j m_{(a+j)} e_j^\perp \sum_\lambda (-1)^{|\lambda|} m_{a^k|\lambda} e_\lambda^\perp \tag{29}$$

Commute the action of $e_j^\perp$ and $m_{a^k|\lambda}$ using Lemma 4.

$$= \sum_{j\geq 0}(-1)^j m_{(a+j)} \sum_\lambda (-1)^{|\lambda|} \sum_\mu r_\mu m_{a^k|\lambda-(\mu)} e_{j-|\mu|}^\perp e_\lambda^\perp \tag{30}$$

$$= \sum_{j\geq 0}\sum_\lambda\sum_\mu (-1)^{|\lambda|+j} r_\mu m_{(a+j)} m_{a^k|\lambda-(\mu)} e_{j-|\mu|}^\perp e_\lambda^\perp \tag{31}$$

The formula for multiplying a monomial symmetric function with one part is given in Lemma 5.

$$= \sum_{j\geq 0}\sum_{\lambda}\sum_{\mu}\sum_{l\geq 0}(-1)^{|\lambda|+j} r_{\mu} n_{a+j+l}(a^k \mid \lambda - (\mu) - (l) + (a+l+j)) \times m_{a^k|\lambda-(\mu)-(l)+(a+l+j)} e^{\perp}_{j-|\mu|} e^{\perp}_{\lambda} \tag{32}$$

The next step is to change the sum over μ so that it includes the part of size l, this is equivalent to making the replacement $\mu \to \mu - (l)$.

$$= \sum_{j\geq 0}\sum_{\lambda}\sum_{\mu}\sum_{l\geq 0}(-1)^{|\lambda|+j} r_{\mu-(l)} n_{a+j+l}(a^k \mid \lambda - (\mu) + (a+l+j)) \times m_{a^k|\lambda-(\mu)+(a+l+j)} e^{\perp}_{j+l-|\mu|} e^{\perp}_{\lambda} \tag{33}$$

Let $i = j + l$, then the sum over j can be converted to a sum over i.

$$= \sum_{\lambda}\sum_{\mu}\sum_{l\geq 0}\sum_{i\geq l}(-1)^{|\lambda|+i-l} r_{\mu-(l)} n_{a+i}(a^k \mid \lambda - (\mu) + (a+i)) \times m_{a^k|\lambda-(\mu)+(a+i)} e^{\perp}_{i-|\mu|} e^{\perp}_{\lambda} \tag{34}$$

Interchange the sum over i and the sum over l. Since $l \geq 0$ and $i \geq l$ then $i \geq 0$ and $0 \leq l \leq i$.

$$= \sum_{\lambda}\sum_{\mu}\sum_{i\geq 0}\sum_{l=0}^{i}(-1)^{l} r_{\mu-(l)} (-1)^{|\lambda|+i} n_{a+i}(a^k \mid \lambda - (\mu) + (a+i)) \times m_{a^k|\lambda-(\mu)+(a+i)} e^{\perp}_{i-|\mu|} e^{\perp}_{\lambda} \tag{35}$$

Notice that since $e^{\perp}_{i-|\mu|}$ is zero for all $i < |\mu|$, then all terms are zero unless $i \geq |\mu|$.

$$= \sum_{\lambda}\sum_{\mu}\sum_{i\geq |\mu|}\sum_{l=0}^{i}(-1)^{l} r_{\mu-(l)} (-1)^{|\lambda|+i} n_{a+i}(a^k \mid \lambda - (\mu) + (a+i)) \times m_{a^k|\lambda-(\mu)+(a+i)} e^{\perp}_{i-|\mu|} e^{\perp}_{\lambda} \tag{36}$$

The sum over l is equal to 0 as long as $|\mu| > 0$ using Lemma 3.

$$= \sum_{\lambda}\sum_{i\geq 0}(-1)^{|\lambda|+i} n_{a+i}(a^k \mid \lambda + (a+i)) m_{a^k|\lambda+(a+i)} e^{\perp}_{i} e^{\perp}_{\lambda} \tag{37}$$

Let the sum over λ include the part of size i, then $\lambda = \lambda + (i)$.

$$= \sum_{\lambda}(-1)^{|\lambda|}\sum_{i\geq 0} n_{a+i}(a^{k+1} \mid \lambda) m_{a^{k+1}|\lambda} e^{\perp}_{\lambda} \tag{38}$$

The sum over i is now independent of λ since $\sum_{i\geq 0} n_{a+i}(a^{k+1} \mid \lambda)$ will always be $k+1$.

$$= (k+1)\sum_{\lambda}(-1)^{|\lambda|} m_{a^{k+1}|\lambda} e_{\lambda}^{\perp} = (k+1) RM_a^{(k+1)} \tag{39}$$

□

It follows that the formula for $RM_a^{(k)}$ can be substituted into Theorem 3 and this provides a more reduced form of the first formula given for RM_a.

Corollary 1 *For $a > 0$, $RM_a = \sum_{k\geq 0}\sum_{\lambda}(-1)^{|\lambda|+k} m_{a^{k+1}|\lambda} e_{\lambda}^{\perp}(h_a^k)^{\perp}$.*

Since the forgotten basis is related to the monomial basis by an application of the involution ω, then the formulas for the symmetric function operator that adds a row to the forgotten symmetric functions follows immediately.

Corollary 2 *For $a > 0$, $RF_a = \sum_{k\geq 0}\sum_{\lambda}(-1)^{|\lambda|+k} f_{a^{k+1}|\lambda} h_{\lambda}^{\perp}(e_a^k)^{\perp}$ has the property that*

$$RF_a f_{\lambda} = f_{\lambda+(a)}$$

There exists an operator $\mathcal{T}_{-X}$ of the same form as the operators that exist already in this paper that has the property that $\mathcal{T}_{-X} P[X] = 0$, if $P[X]$ is a homogeneous symmetric function of degree greater than 0 and $\mathcal{T}_{-X} 1 = 1$. This means that the operator applied to an arbitrary symmetric function has the property that it picks out the constant term of the symmetric function.

Proposition 3 *Define the operator*

$$\mathcal{T}_{-X} = \sum_{\lambda}(-1)^{|\lambda|} s_{\lambda'} s_{\lambda}^{\perp}$$

Then for any dual bases $\{a_\mu\}_\mu$ and $\{b_\mu\}_\mu$ (that is, $\langle a_\mu, b_\lambda\rangle = \delta_{\lambda\mu}$), this is equivalent to

$$\mathcal{T}_{-X} = \sum_{\lambda}(-1)^{|\lambda|}\omega(a_\lambda) b_{\lambda}^{\perp}$$

This operator has the property that $\mathcal{T}_{-X} s_\lambda = 0$ for $|\lambda| > 0$ and $\mathcal{T}_{-X} 1 = 1$.

Proof:

$$\mathcal{T}_{-X} s_\mu = \sum_{\lambda}(-1)^{|\lambda|} s_{\lambda} s_{\lambda'}^{\perp} s_\mu \tag{40}$$

This is exactly the same expression as formula ([3], p. 90, (I.5.23.1)) with the x variables substituted for the y. This expression is 0 unless $s_\mu = 1$.

It requires very little to show that this operator can be given an expression in terms of any dual basis.

$$\mathcal{T}_{-X} = \sum_{\lambda} (-1)^{|\lambda|} \omega(s_\lambda) s_\lambda^\perp \tag{41}$$

$$= \sum_{\lambda} (-1)^{|\lambda|} \sum_{\mu \vdash |\lambda|} \langle \omega(s_\lambda), \omega(b_\mu) \rangle \omega(a_\mu) s_\lambda^\perp \tag{42}$$

$$= \sum_{\mu} \sum_{\lambda \vdash |\mu|} (-1)^{|\mu|} \omega(a_\mu) \langle s_\lambda, b_\mu \rangle s_\lambda^\perp \tag{43}$$

$$= \sum_{\mu} (-1)^{|\mu|} \omega(a_\mu) b_\mu^\perp \tag{44}$$

□

Note that $\mathcal{T}_{-X}$ is actually a special case of a plethystic operator $\mathcal{T}_Z P[X] = P[X + Z]$.

Fix a basis of the symmetric functions, $\{a_\mu\}_\mu$, we may talk about the symmetric function linear operator that sends a_μ to the expression d_μ (where $\{d_\mu\}_\mu$ is any family of symmetric function expressions). We can say that this operator lies in the linear span of the operators $s_\lambda s_\mu^\perp$ and an expression can be given fairly easily.

Corollary 3 (*The everything operator***)** *Let $\{a_\mu\}_\mu$ be a basis of the symmetric functions and $\{b_\mu\}_\mu$ be its dual basis. Then an operator that sends a_μ to the expression d_μ is given by*

$$E_{\{a_\mu\}}^{\{d_\mu\}} = \sum_{\mu} d_\mu \mathcal{T}_{-X} b_\mu^\perp$$

In other words we have that $E_{\{a_\mu\}}^{\{d_\mu\}}$ acts linearly, and on the basis a_μ it has the action $E_{\{a_\mu\}}^{\{d_\mu\}} a_\mu = d_\mu$.

Proof: Note that when $b_\mu^\perp$ acts on a homogeneous polynomial, the result is a homogeneous polynomial of degree $|\mu|$ less. Therefore if $|\mu| > |\lambda|$, then $b_\mu^\perp a_\lambda = 0$. If $|\mu| < |\lambda|$ then $\mathcal{T}_{-X} b_\mu^\perp a_\lambda = 0$ since $\mathcal{T}_{-X}$ kills all non-constant terms. When $|\mu| = |\lambda|$, we have that $b_\mu^\perp a_\mu = \delta_{\lambda\mu}$ and therefore, $\mathcal{T}_{-X} b_\mu^\perp a_\lambda = \delta_{\lambda\mu}$. This also implies that

$$\sum_{\mu} d_\mu \mathcal{T}_{-X} b_\mu^\perp a_\lambda = \sum_{\mu} d_\mu \delta_{\lambda\mu} = d_\lambda \tag{45}$$

□

This operator looks too general to be of much use, but using known symmetric function identities it is possible to reduce and derive expressions for other operators. For instance, the symmetric function operator that adds a column to the monomial symmetric functions is a special case of this.

Theorem 4 *For $a > 0$, let*

$$CM_{a^k} = \sum_{\lambda} (-1)^{|\lambda|} \binom{n_a(\lambda) + k}{k} m_{\lambda+(a^k)} e_{\lambda}^{\perp},$$

then $CM_{a^k} m_{\lambda} = m_{a^k|\lambda}$ with the convention that $m_{a^k|\lambda} = 0$ if $a^k|\lambda$ is undefined.

Proof: We will reduce an expression for $E_{\{m_\lambda\}}^{\{m_{a^k|\lambda}\}}$ to one for CM_{a^k}.

$$E_{\{m_\lambda\}}^{\{m_{a^k|\lambda}\}} = \sum_{\lambda} m_{a^k|\lambda} \sum_{\mu} (-1)^{|\mu|} m_{\mu} e_{\mu}^{\perp} h_{\lambda}^{\perp} \tag{46}$$

Let $r_{\lambda\mu}$ be the coefficient of e_μ in h_λ (by an application of the involution ω it is also the coefficient of h_μ in e_λ). Then the expression is equivalent to

$$= \sum_{\lambda} m_{a^k|\lambda} \sum_{\mu} (-1)^{|\mu|} m_{\mu} e_{\mu}^{\perp} \sum_{\gamma \vdash |\lambda|} r_{\lambda\gamma} e_{\gamma}^{\perp} \tag{47}$$

Rearranging the sums this may be rewritten as

$$= \sum_{\lambda} \sum_{\mu} \sum_{\gamma} (-1)^{|\mu|} m_{a^k|\lambda} r_{\lambda\gamma} m_{\mu} e_{\mu}^{\perp} e_{\gamma}^{\perp} \tag{48}$$

It is possible to group all the terms that skew by the same elementary symmetric function by making the substitution $\mu \to \mu - (\gamma)$ since the sum over μ and γ are over partitions.

$$= \sum_{\lambda} \sum_{\mu} \sum_{\gamma} (-1)^{|\mu|-|\gamma|} m_{a^k|\lambda} r_{\lambda\gamma} m_{\mu-(\gamma)} e_{\mu}^{\perp} \tag{49}$$

Note that $m_{\mu-(\gamma)} = h_{\gamma}^{\perp}(m_\mu)$ and $\sum_{\gamma} (-1)^{|\gamma|} r_{\lambda\gamma} h_{\gamma}^{\perp} = (-1)^{|\lambda|} e_{\lambda}^{\perp}$.

$$= \sum_{\mu} \sum_{\lambda} (-1)^{|\mu|} m_{a^k|\lambda} (-1)^{|\lambda|} e_{\lambda}^{\perp}(m_\mu) e_{\mu}^{\perp} \tag{50}$$

Notice that the first part of this expression is exactly the operator $RM_a^{(k)}$ acting exclusively on m_μ. We may then apply Proposition 2 and note that the expression reduces to the sum stated in the theorem. □

The symmetric function operator that adds a column (or a group of columns) to the forgotten symmetric functions can be found by conjugating the CM_{a^k} operator by the involution ω to derive the following corollary.

Corollary 4 *For $a > 0$, let*

$$CF_{a^k} = \sum_{\lambda} (-1)^{|\lambda|} \binom{n_a(\lambda) + k}{k} f_{\lambda+(a^k)} h_{\lambda}^{\perp},$$

then $CF_{a^k} f_{\lambda} = f_{a^k|\lambda}$ with the convention that $f_{a^k|\lambda} = 0$ if $a^k|\lambda$ is undefined.

Remark Lemma 5 is not true for $a = 0$, therefore the proofs of Proposition 1 and Proposition 2 do not hold for $a = 0$. Something interesting can be said of these operators in this case. By following the calculation carefully, it is possible to see that if we set $RM_0^{(k)} = \sum_{\lambda:\ell(\lambda)\leq k}(-1)^{|\lambda|}m_\lambda e_\lambda^\perp$, then

$$RM_0^{(k)} m_\lambda = \binom{k-\ell(\lambda)}{k} m_\lambda \tag{51}$$

With the convention that $n_0(\lambda) = -\ell(\lambda)$, Theorem 4 and Corollary 4 and their proof make sense.

The operator that adds a sequence of rows to the monomial symmetric functions and the operator that adds a sequence of columns are related by a pair of formulas similar to in the case of formulas (14) and (15). Notice that Proposition 2 and Theorem 4 say that

$$CM_{a^k} = \sum_\lambda (-1)^{|\lambda|} RM_a^{(k)}(m_\lambda) e_\lambda^\perp \tag{52}$$

$$RM_a^{(k)} = \sum_\lambda (-1)^{|\lambda|} CM_{a^k}(m_\lambda) e_\lambda^\perp \tag{53}$$

This is 'eerie coincidence' number two. The relation between these two operators is very similar to the relation between CH_{1^k} and CE_{1^k} but not exactly the same. Once again this is unexpected and unexplained.

5. Schur vertex operators

A symmetric function operator that adds a row to the Schur functions is given in [3] (p. 95–96 I.5.29.d) that is of the same flavor as the other vertex operators presented here.

Theorem 5 (*Bernstein***)** *Let* $RS_a = \sum_{i\geq 0}(-1)^i h_{a+i} e_i^\perp$, *then* $RS_a s_\lambda = s_{\lambda+(a)}$ *if* $a \geq \lambda_1$. *In addition,* $RS_a RS_b = -RS_{b-1} RS_{a+1}$.

Proof: Repeated applications of this operator yields expressions of the Jacobi-Trudi sort. Use the relation $RS_a h_k = h_k RS_a - h_{k-1} RS_{a+1}$ (which follows from [3] example (I.5.29.b.5) and (I.5.29.d)), $RS_a(1) = h_a$ and follow the proof of [3] (I.3.(3.4") p. 43) which does not actually require that the indexing sequence be a partition. It follows then that

$$RS_{s_1} RS_{s_2} \cdots RS_{s_n}(1) = \det|h_{s_j - j + i}|_{1\leq i,j\leq n} \tag{54}$$

□

Conjugating this operator by ω produces an operator that adds a column to a Schur symmetric function. We will show in this section that a nice expression exists for a formula for an operator that adds a column to a Schur function, but with the property that the result is 0 if the partition is longer than the column being added.

It follows from the commutation relation of the RS_a, that there is a combinatorial method for calculating the action of RS_a on a Schur function when $m < \lambda_1$. Let $ht_k(\mu)$ be the integer i such that $\mu\rfloor_k = (\mu_2 - 1, \mu_3 - 1, \ldots, \mu_i - 1, \mu_1 + i - k, \mu_{i+1}, \ldots, \mu_{l(\mu)})$ is chosen to be a partition. This amounts to removing the first k cells from the border of μ. If it is not possible to find such an i such that $\mu\rfloor_k$ is a partition then say that $\mu\rfloor_k$ is undefined.

Corollary 5 *Let $\nu = \lambda + (a + k)$ where $k \geq \lambda_1 - a$ (ν is λ resting on a sufficiently long first row).*

$$RS_a s_\lambda = (-1)^{ht_k(\nu)-1} s_{\nu\rfloor_k}$$

where it is assumed that $s_{\nu\rfloor_k} = 0$ if $\nu\rfloor_k$ does not exist.

The proof of this corollary is not difficult, just a matter of showing that the commutation relation of $RS_a RS_b$ agrees with this definition of $\nu\rfloor_k$ and that the vanishing condition exists because $RS_a RS_{a+1} = 0$. This definition and corollary are useful in showing that an expression for $(RS_a)^k$ can be reduced to a form that is very similar to the other vertex operators presented here.

Lemma 6 *For $a \geq 0$,*

$$(RS_a)^k = \sum_\lambda (-1)^{|\lambda|} s_{a^k|\lambda} s_{\lambda'}^\perp$$

with the convention that $s_{a^k|\lambda}$ is 0 if $a^k|\lambda$ is undefined.

Proof: By induction on k. The statement agrees with Theorem 5 for $k = 1$.

$$RS_a (RS_a)^k = \sum_{i \geq 0} (-1)^i h_{a+i} e_i^\perp \sum_\lambda (-1)^{|\lambda|} s_{a^k|\lambda} s_{\lambda'}^\perp \tag{55}$$

$e_i^\perp$ can be commuted with the Schur function to produce

$$= \sum_{i \geq 0} (-1)^i h_{a+i} \sum_\lambda (-1)^{|\lambda|} \sum_{j=0}^{i} e_j^\perp (s_{a^k|\lambda}) e_{i-j}^\perp s_{\lambda'}^\perp \tag{56}$$

Interchange the order of all of the sums.

$$= \sum_\lambda \sum_{j \geq 0} \sum_{i \geq j} (-1)^{|\lambda|+i} h_{a+i} e_j^\perp (s_{a^k|\lambda}) e_{i-j}^\perp s_{\lambda'}^\perp \tag{57}$$

Make the substitution that $i \to i + j$, changing the sum so that it is over all $i \geq 0$ and expand the product $e_i^\perp s_{\lambda'}^\perp$. The notation that $\gamma/\lambda' \in \mathcal{V}_i$ means that γ differs from λ' by a vertical i strip ($\lambda'_j \leq \gamma_j \leq \lambda'_j + 1$ and $|\gamma| = |\lambda| + i$).

$$= \sum_\lambda \sum_{j \geq 0} \sum_{i \geq 0} (-1)^{|\lambda|+i+j} h_{a+i+j} e_j^\perp (s_{a^k|\lambda}) \sum_{\gamma/\lambda' \in \mathcal{V}_i} s_\gamma^\perp \tag{58}$$

Make the substitution $\gamma \to \gamma'$ so that the sum is over all partitions γ that differ from λ by a horizontal i strip and rearrange the sums.

$$= \sum_{\lambda} \sum_{i \geq 0} \sum_{\gamma/\lambda \in \mathcal{H}_i} (-1)^{|\lambda|+i} \sum_{j \geq 0} (-1)^j h_{a+i+j} e_j^{\perp}(s_{a^k|\lambda}) s_{\gamma'}^{\perp} \tag{59}$$

Now it is only necessary to notice that the sum over j is actually an application of the Schur vertex operator acting exclusively on the symmetric function $s_{a^k|\lambda}$. Switch the order of the sums over the partitions and expression becomes

$$= \sum_{\gamma} (-1)^{|\gamma|} \sum_{i \geq 0} \sum_{\lambda:\gamma/\lambda \in \mathcal{H}_i} RS_{a+i}(s_{a^k|\lambda}) s_{\gamma'}^{\perp} \tag{60}$$

There is a sign reversing involution on these terms so that only one term in the sum over i and λ survives, namely, $s_{a^{k+1}|\gamma}$. If $i = \gamma_1$ then $RS_{a+\gamma_1}(s_{a^k|(\gamma-(\gamma_1))}) = s_{a^{k+1}|\gamma}$.

Take any partition λ in this sum such that γ/λ is a horizontal strip of length less than γ_1. If $RS_{a+i}(s_{a^k|\lambda}) = 0$, then this term does not contribute to the sum. If $RS_{a+i}(s_{a^k|\lambda}) = s_{a^k|\nu}$ then $\nu = \lambda + (i+n)\rfloor_n$, where $n = \gamma_1 - i$. There is a combinatorial statement that can be made about partitions that satisfy this condition, this is a lemma stated in [4] (Lemma 3.15, p. 34).

Lemma 7 *There exists an involution I_γ^n on partitions μ such that μ/γ is a horizontal n strip, $\mu\rfloor_n$ exists and $\gamma \neq \mu\rfloor_n$ with the property that $ht_n(I_\gamma^n(\mu)) = ht_n(\mu) \pm 1$ and $\mu\rfloor_n = I_\gamma^n(\mu)\rfloor_n$.*

This is exactly the situation here. Set $\mu = \lambda + (i+n)$ then μ/γ is a horizontal strip of size $|\mu| - |\gamma| = |\lambda| + i + n - |\gamma| = n$. The result then is that all terms cancel *except* for the terms such that $\gamma = \lambda + (i+n)\rfloor_n$ or $i = \gamma_1$ and $RS_{a+i}(s_{a^k|\lambda}) = s_{a^{k+1}|\gamma}$.

The sum therefore reduces to

$$= \sum_{\gamma} (-1)^{|\gamma|} s_{a^{k+1}|\gamma} s_{\gamma'}^{\perp} \tag{61}$$

□

With this expression for the Schur function vertex operator, it is possible to reduce the expression for the 'everything operator' that adds a column to the Schur functions but is zero when the length of the indexing partition is larger than the height of the column being added.

Theorem 6 *For $a, k \geq 0$, let $CS_{a^k} = \sum_\lambda (-1)^{|\lambda|}(RS_a)^k(s_\lambda)s_{\lambda'}^{\perp}$. This operator has the property that $CS_{a^k}s_\lambda = s_{a^k|\lambda}$ if $l(\lambda) \leq k$ and $CS_{a^k}s_\lambda = 0$ for $l(\lambda) > 0$.*

Proof: Take the expression for the everything operator that adds a columns of height k using the convention that $s_{a^k|\lambda}$ is zero whenever $a^k \mid \lambda$ is undefined.

$$E_{\{s_\lambda\}}^{\{s_{a^k|\lambda}\}} = \sum_\lambda s_{a^k|\lambda} \sum_\mu (-1)^{|\mu|} s_\mu s_{\mu'}^\perp s_{\lambda'}^\perp \tag{62}$$

The coefficients of the expansion of $s_\mu s_\lambda$ in terms of Schur functions are well studied and there exists formulas and combinatorial interpretations for their calculation. The only properties that we require here is that the coefficients in the the expression $s_\mu s_\lambda = \sum_\nu c^\nu_{\lambda\mu} s_\nu$ have the property that $c^\nu_{\lambda\mu} = c^{\nu'}_{\lambda'\mu'}$ and $s_\lambda^\perp s_\nu = \sum_\nu c^\nu_{\mu\lambda} s_\mu$.

$$= \sum_\lambda s_{a^k|\lambda} \sum_\mu (-1)^{|\mu|} s_\mu \sum_\nu c^{\nu'}_{\lambda\mu'} s_{\nu'}^\perp \tag{63}$$

Next, we rearrange the sums and make the substitution $c^{\nu'}_{\lambda\mu'} = c^\nu_{\lambda'\mu}$.

$$= \sum_\lambda s_{a^k|\lambda} \sum_\nu (-1)^{|\nu|-|\lambda|} \sum_\mu c^\nu_{\lambda'\mu} s_\mu s_{\nu'}^\perp \tag{64}$$

Therefore the sum over μ is just an application of $s_{\lambda'}^\perp$ on (s_ν) and the sums can be rearranged.

$$= \sum_\nu (-1)^{|\nu|} \sum_\lambda (-1)^{|\lambda|} s_{a^k|\lambda} s_{\lambda'}^\perp (s_\nu) s_{\nu'}^\perp \tag{65}$$

The sum over λ is now exactly an application of Lemma 6.

$$= \sum_\nu (-1)^{|\nu|} (RS_a)^k (s_\nu) s_{\nu'}^\perp \tag{66}$$

This is the expression given in the statement of the theorem. □

The last of the 'eerie coincidences' of this article is that the CS_{a^k} and $(RS_a)^k$ are related by a pair of formulas similar to the case of formulas (14), (15) and (52), (53).

$$CS_{a^k} = \sum_\lambda (-1)^{|\lambda|} (RS_a)^k (s_\lambda) s_{\lambda'}^\perp \tag{67}$$

$$(RS_a)^k = \sum_\lambda (-1)^{|\lambda|} CS_{a^k} (s_\lambda) s_{\lambda'}^\perp \tag{68}$$

They say that once is happenstance, twice is coincidence and three times is a conspiracy. This relationship can be made more explicit and it explains why these operators come in pairs, but not why column adding operators happen to be related row adding operators for both the Schur and monomial bases and why a similar relation exists with the homogeneous and elementary vertex operators.

Let V be a linear operator from the space of symmetric functions to itself. Define

$$\bar{V} = \sum_{\lambda} (-1)^{|\lambda|} V(s_\lambda) s_{\lambda'}^{\perp} = \sum_{\lambda} (-1)^{|\lambda|} V(a_\lambda)(\omega b_\lambda)^{\perp} \tag{69}$$

where the sum is over all partitions λ and $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$ are any two dual bases.

It is not difficult to show that $\bar{\bar{V}} = V$ and that Eqs. (52), (53), (67), (68) may be summarized as $RM_a^{(k)} = \overline{CM}_{a^k}$ and $(RS_a)^k = \overline{CS}_{a^k}$. The relationship between (14) and (15) is not exactly the same, but it follows that $CH_{1^k}(h_\lambda) = \overline{CE}_{1^k}(h_\lambda)$ for all $\ell(\lambda) \le k$.

6. An application: The tableaux of bounded height

One observation about the operator CS_{a^k} that could have an interesting application is that $CS_{0^k} s_\lambda = 0$ if $l(\lambda) > k$ and $CS_{0^k} s_\lambda = s_\lambda$ if $l(\lambda) \le k$. Knowing this and the commutation relation between RS_a and h_k allows us to calculate the number of pairs of standard tableaux of the same shape of bounded height [1] $\sum_{\lambda \vdash n : l(\lambda) \le k} f_\lambda^2$ (where f_λ is the number of standard tableaux of shape λ).

Proposition 4 *Let $CP(n,k)$ be the collection of sequences of non-negative integers of length k such that the sum is n.*

$$\sum_{\lambda \vdash n : l(\lambda) \le k} f_\lambda^2 = \sum_{s \in CP(n,k)} \binom{n}{s} \frac{\prod_{i<j}(s_j + j - (s_i + i))}{\prod_{i=1}^{k}(s_i + i - 1)!} n!$$

The formula follows by applying CS_{0^k} to the symmetric function h_1^n to arrive at a formula for the symmetric function $\sum_{\lambda \vdash n : l(\lambda) \le k} f_\lambda s_\lambda$.

Lemma 8

$$CS_{0^k}\left(h_1^n\right) = \sum_{\lambda \vdash n : l(\lambda) \le k} f_\lambda s_\lambda = \sum_{s \in CP(n,k)} \binom{n}{s} \det |h_{s_j - j + i}|_{1 \le i,j \le k}$$

Proof: Use the relation $RS_a h_k = h_k RS_a - h_{k-1} RS_{a+1}$, $RS_a 1 = h_a$ and induction to calculate that

$$RS_0^k\left(h_1^n\right) = \sum_{l=0}^{n} \sum_{s \in CP(n-l,k)} (-1)^{n-l} h_1^l \binom{n}{l, s} \det |h_{s_j - j + i}|_{1 \le i,j \le k} \tag{70}$$

Using the relation that $s_\lambda^{\perp}(h_1^n) = \binom{n}{|\lambda|} f_\lambda h_1^{n-|\lambda|}$ we have that

$$CS_{0^k}\left(h_1^n\right) = \sum_{\lambda} (-1)^{|\lambda|} (RS_0)^k (s_\lambda) s_{\lambda'}^{\perp}\left(h_1^n\right) \tag{71}$$

$$= \sum_{\lambda} (-1)^{|\lambda|} (RS_0)^k (s_\lambda) \binom{n}{|\lambda|} f_\lambda h_1^{n-|\lambda|} \tag{72}$$

$$= \sum_{i=0}^{n} \sum_{\lambda \vdash i} (-1)^i \binom{n}{i} (RS_0)^k (f_\lambda s_\lambda) h_1^{n-i} \tag{73}$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (RS_0)^k \left(h_1^i\right) h_1^{n-i} \tag{74}$$

Now using (70) we can reduce this further to

$$= \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{s \in CP(m-l,k)} (-1)^l \binom{n}{m} \binom{m}{l, s} h_1^{n+l-m} \det |h_{s_j - j + i}|_{1 \le i, j \le k} \tag{75}$$

Now switch the sums indexed by l and m and then make the replacement $m \to m + l$

$$= \sum_{l=0}^{n} \sum_{m=0}^{n-l} \sum_{s \in CP(m,k)} (-1)^l \binom{n}{m+l} \binom{m+l}{l, s} h_1^{n-m} \det |h_{s_j - j + i}|_{1 \le i, j \le k} \tag{76}$$

Now switch the sums back and rearrange the binomial coefficients

$$= \sum_{m=0}^{n} \sum_{l=0}^{n-m} \sum_{s \in CP(m,k)} (-1)^l \binom{n}{n-m, s} \binom{n-m}{l} h_1^{n-m} \det |h_{s_j - j + i}|_{1 \le i, j \le k} \tag{77}$$

Now the sum $\sum_{l=0}^{n-m} (-1)^l \binom{n-m}{l}$ will always be zero unless $n - m = 0$ and if $n = m$ then it is 1 and so the entire sum collapses to

$$= \sum_{s \in CP(n,k)} \binom{n}{s} \det |h_{s_j - j + i}|_{1 \le i, j \le k} \tag{78}$$

□

Proof of Proposition 4: The proposition follows from this lemma with a little manipulation. There is a linear and multiplicative homomorphism that sends the symmetric functions to the space of polynomials in one variable due to Gessel defined by $\theta(h_n) = x^n/n!$. This homomorphism has the property that $\theta(s_\lambda) = f_\lambda x^{|\lambda|}/|\lambda|!$. The image of the formula in the lemma is then

$$\theta\left(CS_{0^k}\left(h_1^n\right)\right) = \theta\left(\sum_{\lambda \vdash n : l(\lambda) \le k} f_\lambda s_\lambda\right) = \sum_{\lambda \vdash n : l(\lambda) \le k} f_\lambda^2 \frac{x^n}{n!} \tag{79}$$

Therefore if we set $(a)_0 = 1$ and $(a)_i = a(a-1)\cdots(a-i+1)$ then we have (by making a slight transformation that reverses order of the sequence first... $j \to n+1-j, i \to n+1-i$

and $s_i \to s_{n+1-i}$) that

$$\sum_{\lambda\vdash n: l(\lambda)\le k} f_\lambda^2 = \sum_{s\in CP(n,k)} \binom{n}{s} \det\left|\frac{(s_j+j-1)_{i-1}}{(s_j+j-1)!}\right|_{1\le i,j\le k} n! \tag{80}$$

$$\sum_{\lambda\vdash n: l(\lambda)\le k} f_\lambda^2 = \sum_{s\in CP(n,k)} \binom{n}{s} \frac{\det|(s_j+j-1)_{i-1}|_{1\le i,j\le k}}{\prod_{i=1}^{k}(s_j+j-1)!} n! \tag{81}$$

The determinant is a specialization of the Vandermonde determinant in the variables $s_j + j - 1$ so the formula reduces to the expression stated in the proposition. □

We note that in the case that $k = 1$ this sum reduces to 1 and in the case that $k = 2$ we have that

$$\sum_{\lambda\vdash n\,:\,l(\lambda)\le 2} f_\lambda^2 = \sum_{j=0}^{n} \binom{n}{j} \frac{n-2j+1}{(j)!(n-j+1)!} n! = \sum_{j=0}^{n} \binom{n}{j}^2 \frac{n-2j+1}{n-j+1} \tag{82}$$

And this is an expression for the Catalan numbers. It would be interesting to see if these expressions and equations could be q or q, t analogued.

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