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On the Generation of Some Embeddable GF(2) Geometries

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Abstract. The generating rank is determined for several GF(2)-embeddable geometries and it is demonstrated that their generating and embedding ranks are equal. Specifically, we prove that each of the two generalized hexagons of order $(2, 2)$ has generating rank 14, that the central involution geometry of the Hall-Janko sporadic group has generating rank 28, and that the dual polar space DU(6,2) has generating rank 22. We also include a survey of all instances in which either the generating or embedding rank of an embeddable GF(2) geometry is known.

Keywords: point-line geometry, embeddable geometry, embedding rank, generating rank

1. Introduction

An *incidence system* is a triple (P, L, I) consisting of a set P whose elements are called *points*, a set L whose members are called *lines*, and a symmetric relation $I \subset (P \times L) \cup (L \times P)$. If $p \in P, l \in L$ and $(p, l) \in I$ then we say p is incident or on l. (P, L, I) is said to be a *linear incidence system* or a *point-line geometry* if two points are incident with at most one line. In this case we may identify each line with the set of points with which it is incident and replace I with the symmetrization of the relation $\in$ and then we will write (P, L) in place of (P, L, I). (P, L) is a said to be an *geometry of order 2* or, alternatively for the purpose of this paper, a GF(2) geometry if every one of its lines has three points. For a finite GF(2)-geometry a projective embedding is an injective mapping $e : P \to \mathbb{PG}(n-1, 2) = \Pi$ such that

(1) $\langle e(P)\rangle_{\mathbb{F}_2} = \Pi$ and
(2) for any line $l = \{x, y, z\}, e(x) + e(y) + e(z) = 0$.

The latter condition is equivalent to $\langle e(l)\rangle_{\mathbb{F}_2}$ is a projective line of Π. Assume that (P, L) is a GF(2)-geometry. Let $[P]$ be the vector space over GF(2) with basis P. For a line $l = \{x, y, z\}$ let $\bar{l} = x + y + z \in [P]$ and set $[L] = \langle \bar{l} \mid l \in L\rangle$ a subspace of $[P]$. Let $U(P)$ be the quotient $[P]/[L]$ and for $x \in P$ set $\bar{e}(x) = x + [L]$. Then an embedding exists for (P, L) if and only if the the map $\bar{e}$ is injective. In this case this embedding is called

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the *universal embedding*. For such a geometry we define the *embedding rank* of (P, L), $\mathrm{er}(P, L)$, equal to be the dimension of $U(P)$.

By a *subspace* of an incidence system $\Gamma = (P, L)$ we mean a subset X of the point set P with the property that if a line meets X in at least two points then the line is entirely contained in X. Clearly the intersection of subspaces is a subspace. Consequently, for an arbitrary subset X of P we can define the *subspace generated* by X to be the intersection of all subspaces containing X and will be denoted by $\langle X\rangle_\Gamma$. This is the unique minimal element among the collection of subspaces which contain X. We will say that a subset X *generates* P if $\langle X\rangle_\Gamma = P$ and we define the *generating rank* of (P, L), $\mathrm{gr}(P, L)$ to be the size of a generating set of minimal cardinality. It is an immediate consequence of these definitions that if $\Gamma = (P, L)$ is an embeddable GF(2)-geometry then $\mathrm{gr}(\Gamma) \geq \mathrm{er}(\Gamma)$.

This paper is part of a larger project to determine the generating rank of highly regular geometries with three points on a line and more generally to investigate the relationship between the embedding and generating rank. Elsewhere we conjectured the generating and embedding ranks are equal for GF(2) geometries but that has been shown to be false ([20]). His counterexample, however, is not a common geometry so the question remains whether for Lie type geometries or geometries for sporadic groups the embedding and generating ranks are equal. Here we will determine the generating rank of four GF(2) embeddable geometries which are all generalized hexagons or near polygons. By a near-2*n* gon we mean a geometry (P, L) in which the collineairity graph has diameter *n* and which has the property that for each point line pair (p, l) there is a unique closest point to p on l. ([25]). A *generalized hexagon* is a near hexagon (6-gon) with the additional property that for two points at distance two there is a unique common neighbor. ([8]). The particular geometries studied here are the smallest Lie geometries for which the generating rank was, heretofore, unknown. For each we will determine its generating rank and, in particular, show that it is equal to the previously determined embedding rank. Specifically, we prove the following

Theorem A

(a) *The usual* $G_2(2)$ *generalized hexagon has generating rank* 14.
(b) *The dual* $G_2(2)$ *generalized hexagon has generating rank* 14.
(c) *The involution geometry of the Hall-Janko simple group has generating rank* 28.
(d) *The unitary dual polar space* $\mathrm{DU}(6, 2)$ *has generating rank* 22. *In each instance the generating rank is equal to the universal embedding rank.*

By the "usual" $G_2(2)$ generalized hexagon we mean the duality class which has an embedding in $PG(5, 2)$. The outline of this paper is as follows: In Section two we determine the generating rank of the usual $G_2(2)$ generalized hexagon. Section three is devoted to its dual. Section four treats the involution geometry of the Hall-Janko group. In Section five we study the last of our four geometries, the dual polar space of unitary type, *DU*(6, 2). Finally, in Section six we include a survey of generating and embedding ranks of GF(2) embeddable geometries.

2. The usual $G_2(2)$ generalized hexagon

The purpose of this section is to prove the following

Proposition 2.1 *The usual $G_2(2)$ generalized hexagon has generating rank 14.*

Let $DSP(2n, q)$ denote the dual polar space of symplectic type in dimension $2n$ over the field $\mathbb{F}_q$ (see [12]) for a description of this geometry.

It is well known that the usual $G_2(2)$ generalized hexagon is a geometric hyperplane of $DSP(6, 2)$ ([24]). We describe this inclusion: Let $(\mathcal{P}, \mathcal{L})$ be the dual polar space $DSP(6, 2)$ and for $i \leq 3$ let Δ_i denote the pairs of points at distance i and let $d(,)$ denote the distance function. For a point $a \in \mathcal{P}$ denote by H_a the geometric hyperplane consisting of all points at distance at most two from a:

$$H_a = \Delta_{\leq 2}(a) = \{b \in \mathcal{P} \mid d(a, b) \leq 2\}.$$

Then for any pair of points $a, b \in \Delta_3$ the sum

$$H_a \oplus H_b = [H_a \cap H_b] \cup [(\mathcal{P} \backslash H_a) \cap (\mathcal{P} \backslash H_b)]$$

is a geometric hyperplane isomorphic to the usual $G_2(2)$ generalized hexagon.

In ([15]) the following method for constructing a new GF(2) geometry from a given GF(2) geometry $\Gamma = (P, L)$ is introduced:

Let $I = \{1, 2, 3\}$. Set $Y = I \times P$, $Z = \{\sigma : I \to P \mid Im(\sigma) \in L\}$. Then $\tilde{P} = Y \cup Z$ is the new set of points.

The lines are of four types:

(i) For $i \in I, l \in L, \{i\} \times l \in \tilde{L}$;
(ii) For $x \in P, I \times \{x\} \in \tilde{L}$;
(iii) For $(i, x) \in Y, \sigma \neq \tau \in Z$; $\{(i, x), \sigma, \tau\} \in \tilde{L}$ if $Im(\sigma) = Im(\tau), \sigma(i) = \tau(i) = x$; and
(iv) $\{\sigma_1, \sigma_2, \sigma_3\} \subset Z$ is in $\tilde{L}$ if the $Im(\sigma_i)$ are distinct and if each i

$$\{\sigma_1(i), \sigma_2(i), \sigma_3(i)\} \in L.$$

It is further shown in ([15]) that if $\Gamma = (P, L)$ is a generalized quadrangle of order 2 then $\tilde{\Gamma} = (\tilde{P}, \tilde{L})$ is $DSP(6, 2)$. We will make use of this model. Before doing so, however, we also need a useable description of a generalized quadrangle of order 2. Thus, let $V = \langle x_1, x_2, y_1, y_2 \rangle$ be a four dimensional symplectic space over GF(2) and assume that $x_1 \perp x_2 \perp y_2 \perp y_1 \perp x_1$.

Let $a_1 = x_1, a_2 = y_2, a_3 = x_1 + y_2, b_1 = x_2, b_2 = y_1, b_3 = x_2 + y_1$ and set $a = (1, a_1), b = (2, a_2)$ so that $d_{\tilde{\Gamma}}(a, b) = 3$. We first enumerate the points in $(H_a \oplus H_b) \cap Y$ which we shall denote by A:

$$\begin{aligned} A &= (H_a \oplus H_b) \cap Y \\ &= (1, a_2), (1, b_1), (1, a_2 + b_1), (1, b_2), (1, a_2 + b_2), (1, b_3), (1, a_2 + b_3), \end{aligned}$$

$$(2, a_1), (2, b_1), (2, a_1 + b_1), (2, b_2), (2, a_1 + b_2), (2, b_3), (2, a_1 + b_3),$$
$$(3, a_3), (3, b_1), (3, a_3 + b_1), (3, b_2), (3, a_3 + b_2), (3, b_3), (3, a_3 + b_3).$$

This set of 21 points is generated by the following nine points:

$$(1, a_2), (2, a_1), (3, a_3), (i, b_j), i = 1, 2;\ j = 1, 2, 3.$$

Now we claim that $H_a \oplus H_b$ is generated by these nine points together with the following five points from $(H_a \oplus H_b) \cap Z$:

$$\sigma_1 = (a_1, b_1, a_1 + b_1), \sigma_2 = (b_2, a_2, a_2 + b_2), \sigma_3 = (a_3, a_3 + b_3, b_3),$$
$$\sigma_4 = (a_2 + b_3, a_1 + b_2, a_3 + b_1), \sigma_5 = (a_2 + b_1, a_1 + b_3, a_3 + b_2).$$

Let X denote the subspace of $\tilde{\Gamma}$ generated by these 14 points so that X contains $(H_a \oplus H_b) \cap Y$.

$Z \cap (H_a \oplus H_b)$ has 42 points and each of these is of the form (x, y, z) where $\{x, y, z\}$ is a line of Γ. There are, of course, 15 lines in Γ and 9 of them are of the form $\{a_i, b_j, a_i + b_j\}$ for $i, j \in I$. For each l in this latter set there are exactly two $\sigma \in Z \cap (H_a \oplus H_b)$ with $Im(\sigma) = l$. Moreover, if one such σ is in X then there will be a unique point in A collinear with σ and then the third point, τ on the line joining these two will also have $Im(\tau) = l$ and hence also $\tau \in X$. For each l among the remaining six lines there are four $\sigma \in Z \cap (H_a \oplus H_b)$ with $Im(\sigma) = l$. In fact, for each such line l there is a unique $\sigma \in Z$ with $Im(\sigma) = l$ such that $(i, \sigma(i)) \in A$ for $i = 1, 2, 3$. If we set τ_i equal to the third point on the line joining σ and $(i, \sigma(i))$ then $\tau_i \in Z \cap (H_a \oplus H_b)$ and $Im(\tau_i) = l$. It therefore suffices to show that for each line l of Γ there is some $\sigma \in X$ such that $Im(\sigma) = l$.

For a permutation $\pi \in S_3$ and a $\sigma \in Z$ we shall denote by $\pi\sigma$ the effect of permutating the three entries of σ by π. When π is a transposition, (ij) then the resulting point of Z is the point on the line joining σ to $(k, \sigma(k))$ where $\{1, 2, 3\} = \{i, j, k\}$. So, for example, $(13)\sigma_1 = (a_1 + b_1, b_1, a_1)$ is the point of Z on the line joining σ_1 to $(2, b_1)$. Since these are both in X it follows that $(13)\sigma_1 \in X$. In each of the cases below the permutations of the σ_i are obtained by joining σ_i to a point of A. We shall also indicate that two points of $\tilde{P}$ are collinear by writing $x \sim y$ and in this case the third point on the line will be denoted by $x + y$.

$\sigma_1 \sim \sigma_2$ and $\sigma_1 + \sigma_2 = (a_1 + b_2, a_2 + b_1, a_3 + b_3)$.

$(13)\sigma_1 \sim (12)\sigma_3$ and $(13)\sigma_1 + (12)\sigma_3 = (a_2 + b_2, a_3 + b_1, a_1 + b_3)$.

$(13)\sigma_1 \sim (23)\sigma_4$ and $(13)\sigma_1 + (23)\sigma_4 = (a_3 + b_2, a_3, b_2)$.

$\sigma_1 \sim (12)\sigma_5$ and $\sigma_1 + (12)\sigma_5 = (b_3, a_2, a_2 + b_3)$.

$(23)\sigma_2 \sim \sigma_3$ and $(23)\sigma_2 + \sigma_3 = (a_3 + b_2, a_1 + b_1, a_2 + b_3)$.

$\sigma_2 \sim (12)\sigma_4$ and $\sigma_2 + (12)\sigma_4 = (a_1, b_3, a_1 + b_3)$.

$(23)\sigma_2 \sim (13)\sigma_5$ and $(23)\sigma_2 + (13)\sigma_5 = (a_3, a_3 + b_1, b_1)$.

$\sigma_3 \sim (13)\sigma_4$ and $\sigma_3 + (13)\sigma_4 = (b_1, a_2 + b_1, a_2)$.

$(12)\sigma_3 \sim (23)\sigma_5$ and $(12)\sigma_3 + (23)\sigma_5 = (a_1 + b_2, b_2, a_1)$.
$(12)\sigma_4 \sim (23)\sigma_5$ and $(12)\sigma_4 + (23)\sigma_5 = (a_3 + b_3, a_1 + b_1, a_2 + b_2)$.

These ten lines complement the five lines which are the images of $\sigma_i, 1 \le i \le 5$. We have therefore shown that for every line l there is a $\sigma \in X$ such that $Im(\sigma) = l$. From the above argument it then follows that $X = H_a \oplus H_b$ and consequently, the usual generalized hexagon of order 2 is generated by 14 points. By ([17]) we know as well that the universal embedding rank is 14. This completes the results of this section.

3. The dual $G_2(2)$ generalized hexagon

In this section we prove part (b) of our main theorem. Throughout this section $\Gamma = (P, L)$ will be the dual $G_2(2)$ generalized hexagon and we will denote by $d(,)$ the distance function in the point-collinearity graph. For $x \in P, i \le 3$ an integer we let $\Delta_i(x)$ be the set of points at distance i from x. We will also denote by $\Delta_{\le i}(x)$ the collection of points y such that $d(x, y) \le i$. As with the usual $G_2(2)$ generalized hexagon every line contains three points and every point lies on three lines and then it is trivial to compute that $|\Delta_1(x)| = 6$, $|\Delta_2(x)| = 24$, $|\Delta_3(x)| = 32$.

The incidence system of this section can be realized as the geometry of reflection centers for the group $G_2(2)$ represented in a six space over the quaternions. This geometry is the same as the long root subgroup geometry of the group $G_2(2)$ and in this section we make this identification. As a reference see ([11]). For the remainder of this section we let G denote a group isomorphic to $G_2(2)$. P is then the 63 central involutions of the group G (those which belong to the commutator subgroup which is isomorphic to $U_3(3)$). For a subgroup Y of G we will let $P(Y)$, $L(Y)$ denote the points and lines contained in Y. We remark that $P(Y)$ is a subspace of P. The possible relations between a pair $x, y \in P$ of distinct central involutions are as follows:

1. x and y are collinear, that is, $d(x, y) = 1$. In this case $xy = yx \in P$ and the line on x and y is $\{x, y, xy\}$. For $x \in P$ we denote by $\Delta_1(x)$ the points at distance one from x.
2. $d(x, y) = 2$. In this case $\langle x, y\rangle$ is a dihedral group of order 8 and the unique point which is collinear to both x and y is $(xy)^2 = Z(\langle x, y\rangle) = \langle x, y\rangle'$.
3. $d(x, y) = 3 = |xy|$ and therefore $\langle x, y\rangle$ is isomorphic to S_3.

We next record as a lemma the fact that $G_2(2)$ contains $SL(3, 2)$ as a subgroup generated by long root involutions:

Lemma 3.1 $G = G_2(2)$ *contains a single class* $\mathcal{S}$ *of subgroups isomorphic to* $SL(3, 2)$ *generated by long root involutions. The set of long root involutions contained in* S *form a subhexagon with parameters* $(2, 1)$. *For* S *such a subgroup* $N_G(S)$ *is isomorphic to* $Aut(SL(3, 2))$ *and consequently there are* 36 *such subgroups.*

Proof: This is easily deduced from ([10]). □

Lemma 3.2 *Let $S \in \mathcal{S}$. Then for every $x \in P \backslash P(S)$, $|P(S) \cap \Delta_1(x)| = 1$.*

Proof: Note that if $y, z \in P(S)$ then $\langle y, z\rangle \subset S$ and consequently $P(\langle y, z\rangle) \subset P(S)$. Suppose now that $y, z \in P(S)$ and $d(y, z) = 2$. Then $\Delta_1(y) \cap \Delta_1(z) = \{(yz)^2\} \in P(S)$. From this it follows that for any $x \in P \backslash P(S)$, $|\, \Delta_1(x) \cap P(S)\,| \leq 1$. On the other hand there are 21 points in $P(S)$. For each point $y \in P(S)$ there is a unique line l_y on y which is not contained in $P(S)$ and each such line contains 2 points which are not in $P(S)$. Consequently, there are $21 \times 2 = 42$ points $x \in P \backslash P(S)$ such that $P(S) \cap \Delta_1(x) \neq \emptyset$. Since $|P(S)| = 21$ this accounts for all points. □

Lemma 3.3 *Let $y, z \in P$, $d(y, z) = 2$. Then $|\{S \in \mathcal{S} \mid \langle y, z\rangle \leq S\}| = 4$.*

Proof: G is transitive on such pairs of which there are $\frac{63\times 24}{2} = 756$. G is also transitive on $\mathcal{S}$. Moreover, for $S \in \mathcal{S}$ there are $\frac{21\times 8}{2} = 84$ such pairs and $N_G(S)$ is transitive on the distance two pairs contained in S. It then follows that for any such pair $\{y, z\}$ the number $\alpha(y, z) = |\{S \in \mathcal{S} \mid \langle y, z\rangle \leq S\}|$ is independent of the pair y, z. Letting α denote this common value we have $36 \times 84 = 756\alpha$ and hence $\alpha = 4$. □

Lemma 3.4 *Let $y, z \in P$, $d(y, z) = 2$ and set $x = (yz)^2$ the unique point collinear with both y and z. Let $u \in \Delta_1(y) \cap \Delta_2(x)$. Then there are precisely two $S \in \mathcal{S}$ containing u, y, x, z.*

Proof: Since $d(u, x) = 2$ and $\Delta_1(u) \cap \Delta_1(x) = y = (ux)^2$ it follows that $d(u, y) = 3$, $\langle u, y, x, z\rangle \cong S_4$ and $G_2(2)$ is transitive on such quadruples (u, y, x, z) as can be seen from Section 3 of ([11]). The number of such quadruples is $63 \times 24 \times 4$. On the other hand the number of such quadruples lying in an element of $\mathcal{S}$ is $21 \times 8 \times 2$. Since $|\mathcal{S}| = 36$ it now follows that each such quadruple is contained in two elements of $\mathcal{S}$ as required. □

Before proceeding to our main proposition we require one last lemma:

Lemma 3.5 *$G_2(2)$ acts via conjugation as a rank three group on $\mathcal{S}$ with subdegrees 14 and 21. Moreover, for $S \neq S' \in S$, $S \cap S' = \langle P(S \cap S')\rangle$ and is either isomorphic to D_8 or S_4.*

Proof: Let $S \in \mathcal{S}$. S contains 21 two-Sylow subgroups isomorphic to D_8 and 14 subgroups isomorphic to S_4. Let X be a subgroup of S isomorphic to S_4. Then X is maximal in S and by (3.4) there are two members of $\mathcal{S}$ containing X one of which is S. In this way we obtain 14 elements $S' \in \mathcal{S}$ with $S \cap S' = \langle P(S \cap S')\rangle \cong S_4$. Now each two-Sylow of S is contained in two subgroups isomorphic to S_4 and these are the unique maximal subgroups of S containing the two-Sylow. Let T be a two-Sylow of S. From (3.3) T is contained in four elements of $\mathcal{S}$ of which S is one. By the above there are subgroups $T_1, T_2 \cong S_4$ of S containing T. By (3.4) each of T_i is contained in two members of $\mathcal{S}$ one of which is S. In this way we obtain two members $S_1, S_2 \neq S \in \mathcal{S}$ containing T such that $S_i \cap T \cong S_4$, $i = 1, 2$. Consequently, there is one more element $S' \in \mathcal{S}$ containing T and for this S', $S \cap S'$ is not isomorphic to S_4. Therefore, $S \cap S' = T \cong D_8$. This accounts for 21 subgroups in $\mathcal{S}$ and hence the remaining ones since $1 + 14 + 21 = 36$. □

We can now prove our main proposition which is part (b) of Theorem A:

Proposition 3.6 *The dual $G_2(2)$ generalized hexagon can be generated by* 14 *points.*

Proof: Let $y, z \in P$, $d(y, z) = 2$ and set $x = (yz)^2$. By (3.5) there are $S_1, S_2 \in \mathcal{S}$ with $S_1 \cap S_2 = \langle y, z\rangle$. Now we can generate each of $P(S_i)$, $i = 1, 2$ by 8 points (for example take $\Delta_3(a) \cap P(S_i)$ for any $a \in P(S_i)$). Moreover, we can take three of those points to be y, x, z. Therefore the subspace of $\Gamma = (P, L)$ generated by $P(S_1) \cup P(S_2)$ can be generated by 13 points. We now determine the points of this subspace. We claim that it contains all points in $\Delta_{\leq 2}(x)$. □

Note that $|P(S_i) \cap \Delta_2(x)| = 8$, $i = 1, 2$. As indicated above there are 6 points in $\Delta_1(x)$ and 24 points in $\Delta_2(x)$. We further note that $\Delta_1(y)$ is already contained in $P(S_1) \cup P(S_2)$ since each $P(S_i)$ contains two lines on y and they have in common only the line $P(\langle x, y\rangle)$. This is also true for the points xy, z and xz. Now suppose a is one of the 8 points in $P(S_1) \cap \Delta_3(x)$. By (3.2) there is a unique point $b \in P(S_2) \cap \Delta_1(a)$. Since $d(a, x) = 3$ $b \notin \Delta_1(x) \cap P(S_2) = P(\langle y, z\rangle)\backslash\{x\}$. Also, $b \notin \Delta_2(x)$. For if $b \in \Delta_2(x)$ then $c = (bx)^2 \in P(S_2) \cap \Delta_1(x) \subset P(S_1) \cap P(S_2)$. However, b is collinear with a unique point in $P(S_1)$ by (3.2) and this contradicts the fact that b is collinear with a. Thus $b \in \Delta_3(x)$. Then the point $ab \in \Delta_2(x)$, but $ab \notin [P(S_1) \cup P(S_2)]$. In this way we obtain eight points in $\Delta_2(x)$ not contained in $P(S_1) \cup P(S_2)$. This now accounts for all 24 points in $\Delta_2(x)$. Since every point in $\Delta_1(x)$ is contained on a line containing two points of $\Delta_2(x)$ it follows that this subspace contains $\Delta_1(x)$ and consequently, $\Delta_{\leq 2}(x)$.

Thus, altogether this subspace contains x, the six points in $\Delta_1(x)$, the 24 points in $\Delta_2(x)$, and 16 points from $\Delta_3(x)$. It is known (see, for example, figure 1 in [9]) that there are only three subspaces properly containing $\Delta_{\leq 2}(x)$: two subspaces with cardinality 47 of which the present one is an example, and all of P. It follows that the subspace $\langle P(S_1), P(S_2)\rangle_\Gamma$ is a maximal subspace and therefore with one further point chosen from $\Delta_3(x)$ we can generate all of P. Finally, we remark that in ([17]) it is shown that the universal embedding rank of the dual $G_2(2)$ generalized hexagon is 14.

Before we move on we collect as corollaries some generation results which will prove useful in the subsequent section.

Corollary 3.7 *Let $x \in P, S \in \mathcal{S}$, and $x \in P(S)$. Then $\langle P(S), \Delta_{\leq 2}(x)\rangle$ is a subspace with* 47 *points.*

Proof: As in the proof of (3.6) assume we have $S, S' \in \mathcal{S}$ with $S \cap S' \cong D_8$ so that the subspace X generated by $P(S) \cup P(S')$ has 47 points. Now let $x = Z(S \cap S')$. By the proof of (3.6), $X \supset \Delta_{\leq 2}(x)$ and consequently, $\langle P(S), \Delta_{\leq 2}(x)\rangle \subset X$. Because we are transitive on pairs $(S, x) \in \mathcal{S} \times P$, $x \in P(S)$ it suffices to prove that we have equality. Now there are 21 points in $P(S)$ and 31 points in $\Delta_{\leq 2}(x)$. The intersection has 13 points and so the union has 39 points. Now by (3.2) every point of $\Delta_{\leq 2}(x)\backslash P(S)$ is collinear to a unique point of

$P(S)$. Of the 16 points in $\Delta_2(x)\backslash P(S)$, eight are collinear with a point in $P(S) \cap \Delta_1(x)$. The remaining eight points are collinear with points in $P(S) \cap \Delta_3(x)$ and then the eight points on these lines are in $\Delta_3(x)$. In this way the subspace generated by $P(S)$ and $\Delta_{\leq 2}(x)$ contains at least another eight points and hence at least 47 altogether. It follows that we have equality. □

Corollary 3.8 *Let $x \in P$ and $\{y_1, y_2, y_3\}$ be points from the three different lines on x. Then $\{x, y_1, y_2, y_3\}$ can be extended to a generating set.*

Proof: As in the proof of (3.6) we construct a generating set by taking a pair $S, S' \in \mathcal{S}$ with $S \cap S' \cong D_8$, a generating set for $P(S \cap S')$ and extending it to generating sets for each of S, S'. If x is not the center of $S \cap S'$, $x \in P(S \cap S')$ then the generating set contains representatives of each of the three lines on x. Since we are transitive on points the result follows. □

4. The involution geometry of the Hall-Janko group

In this section $\Gamma = (P, L)$ will be the central involution geometry of the Hall-Janko group, which we denote by *HJ*. Thus, P consists of the 315 central involutions of the *HJ* group and L is the collection of 525 elementry abelian subgroups of order 4 all of whose involutions are central. This geometry is a near-octagon in the sense of Shult and Yanushka ([25]). We let $d(,)$ denote the distance function for the point-collinearity graph of (P, L) and as previously, we let $\Delta_i(x)$ be the set of points at distance i from x where $i \leq 4$. Now, the possible relations between a pair $x, y \in P$ of distinct central involutions are as follows:

1. $d(x, y) = 1$ in which case, as defined above, $xy = yx \in P$. $|\Delta_1(x)| = 10$, there are five lines on x and the centralizer, $C_{HJ}(x)$, induces the alternating group A_5 on these lines and therefore permutes them three-transitively.
2. $d(x, y) = 2$. $|\Delta_2(x)| = 80$. In this case $\langle x, y\rangle$ is a dihedral group of order 8. There is a unique point collinear to both x and y which is $(xy)^2 = Z(\langle x, y\rangle) = \langle x, y\rangle'$.
3. $d(x, y) = 3 = |xy|$ and therefore $\langle x, y\rangle$ is isomorphic to S_3. $|\Delta_3(x)| = 160$. If $y \in \Delta_3(x)$ then there is a unique line on y which contains elements of $\Delta_4(x)$.
4. $d(x, y) = 4$, $|xy| = 5$, $\langle x, y\rangle \cong D_{10}$ a dihedral group of order 10. $|\Delta_4(x)| = 64$.

For a subgroup X of *HJ* we denote by $P(X)$ the set of central involutions contained in X. We point out that $P(X)$ is a subspace of P. We next record as a lemma the fact that *HJ* contains a subgroup $U_3(3)$ generated by central involutions:

Lemma 4.1 *HJ contains a single class of subgroups isomorphic to $U_3(3)$ generated by central involutions. The number of such subgroups is 100 and the action of HJ on this class of subgroups has permutation rank three with subdegrees 36 and 63. If G is a subgroup of HJ isomorphic to $U_3(3)$ then for every $x \in P(G)$ there is a unique $G' \cong U_3(3)$, $G' \neq G$ such that $P(G) \cap P(G') = \{x\} \cup [P(G) \cap \Delta_1(x)]$ (this generates the subgroup $2_-^{1+5} : A_5$). Moreover, for every subgroup S of G, $S = \langle P(S)\rangle \leq G$, $S \cong SL(3,2)$ there is a unique G^* such that $G \cap G^* = S$.*

Proof: This is easily deduced from ([10]). □

We will let $\mathcal{G}$ denote the conjugacy class of subgroups isomorphic to $U_3(3)$.

Lemma 4.2 *Let $G \in \mathcal{G}$. Then for every $x \in P \backslash P(G), |P(G) \cap \Delta_1(x)| = 1$.*

Proof: Note that if $y, z \in P(G)$ then $\langle y, z\rangle \subset G$ and consequently $P(\langle y, z\rangle) \subset P(G)$. Suppose now that $y, z \in P(G)$ and $d(y, z) = 2$. Then $\Delta_1(y) \cap \Delta_1(z) = \{(yz)^2\} \in P(G)$. From this it follows that for any $x \in P\backslash P(G)$, $|\Delta_1(x) \cap P(G)| \leq 1$. On the other hand there are 63 points in $P(G)$. For each point $y \in P(G)$ there are two lines l on y which are not contained in $P(G)$ and each such line contains 2 points which are not in $P(G)$. There are therefore $63 \times 4 = 252$ points $x \in P\backslash P(G)$ such that $P(G) \cap \Gamma_1(x) \neq \emptyset$. Since $|P(G)| = 63$ this accounts for all points. □

We can now prove our main result of this section which is part (c) of Theorem A:

Proposition 4.3 *The central involution geometry of HJ can be generated by* 28 *points.*

Proof: Let G_1 and let $x \in P(G_1)$. By (4.1) there is a unique $G_2 \in \mathcal{G}$ such that $P(G_1) \cap P(G_2) = [P(G_1) \cap \Delta_{\leq 1}(x)]$. Set $A = \langle P(G_1), P(G_2)\rangle_\Gamma$. We first show that A contains $\Delta_{\leq 2}(x)$. Suppose $y \in P(G_1) \cap \Delta_2(x)$. By (4.2), $P(G_2) \cap \Delta(y)$ consists of a single point. However, $(xy)^2 = \Delta_1(x) \cap \Delta_1(y) \in P(G_2)$. It therefore follows that if $z \in P(G_1) \cap \Delta_3(x)$ then the unique point in $P(G_2) \cap \Delta(z)$ has distance 3 from x. Thus each point of $P(G_1) \cap \Delta_3(x)$ is collinear with a unique point of $P(G_2) \cap \Delta_3(x)$ and conversely. Now if $z_i \in P(G_i) \cap \Delta_3(x), i = 1, 2$ are collinear points then it must be the case that the third point on the line z_1z_2 is in $\Delta_2(x)$. Since there are 32 points in each of $P(G_i) \cap \Delta_3(x)$ in this way we obtain 32 points in $\Delta_2(x)$ which are not in $P(G_i) \cap \Delta_2(x)$. On the other hand each $P(G_1) \cap \Delta_2(x)$ contains 24 points and this accounts for $2 \times 24 + 32 = 80$ points, hence all of $\Delta_2(x)$. Since each point in $\Delta_1(x)$ lies on four lines l with two points from $\Delta_2(x)$ it follows that $\langle P(G_1), P(G_2)\rangle_\Gamma$ contains $\Delta_{\leq 2}(x)$. Set $Z_1 = \langle P(G_1), P(G_2)\rangle_\Gamma$.

Now set $\Omega = \{1, 2, 3, 4, 5\}$ and set $\Pi = \Omega^{\{3\}}$ the collection of three element subsets from Ω. Let $x \in P$ and let $l_i, i \in \Omega$, be the the five lines on x. For $\alpha \in \Pi$ set $Y_\alpha = \cup_{i\in\alpha} l_i$. By (4.1) there are two subgroups $G_1, G_2 \in \mathcal{G}$ which contain Y_α and for these two groups $P(G_1) \cap P(G_2) = Y_\alpha$. We will denote these by $G_i^\alpha, i = 1, 2$.

Let $\alpha = \{1, 2, 3\}, \beta = \{1, 2, 4\}, \gamma = \{1, 2, 5\}$. Suppose $i, j, k \in \{1, 2\}$. Then $G_i^\alpha \cap G_j^\beta, G_i^\alpha \cap G_k^\gamma$ are subgroups of $G_i^\alpha \cong U_3(3)$ generated by root elements and isomorphic to $SL(3, 2)$. Moreover, $P(G_i^\alpha) \cap P(G_j^\beta) \cap P(G_k^\gamma)$ contains $l_1 \cup l_2$ and by (3.1) we have $\langle P(G_i^\alpha) \cap P(G_j^\beta) \cap P(G_k^\gamma)\rangle \cong S_4$ or D_8. In either case it follows that $\Delta_3(x) \cap [P(G_i^\alpha) \cap P(G_j^\beta) \cap P(G_k^\gamma)] = \emptyset$.

Set $B = \langle P(G_1^\alpha), P(G_2^\alpha), P(G_1^\beta)\rangle_\Gamma$. We claim that $P(G_i^\delta) \subset B$ for all $\delta \in \{\alpha, \beta, \gamma\}, i = 1, 2$. Since $B \supset \langle P(G_1^\alpha), P(G_2^\alpha)\rangle_\Gamma$ it follows by the above argument that $B \supset \Delta_{\leq 2}(x)$. Now $\Delta_3(x) \cap P(G_1^\alpha) \cap P(G_2^\alpha) = \emptyset$. On the other hand $G_i^\alpha \cap G_j^\gamma \cong SL(3, 2)$ and therefore $P(G_i^\alpha) \cap P(G_j^\gamma)$ contains 8 points of $\Delta_3(x)$ for $i, j = 1, 2$. Similarly, $P(G_1^\beta) \cap$

$P(G_j^\gamma)$ meets $\Delta_3(x)$ in eight points and these are disjoint from the points in $[P(G_1^\alpha) \cup P(G_2^\alpha)] \cap P(G_j^\gamma)$. Since $\Delta_{\leq 2}(x)$ is contained in B it now follows from (3.7) that $P(G_j^\gamma) \subset B$ for $j = 1, 2$. But then by interchanging the roles of β, γ we also get $P(G_2^\beta) \subset B$.

Because $C_{HJ}(x)$ acts 3-transitive on L_x, the set of lines on x, it follows that for any permutation π of Ω that $P(G_j^{\pi(\delta)}) \subset \langle P(G_1^{\pi(\alpha)}), P(G_2^{\pi(\alpha)}), P(G_1^{\pi(\beta)})\rangle_\Gamma$.

Set $\delta = \{1, 3, 4\}$ and $C = \langle B, P(G_1^\delta)\rangle_\Gamma$. Let $T = \{\tau \in \Pi \mid P(G_i^\tau) \subset C$ for $i = 1, 2\}$. Suppose $\tau \in \Pi$ with $1 \in \tau$. We claim that $\tau \in T$. By the above argument we know that $\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\} \in T$. Also, by the same argument $\{1, 3, 4\}, \{1, 3, 5\} \in T$. Since $\{1, 2, 4\}, \{1, 3, 4\} \in T$ this argument also implies that $\{1, 4, 5\} \in T$ completing the claim.

Now set $\gamma = \{2, 3, 4\}$ and $D = \langle C, P(G_1^\gamma)\rangle_\Gamma$. We claim that D contains $\Delta_{\leq 3}(x)$. Towards this end, set $S = \{\sigma \in \Pi \mid P(G_i^\sigma) \subset D$ for $i = 1, 2\}$. By the argument of the previous paragraph if $\sigma \in \Pi, \sigma \cap \{1, 2\} \neq \emptyset$ then $P(G_i^\tau) \subset D$ for $i = 1, 2$. This includes all σ except $\{3, 4, 5\}$. However, since $\{1, 4, 5\}, \{2, 4, 5\} \in S$ it then follows that $\{3, 4, 5\} \in S$. Consequently, for every $G \in \mathcal{G}$ which contains x we have $P(G) \subset D$. Since HJ is transitive on pairs (x, z) with $z \in \Delta_3(x)$ it follows that there is a $G \in \mathcal{G}$ containing x and z. It then follows that $z \in D$.

We can now complete the proof. $P(G_1^\alpha) \cap P(G_2^\alpha) = Y_\alpha$ which can be generated by four points: x together with one further point from each of the lines l_i, $i \in \alpha$. Now each of $P(G_i^\alpha)$ can be generated by Y_α together with 10 other points by (3.8). Thus $A = \langle P(G_1^\alpha), P(G_2^\alpha)\rangle_\Gamma$ can be generated with 24 points. Now $G_i^\alpha \cap G_1^\beta \cong SL(3, 2)$ and $|P(G_1^\beta) \cap [P(G_1^\alpha) \cup P(G_2^\alpha)] \cap \Delta_3(x)| = 16$. A contains $\Delta_{\leq 2}(x)$ and therefore $P(G_1^\beta) \cap \Delta_{\leq 2}(x)$. If $z_1 \in \Delta_3(x) \cap P(G_1^\beta)$ but is not contained in $(P(G_1^\beta) \cap [P(G_1^\alpha) \cup P(G_2^\alpha)] \cap \Delta_3(x))$ then $P(G_1^\beta) \subset \langle A, z_1\rangle_\Gamma$ and then we get $\langle A, z_1\rangle_\Gamma = B$ and B can be generated by $24 + 1 = 25$ points. Arguing similarly, C can be generated by 26 points and D by 27 points. By what we have shown above D contains $\Delta_{\leq 3}(x)$. By ([9]), $\Delta_4(x)$ has a single connected component which implies that for any point $u \in \Delta_4(x)$, $\langle D, u\rangle_\Gamma = P$ and P can be generated by 28 points. Finally, Frohardt and Smith ([18]) have shown that the embedding rank for the HJ central involution geometry is 28. □

5. The unitary dual polar space DU(6, 2)

We refer to ([13]) for a definition and properties of the dual polar spaces of unitary type and make use of the notation introduced there. Thus, we let V be a space of dimension 6 over the field $\mathbb{F}_4$ with basis x_i, y_i, $i = 1, 2, 3$ and let $h : V \times V \to \mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ be the Hermitian form with $h(x_i, y_j) = \delta_{ij}$, $h(x_i, x_j) = h(y_i, y_j) = 0$ for $i, j \in \{1, 2, 3\}$. We denote the set of isotropic one spaces by P and the collection of maximal totally isotropic subspaces by $\mathcal{P}$. For a totally isotropic subspace A we let $U(A) = \{m \in \mathcal{P} \mid A \subset m\} = \mathcal{P}(A^\perp)$. The set of lines, $\mathcal{L} = \{U(A) \mid dim\ A = 2, A$ totally isotropic$\}$. When x is an isotropic point, $U(x)$ is a convex subspace and a generalized quadrangle (a quad of the the near hexagon formed by the dual polar space). This generalized quadrangle is dual to the generalized quadrangle induced on $x^\perp/x$ which is a $U(4, 2)$ generalized quadrangle with parameters (4, 2). We therefore have

Lemma 5.1 *For an isotropic point of the unitary space V the geometry induced on $U(x)$ is a* $(2,4)$*generalized quadrangle isomorphic to the singular points and totally singular lines in an orthogonal space* $O^-(6,2)$.

Proof: It is well known that the dual to the unitary quadrangle $U(4,2)$ is the orthogonal generalized quadrangle, $O^-(6,2)$. For example, see ([22]). □

We require one more lemma before proceeding to our main result:

Lemma 5.2 *Let l, m be two disjoint lines in a generalized quadrangle (P, L) isomorphic to* $O^-(6,2)$. *Then P can be generated by $l \cup m$ together with* 2 *further points. In particular, P can be generated by* 6 *points.*

Proof: A pair of opposite lines generates a grid and the automorphism group of (P, L) is transitive on such grids. From the orthogonal geometry $O^-(6,2)$ it is clear the subspace generated by the grid and any other point is a (2, 2) generalized quadrangle and that there are three such subspaces. This follows since the stablizer of a grid fixes its orthogonal complement (an elliptic space of dimension two, $O^-(2,2)$), and is transitive on its three points, whence the three hyperplanes containing the grid. Moreover, any one of these must be maximal as follows: Let G denote the set of points of the grid. Suppose l_i, $i = 1, 2, 3$ are three disjoint lines which cover G and $x \in P\backslash G$. Then x is collinear with a unique point on each l_i and hence with precisely three (non-collinear) points of G. Now suppose $x \in P\backslash G$ and set $P_x = \langle G, x\rangle$. We claim that every point $y \in P\backslash P_x$ is collinear with 5 points of P_x. Each point of P_x is collinear with $2 \times 2 = 4$ points of $P\backslash P_x$. On the other hand each of the 12 points of $P\backslash P_x$ are collinear with at most 5 points of P_x since for such a point y the points in $P_x \cap \Delta_1(y)$ are pairwise non-collinear and so a partial ovoid. However, since $15 \times 4 = 12 \times 5$ we must have that for every point $y \in P\backslash P_x$, $P_x \cap \Delta_1(y)$ is an ovoid. Now $\Delta_1(y) \cap G$ is three points and consequently y is collinear with two points of $P_x\backslash G$. Without loss of generality we may assume that x and y are collinear. Now the third point z on the line $\langle x, y\rangle$ belongs to neither P_x or $P_y = \langle G, y\rangle$. But then $\langle P_x, y\rangle$ contains P_x, P_y and $P_z = \langle G, z\rangle$ and consequently all points of P. It now follows that P can be generated by 6 points. □

We can now prove our main result which is part (d) of Theorem A:

Proposition 5.3 *The dual unitary polar space* $DU(6,2)$ *can be generated by* 22 *points.*

Proof: For an isotropic point u let τ_u be the unique transvection with center u and axis $u^\perp$ contained in $G = \{\sigma : V \to V \mid h(\sigma(u), \sigma(v)) = h(u, v), \forall u, v \in V\}$. Also, let $\mathcal{H}$ denote the set of hyperbolic lines in V, that is, the sets $P(A)$ where A is a non-degenerate subspace of V of dimension two. For a subset X of isotropic points we will denote by $\langle X\rangle_{\mathcal{H}}$ the subspace of $(P, \mathcal{H})$ spanned by this set of points.

Let $U = \langle x_1, x_2, y_1, y_2\rangle$, a non-degenerate subspace of V of dimension four. Set $u_1 = \langle x_1\rangle$, $u_2 = \langle y_1\rangle$, $u_3 = \langle \omega^2 x_1 + \omega y_1 + \omega^2 x_2 + \omega y_2\rangle$, $u_4 = \langle x_1 + x_2\rangle$, $u_5 = \langle x_2\rangle$, and let $\tau_i = \tau_{u_i}$. Now note that $\{u_1, u_2, u_3, u_k\}$ is independent for either $k = 4$ or 5 and hence spans U. Also note that $\langle u_1, u_2, u_3\rangle$ is a non-degenerate three subspace and hence for $i \neq j \in \{1, 2, 3\}$ u_i, u_j are non-orthogonal. We now claim that $\langle u_i \mid 1 \leq i \leq 5\rangle_{\mathcal{H}} = P(U)$. Let $Y = \langle u_i \mid 1 \leq i \leq 5\rangle_{\mathcal{H}}$. The group $T = \langle \tau_i \mid 1 \leq i \leq 5\rangle$ leaves Y invariant which can be seen as follows: Let $a \in Y$. If $u_i \perp a$ then $\tau_i(a) = a$. On the other hand, if u_i and a are non-orthogonal then they span a hyperbolic line which contains three isotropic points. Two of these are a, u_i. Also, $\tau_i(a)$ is isotropic and lies in $\langle a, u_i\rangle$ and hence this is the third point. But $P(\langle a, u_i\rangle)$ is the hyperbolic line on a and u_i and as $a, u_i \in Y$ if follows that $P(\langle a, u_i\rangle) \subset Y$. Thus, $\tau_i(a) \in Y$ and T leaves Y invariant as claimed.

We next claim that $T = N_G(U) \cap C_G(U^{\perp}) \cong SU(4, 2)$. This is easily deduced from ([21]): $\langle \tau_1, \tau_2, \tau_3\rangle$ is a group of order 54. Then the group obtained by adjoining τ_4 is an extension of an elementary abelian group of order 27 by the symmetric group S_4 which is a maximal subgroup in SU(4, 2). Consequently, with the addition of τ_5 the entire group is generated. Now since $N_G(U) \cap C_G(U^{\perp})$ is transitive on $P(U)$ it follows that $Y = P(U)$.

Now as in section three of ([13]) if X is a set of isotropic points then the subspace of $\Gamma = (\mathcal{P}, \mathcal{L})$ generated by $U(x)$, $x \in X$ is equal to $\cup_{y \in \langle X\rangle_{\mathcal{H}}} U(y)$. Now set $Z = \langle U(u_i) \mid 1 \leq i \leq 5\rangle_{\Gamma}$. By what we have shown $Z = \cup_{y \in P(U)} U(y)$. However, suppose m is a maximal isotropic subspace. Then $m \cap U \neq 0$ and if y is a point in $m \cap U$ then $m \in U(y) \subset Z$. Since m is arbitary, it follows that $Z = \mathcal{P}$. It now remains to show that we can generate $U(u_i)$, $1 \leq i \leq 5$ with 22 points.

By (5.2) we can generate each $U(u_i)$, $i = 1, 2, 3$ with 6 points. Now consider $U(u_4)$. $u_4^{\perp} \cap \langle u_1, u_2, u_3\rangle = \langle u_1, u_3\rangle$ is a hyperbolic line. Each of $U(\langle u_4, u_1\rangle)$, $U(\langle u_4, u_3\rangle)$ is a line in the generalized quadrangle $U(u_4)$ and these lines are disjoint since u_1 is not perpendicular to u_3. Moreover, these two lines are already contained in the subspace of Γ generated by $U(u_i)$, $i = 1, 2, 3$ since $U(\langle u_4, u_1\rangle) \subset U(u_1)$ and $U(\langle u_4, u_3\rangle) \subset U(u_3)$. Therefore by (5.2) we need two further points to generate $U(u_4)$. In exactly the same fashion, in $U(u_5)$ we have the two disjoint lines, $U(\langle u_5, u_1\rangle)$ and $U(\langle u_5, u_2\rangle)$ which are already contained in $U(u_1)$ and $U(u_2)$, respectively. So, again by (5.2), we need two further points to generate $U(u_5)$. Thus, altogether we can generate $\langle U(u_i) \mid 1 \leq i \leq 5\rangle_{\Gamma}$ by $3 \times 6 + 2 \times 2 = 22$ points. However, by ([29]) $DU(6, 2)$ has a projective embedding in $\mathbb{PG}(21, 2)$ and consequenlty 22 is the minimal possible size for a generating set. This completes the proposition and the proof of Theorem A. □

6. A survey of embedding and generating ranks of GF(2) embeddable geometries

The following table summarizes many other instances in which the generating rank and/or the embedding rank of a GF(2) geometry is known but we make no pretense to stating that it is complete. In particular, we have only included geometries which have embeddings in $\mathbb{PG}(n, 2)$ for some n and consequently have excluded geometries with affine embeddings, e.g. those Fischer spaces which are not cotriangular spaces. We have made an attempt to include all instances of Lie geometries which are known. The last three entries in the table refer to the central involution geometries of the group $U_4(3)$, and the sporadic groups *Suz* and Co_1. ${}^{nh}U_4(3)$ refers to the near-hexagon on which the group $U_4(3)$ acts as automorphism and ${}^{nh}M_{24}$ refers to the near-hexagon on 759 points on which M_{24} acts. The notation $X_{n,k}$ refers to a Lie incidence geometry arising from the group X_n acting on the parabolic subgroup

Geometry Γ	Embedding rank	Generating rank
$A_{n,1}(2) = \mathbb{PG}(n-1, 2)$	n	n
$A_{n-1,k}, 2 \leq k \leq n-2$	$\binom{n}{k}$ [28]	$\binom{n}{k}$ [5], [16], [23]
$A_{n-1;1,n}(2), n \geq 3$	$n^2 - 1$ [27]	$n^2 - 1$ [14]
$B_{n,1}(2), n \geq 2$	$2n + 1$ [26]	$2n + 1$ [14]
$B_{n,2}(2), n \geq 3$	$\binom{2n+1}{2}$ [27]	$\binom{2n+1}{2}$ [14]
$B_{n,n}(2), 2 \leq n \leq 5$	$\frac{(2^n+1)(2^{n-1}+1)}{3}$ [6]	$\frac{(2^n+1)(2^{n-1}+1)}{3}$ [12]
${}^cC_n(2)$	$2n + 1$ [19]	$2n + 1$ [19]
${}^cD_n^{\pm}(2)$	2n [19]	2n [19]
${}^c\Sigma_n$	$n - 1$ [19]	$n - 1$ [19]
$D_{n,1}(2), n \geq 4$	$2n$ [26]	$2n$ [14]
$D_{n,2}(2), n \geq 4$	$\binom{2n}{2}$ [27]	$\binom{2n}{2}$ [14]
$D_{n,k}(2), k = n-1, n, n \geq 5$	2^{n-1} [28]	2^{n-1} [5], [16], [23]
${}^2D_{n,1}(2), n \geq 3$	$2n$ [26]	$2n$ [14]
${}^2D_{n,2}(2), n \geq 4$	$\binom{2n}{2}$ [27]	$\binom{2n}{2}$ [14]
$DU(6, 2)$	22 [29]	22
$E_{6,1}(2)$	27 [5], [16], [23]	27 [5], [16], [23]
$E_{7,1}(2)$	56 [5], [16], [23]	56 [5], [16], [23]
${}^3D_4(2)$	28 [18]	?
$G_2(2)$	14	14
$G_2(2)^d$	14	14
${}^{nh}M_{24}$	23 [7]	?
HJ	28 [18]	28
${}^{nh}U_4(3)$	21 [1], [7], [29]	?
$U_4(3)$	70 [2]	?
Suz	143 [3]	?
Co_1	300 [4]	?

corresponding to removing the k^{th} node. For example, $A_{n-1,k}$ is the Grassmannian of k dimensional vector subspaces of an n dimensional vector space. By ${}^cC_n(2)$ we mean the co-triangular space whose points are the non-zero vectors and hyperbolic lines in a non-degenerate symplectic space over GF(2), ${}^cD_n^{\epsilon}(2)$ is the geometry of nonsingular vectors in a non-degenerate orthogonal space of dimension 2n over GF(2) of type $\epsilon \in \{+, -\}$ and ${}^c\Sigma_n$ is the geometry whose points are the pairs from $\Omega = \{1, 2, \ldots, n\}$ and whose lines are the the triples with incidence given by inclusion. Finally, $A_{n-1;1,n-2}$ refers to the geometry whose points are the full transvections groups $\chi(p, H)$ for a given center p and axis H acting on an n–dimensional vector space, where two points are collinear if they have a common center or a common axis. We have also included the geometries treated in this paper.

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