Journal of Algebraic Combinatorics **13** (2001), 29–40

Quotients of Poincaré Polynomials Evaluated at -1

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Received November 25, 1998; Revised November 22, 1999

Abstract. For a finite reflection group W and parabolic subgroup W_J, we establish that the quotient of Poincaré polynomials $\frac{W(t)}{W_J(t)}$, when evaluated at $t = -1$, counts the number of cosets of W_J in W fixed by the longest element. Our case-by-case proof relies on the work of Stembridge (Stembridge, *Duke Mathematical Journal*, **73** (1994), 469–490) regarding minuscule representations and on the calculations of $\frac{W(-1)}{W_J(-1)}$ of Tan (Tan, *Communications in Algebra*, **22** (1994), 1049–1061).

Keywords: reflection groups, Poincaré polynomials, longest element, minuscule representations

1. Introduction

The $t = -1$ phenomenon, which has been studied by Stembridge [4, 5], is said to occur when one has a finite set X equipped with an involution $\theta : X \to X$ and an integer-valued function $|\ | : X \to \mathbf{Z}$ such that the generating function polynomial

$$\sum_{x \in X} t^{|x|}$$

counts the number of fixed points of θ when evaluated at $t = -1$. Stembridge studied the $t = -1$ phenomenon while working in the context of representations of Lie algebras. His main result [5] shows that certain specialization polynomials when evaluated at -1 count the number of weight vectors fixed by a map called Lusztig's involution, which is intimately connected to the longest element of the corresponding Weyl group. This paper presents a result similar to that of Stembridge in the setting of reflection groups. An understanding of reflection groups and representations of Lie algebras is all we assume. For reference, the reader may refer to Chapter 1 of Humphreys [2] and Chapters 3 and 6 of Humphreys [1].

Let (W, S) be a finite reflection group with simple reflections S, and let Δ denote the set of corresponding simple roots. Let Φ and Φ^+ denote the set of all roots and the set of all positive roots respectively. For any subset $J \subseteq S$, one may form the parabolic subgroup W_J generated by the simple reflections in J. Now each left coset wW_J has a unique element of minimal length. These coset representatives of minimal length are called *distinguished representatives*. Let W^J be the set of distinguished representatives for $W_J \subseteq W$. Let P_J and N_J denote the number of elements in W^J of even and odd length respectively, and let $D_J = P_J - N_J$.

One can now form the Poincaré polynomials

$$W^J(t) = \sum_{w \in W^J} t^{l(w)}$$
$$W_J(t) = \sum_{w \in W_J} t^{l(w)}$$

where $l(w)$ denotes the length of w. It is well-known [2] that $W^J(t)$ is the quotient of Poincaré polynomials corresponding to J and S,

$$W^J(t) = \frac{W_S(t)}{W_J(t)}.$$

Recall from [2] that the *degrees* of a finite reflection group W are the degrees of a set of algebraically independent generators of the ring of W-invariant polynomials. Now let $d_1, d_2, \ldots, d_k$ denote the degrees of W and $d'_1, \ldots, d'_m$ the degrees of W_J. Note $m \leq k$. Factorization of the Poincaré polynomials [2] yields

$$W^J(t) = \frac{\prod_{i=1}^{k}\left(t^{d_i} - 1\right)}{(t-1)^{k-m}\prod_{i=1}^{m}\left(t^{d'_i} - 1\right)}.$$

By definition $W^J(-1) = D_J$. Observe in the unexpanded factorization above that there cannot be more even powers of t in the denominator than in the numerator, for that case would imply $\lim_{t \to -1} W^J(t) = \pm\infty$, an impossibility. If there are more even powers of t in the numerator than in the denominator, then $W^J(-1) = 0$. If there is an equal number of even powers of t in numerator and denominator, the factors which have odd powers of t cancel to 1 and by L'Hopital's rule, $W^J(-1) > 0$. In any case, $W^J(-1)$ is a non-negative integer. Given one has encountered previous examples of the $t = -1$ phenomenon, the natural question to ask is whether $W^J(-1)$ is counting the fixed points of an involution on W^J. The goal of this article will be to prove the theorem below and show that this is in fact the case. The proper involution to consider is $u \mapsto w_0 u w_0^J$ where w_0 and w_0^J are the longest elements in W and W_J respectively.

Theorem 1 *Let (W,S) be a finite reflection group with simple reflections S. Let $J \subseteq S$, and W^J be the distinguished left coset representives for $W_J \subseteq W$. Let w_0 and w_0^J be the longest elements of W and W_J respectively. The map $\Theta : u \mapsto w_0 u w_0^J$ is a well-defined involution from W^J to W^J. Let $W^J(t) = \sum_{w \in W^J} t^{l(w)}$. Then*

$$\begin{aligned} W^J(-1) &= |\{uW_J \mid u \in W \text{ and } w_0 u W_J = u W_J\}| \\ &= |\{u \in W^J \mid \Theta u = u\}|. \end{aligned}$$

Stembridge [4] essentially proved a special case of this theorem, and the work in this article relies on it. However, his related later work [5] on the canonical basis, which is deep and general, does not appear to include Theorem 1 in any obvious way, for in working with the canonical basis and quantum groups, one assumes the crystallographic condition on the root system. At any rate, no knowledge of quantum groups is assumed here.

2. Reduction to the irreducible maximal case

To begin, we first show that this involution is well-defined and that it fixes a distinguished element if and only if the longest element fixes the corresponding coset.

Claim 1 The map $\Theta : u \mapsto w_0 u w_0^J$ is a well-defined map from W^J to W^J.

Proof: As $u \in W^J$, $l(uv) = l(u) + l(v)$ for all $v \in W_J$. See Humphreys [2], p. 19. Consider the coset $w_0 u W_J$. We will show that $w_0 u w_0^J$ is the distinguished representative for this coset. Since w_0 sends the set of positive roots to the set of negative roots, every positive root sent to a negative root by $y \in W$ remains positive when $w_0 y$ is applied to it. Similarly, every positive root that stays positive under y is sent to a negative root by $w_0 y$. Thus $l(w_0 y) = |\Phi^+| - l(y)$ for all $y \in W$. Now let $w_0 u v$ be an arbitrary element of $w_0 u W_J$. Then

$$\begin{aligned} l(w_0 u v) &= |\Phi^+| - l(uv) \\ &= |\Phi^+| - l(u) - l(v) \\ &\geq |\Phi^+| - l(u) - l\big(w_0^J\big) \\ &= |\Phi^+| - l\big(u w_0^J\big) \\ &= l\big(w_0 u w_0^J\big). \end{aligned}$$

□

Corollary 1 *Let $u \in W^J$. Then $\Theta u = u$ if and only if $w_0 u W_J = u W_J$.*

Proof: The direction $\Rightarrow$ is obvious.

If $w_0 u W_J = u W_J$, then by the above claim, $w_0 u w_0^J$ and u are both the distinguished representative in $u W_J$. Thus $w_0 u w_0^J = u$. □

Let (W, S) be an irreducible finite reflection group with simple system S. Let $J \subseteq S$ be a maximal subset, that is, $|J| = |S| - 1$. In this situation Tan [6] explicitly describes the elements of the set W^J and as an application computes $D_J = W^J(-1)$. Tan then uses D_J to compute the differences in signs appearing in the Laplace expansion for the determinant. However, Tan does not observe the connection between $W^J(-1)$ and the fixed points of Θ. This connection will be made later in this article using case-by-case techniques. Right now, let us show it suffices to reduce to the case that (W, S) is irreducible with $J \subseteq S$ maximal by means of some multiplicativity lemmas.

Suppose (W, S) is a finite reflection group. Then W decomposes as the internal direct sum $W = W_{S_1} \dot{\times} W_{S_2} \dot{\times} \cdots \dot{\times} W_{S_k}$ where each (W_{S_i}, S_i) is an irreducible finite reflection group. Let $w_0^{S_i}$ denote the longest element of the parabolic subgroup W_{S_i}, and form the polynomials $W(t) = \sum_{w \in W} t^{l(w)}$ and $W_{S_i}(t) = \sum_{w \in W_{S_i}} t^{l(w)}$.

Adopting the notation found in Tan [6], we let W^{JI} denote the set of distinguished coset representatives for W_I in W_J, whenever $I \subseteq J \subseteq S$. Let P_{JI} and N_{JI} denote the number of elements in W^{JI} of even and odd length respectively and set $D_{JI} = P_{JI} - N_{JI}$. In particular, $W^{SJ} = W^J$. For $J \subseteq S$, consider the sets $J \cap S_i \subseteq S_i$, and form $W_{J\cap S_i}$ and $W^{S_i, J\cap S_i}$. We have the following claim, whose proof is left to the reader.

Claim 2 Using the notation just introduced, the following hold:

(a) $W(t) = \prod_{i=1}^k W_{S_i}(t)$.
(b) $\sum_{w\in W^J} t^{l(w)} = \prod_{i=1}^k (\sum_{w \in W^{S_i, J\cap S_i}} t^{l(w)})$.
(c) The number of cosets of W_J in W fixed by w_0 equals

$$\prod_{i=1}^k \left(\textit{number of cosets of } W_{J\cap S_i} \textit{ in } W_{S_i} \textit{ fixed by } w_0^{S_i}\right).$$

The following are the key multiplicative lemmas that will allow reduction to the case $J \subseteq S$ with J maximal.

Lemma 1 (Tan [6]) *Suppose $I \subseteq J \subseteq S$. Then $W^{SI} = W^{SJ} W^{JI}$ and $D_{SI} = D_{SJ} D_{JI}$.*

Let w_0, w_0^J, w_0^I denote the longest elements of W, W_J, and W_I respectively. Let

$$\Omega_{SI} = \left|\{uW_I \mid u \in W \text{ and } w_0 u W_I = uW_I\}\right|$$
$$\Omega_{SJ} = \left|\{uW_J \mid u \in W \text{ and } w_0 u W_J = uW_J\}\right|$$
$$\Omega_{JI} = \left|\left\{uW_I \mid u \in W_J \text{ and } w_0^J u W_I = uW_I\right\}\right|.$$

Lemma 2 *Suppose $I \subseteq J \subseteq S$. Then $\Omega_{SI} = \Omega_{SJ} \cdot \Omega_{JI}$.*

Proof: Let us show that the map

$$\phi : \Omega_{SJ} \times \Omega_{JI} \to \Omega_{SI}$$

given by $(xW_J, yW_I) \mapsto xyW_I$ is a bijection, where x and y are chosen in W^{SJ} and W^{JI} respectively. Now $w_0 x W_J = xW_J$, which implies $w_0 x w_0^J = x$ by Corollary 1. Similarly $w_0^J y w_0^I = y$. Then $w_0 xy w_0^I = w_0 x w_0^J w_0^J y w_0^I = xy$, giving $w_0 xy W_I = xyW_I$. Thus ϕ maps to the right place.

Suppose $\phi(xW_J, yW_I) = \phi(x'W_J, y'W_I)$ where $x, x' \in W^{SJ}$ and $y, y' \in W^{JI}$. Then $xyW_I = x'y'W_I$. From Lemma 1, xy and $x'y'$ are both the distinguished representative in xyW_I for $W_I \subseteq W$. Then $xy = x'y'$. Then $xW_J = x'W_J$, giving $x = x'$. And $y = y'$ follows. Thus ϕ must be one-to-one.

Suppose $w_0 u W_I = uW_I$, where $u \in W^{SI}$. Write $u = ab$ with $a \in W^{SJ}$ and $b \in W^{JI}$ using Lemma 1. Since $w_0 ab W_I = abW_I$, then $w_0 a W_J = aW_J$. Both $w_0 a w_0^J = a$ and $w_0 ab w_0^I = ab$ hold by Corollary 1. Now use $a^{-1} \cdot ab = b$ to see that $w_0^J b w_0^I = b$. Thus $w_0^J b W_I = bW_I$. It follows that $\phi(aW_J, bW_I) = uW_I$, and hence that ϕ is onto. □

Proposition 1 *Suppose $W^J(-1) = \Omega_{SJ}$ for every finite irreducible reflection group (W, S) with $J \subseteq S$ maximal. Then $W^J(-1) = \Omega_{SJ}$ for any finite reflection group (W, S) and any subset $J \subseteq S$.*

Proof: Consider first an arbitrary finite reflection group (W, S) with $J \subseteq S$ maximal. Then W decomposes as the internal direct sum $W = W_{S_1} \dot{\times} \cdots \dot{\times} W_{S_k}$ where each (W_{S_i}, S_i) is irreducible. Now $J \cap S_i = S_i$ for all i except one, say $i = m$. Let $w_0^{S_i}$ denote the longest element of W_{S_i}. By Claim 2,

$$\begin{aligned}
W^J(-1) &= \prod_{i=1}^{k} \left(\sum_{w \in W^{S_i, J\cap S_i}} t^{l(w)} \right) \Bigg|_{t=-1} \\
&= \sum_{w \in W^{S_m, J\cap S_m}} t^{l(w)} \Bigg|_{t=-1} \\
&= \text{number of cosets of } W_{J\cap S_m} \text{ in } W_{S_m} \text{ fixed by } w_0^{S_m} \\
&= \prod_{i=1}^{k} \left(\text{number of cosets of } W_{J\cap S_i} \text{ in } W_{S_i} \text{ fixed by } w_0^{S_i}\right) \\
&= \text{number of cosets of } W_J \text{ in } W \text{ fixed by } w_0 \\
&= \Omega_{SJ}.
\end{aligned}$$

Thus $W^J(-1) = \Omega_{SJ}$ holds for any finite reflection group (W, S) with $J \subseteq S$ maximal.

Finally, let (W, S) be any finite reflection group and $J \subseteq S$ and subset. There is a sequence of subsets

$$J = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_t = S$$

with J_i maximal in J_{i+1}. Then

$$\begin{aligned}
W^J(-1) &= D_{SJ} \\
&= D_{J_t J_{t-1}} \cdot D_{J_{t-1} J_{t-2}} \cdots D_{J_1 J_0} \\
&= \Omega_{J_t J_{t-1}} \cdot \Omega_{J_{t-1} J_{t-2}} \cdots \Omega_{J_1 J_0} \\
&= \Omega_{J_t J_0} \\
&= \Omega_{SJ}.
\end{aligned}$$

□

3. Minuscule representations

This paragraph summarizes some of the notation and well-known facts from the representation theory of Lie algebras. Let L denote a simple Lie algebra over the complex numbers **C**. Then its corresponding Weyl group W is irreducible. This Weyl group may be viewed as reflection group acting on a real Euclidean vector space with bilinear form $(,)$. Fix a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ for the Weyl group and corresponding fundamental weights $\{\omega_1, \ldots, \omega_r\}$, such that $(\alpha_i, \omega_j) = 0$ if $i \neq j$ and $(\frac{2\alpha_i}{(\alpha_i,\alpha_i)}, \omega_i) = 1$. Throughout the rest of the article, let us assume the numbering of the simple roots as in Humphreys [1]. For any root $\beta \in \Phi$, let $\check{\beta}$ denote $\frac{2\beta}{(\beta,\beta)}$. Let λ be a dominant weight, and let $W_J = W_\lambda$ be the parabolic subgroup of W that fixes λ. Let $V(\lambda)$ be the irreducible representation with highest weight λ, $\Pi(\lambda)$ the set of weights appearing in $V(\lambda)$, and $V(\lambda)_\mu$ the weight space of weight μ in $V(\lambda)$. The real Euclidean vector space may be thought of as the real span of the fundamental weights. Let $\check{\rho}$ denote the expression $\sum_{i=1}^r \frac{2\omega_i}{(\alpha_i,\alpha_i)}$. At a later point in this section, we shall need the following length lemma:

Lemma 3 (*Length Lemma***)** (Humphreys [2], p. 12) *Let $\alpha \in \Delta$ and $w \in W$. Then*
(a) $w(\alpha) > 0$ *iff* $l(ws_\alpha) = l(w) + 1$.
(b) $w(\alpha) < 0$ *iff* $l(ws_\alpha) = l(w) - 1$.
(c) $w^{-1}(\alpha) > 0$ *iff* $l(s_\alpha w) = l(w) + 1$.
(d) $w^{-1}(\alpha) < 0$ *iff* $l(s_\alpha w) = l(w) - 1$.

Recall the polynomial $W^J(t) = \sum_{w \in W^J} t^{l(w)}$ and form the expression $f_\lambda(t) = \sum_{\mu \in \Pi(\lambda)} \dim(V(\lambda)_\mu) t^{(\lambda - w_0\mu, \check{\rho})}$, which Stembridge introduced in [4]. Observe that if μ appears in $\Pi(\lambda)$, then $w_0\mu \in \Pi(\lambda)$ and $\lambda - w_0\mu$ is a $\mathbf{Z}_{\geq 0}$-linear combination of simple roots. It follows that $(\lambda - w_0\mu, \check{\rho}) \in \mathbf{Z}_{\geq 0}$ since $(\alpha_i, \check{\rho}) = 1$ for each simple root α_i. Thus $f_\lambda(t)$ is a polynomial. It will turn out that for certain λ, namely the minuscule weights, we have $f_\lambda(t) = W^J(t)$ for some subsets J.

A dominant weight is called *minuscule* if $\Pi(\lambda)$, the set of weights appearing in $V(\lambda)$, forms a single Weyl orbit. Equivalent formulations, which can be found in Humphreys [1], include a) $(\lambda, \check{\alpha}) = 0, 1,$or -1 for all roots α; and b) if whenever μ is a dominant weight with $\lambda - \mu \in \mathbf{Z}_{\geq 0}$-linear combination of simple roots, $\lambda = \mu$. The representation corresponding to a minuscule weight is also called minuscule. Certainly the trivial representation, corresponding to $\lambda = 0$, is minuscule. The remaining minuscule weights have all been classified [1] for simple Lie algebras L as follows:

$A_r : \omega_1, \ldots, \omega_r$
$B_r : \omega_r$
$C_r : \omega_1$
$D_r : \omega_1, \omega_{r-1}, \omega_r$
$E_6 : \omega_1, \omega_6$
$E_7 : \omega_7$.

The algebras E_8, G_2, and F_4 have no nontrivial minuscule weights. From now on, minuscule weights will be assumed nonzero unless stated otherwise. Notice that since $\Pi(\lambda)$ forms a single Weyl orbit, the dimension of each weight space of $V(\lambda)$ is 1. Since the minuscule weights in the table are all fundamental weights, $W_J = W_\lambda$ will be such that $J \subseteq S$ is maximal.

Let us return for the moment to (W, S), a not necessarily irreducible finite reflection group.

Lemma 4 *Let $I \subseteq S$. For the parabolic subgroup $W_I \subseteq W$, form W^I, the set of distinguished representatives. From all elements in W^I, choose u so $l(u)$ is maximal. Let w_0 and w_0^I be the longest element of W and W_I respectively. Then $w_0 = uw_0^I$ and $l(u) = l(w_0) - l(w_0^I)$.*

Proof: First observe that $l(w_0) \geq l(uw_0^I) = l(u) + l(w_0^I)$. Write $w_0 = xy$ for $x \in W^I$ and $y \in W_I$. Then $l(w_0) = l(x) + l(y) \leq l(u) + l(w_0^I)$. Thus $l(w_0) = l(u) + l(w_0^I) = l(uw_0^I)$. Hence $w_0 = uw_0^I$ and $l(u) = l(w_0) - l(w_0^I)$. □

Lemma 5 *For I and w_0^I as in the previous lemma, set $m = l(w_0) - l(w_0^I)$ and choose i so that $0 \leq i \leq m$. Let $T = \{a \in W^I \mid l(a) = i\}$,$U = \{b \in W^I \mid l(b) = m - i\}$. Then $|T| = |U|$.*

Proof: Define a map $\phi : T \to U$ by $a \mapsto w_0 a w_0^I$. This is well-defined since $l(w_0 a w_0^I) = l(w_0) - l(a w_0^I) = l(w_0) - l(w_0^I) - l(a) = m - i$, and from the well-definedness of Θ we know that $w_0 a w_0^I$ is indeed in W^I. Moreover ϕ^2 =identity. Thus $|T| = |U|$. □

Lemma 6 *Let λ be minuscule, and suppose $w \in W$. Then $(w\lambda, \check{\alpha}) = 0,1,$ or -1 for any root $\alpha \in \Phi$.*

Proof: From the equivalent formulations of minuscule found in [1] we have that for any $\beta \in \Phi$, that $(\lambda, \check{\beta}) = 0$, 1, or -1. Set $\beta = w^{-1}\alpha \in \Phi$. Then

$$\begin{aligned}(w\lambda, \check{\alpha}) &= (\lambda, w^{-1}(\check{\alpha})) \\ &= \left(\lambda, \frac{2w^{-1}\alpha}{(\alpha, \alpha)}\right) \\ &= \left(\lambda, \frac{2w^{-1}\alpha}{(w^{-1}\alpha, w^{-1}\alpha)}\right) \\ &= (\lambda, \check{\beta}) = 0,\ 1,\ \text{or}\ -1.\end{aligned}$$ □

Lemma 7 *Let $\check{\rho} = \frac{1}{2}\sum_{\alpha\in\Phi^+} \frac{2\alpha}{(\alpha,\alpha)} = \sum_{i=1}^{r} \frac{2\omega_i}{(\alpha,\alpha)}$. Let λ be minuscule, and assume $w \in W^J$ where $W_J = W_\lambda = \{\sigma \in W \mid \sigma\lambda = \lambda\}$. Then $(w\lambda, \check{\rho}) = (\lambda, \check{\rho}) - l(w)$.*

Proof: Induct on $l(w)$. If $l(w) = 0$, then w is the identity and the result is obvious.

Suppose $l(w) = n > 0$. Take a reduced expression of simple reflections for w, $w = s_{i_1}s_{i_2}\cdots s_{i_n}$, where s_{i_j} denotes the simple reflection corresponding to the simple root α_{i_j}. Then $s_{i_j}\cdots s_{i_n} \in W^J$ and $l(s_{i_j}\cdots s_{i_n}) = n - j + 1$ for all $1 \leq j \leq n$. Now for any simple root α, $(\check{\rho}, \alpha) = (\sum_{i=1}^{r} \frac{2\omega_i}{(\alpha_i,\alpha_i)}, \alpha) = 1$ and $s_\alpha\ \check{\rho} = \check{\rho} - \check{\alpha}$. Then

$$\begin{aligned}(w\lambda, \check{\rho}) &= \left(s_{i_1}\cdots s_{i_n}\lambda, \check{\rho}\right) \\ &= \left(s_{i_2}\cdots s_{i_n}\lambda, s_{i_1}\check{\rho}\right)\end{aligned}$$

$$= \left(s_{i_2} \cdots s_{i_n}\lambda,\, \check{\rho} - \check{\alpha}_{i_1}\right)$$
$$= \left(s_{i_2} \cdots s_{i_n}\lambda,\, \check{\rho}\right) - \left(s_{i_2} \cdots s_{i_n}\lambda,\, \check{\alpha}_{i_1}\right).$$

But by the length lemma, $(s_{i_2} \cdots s_{i_n})^{-1}\check{\alpha}_{i_1} > 0$. Then $(\lambda, (s_{i_2} \cdots s_{i_n})^{-1}\check{\alpha}_{i_1}) \geq 0$. But if $(\lambda, (s_{i_2} \cdots s_{i_n})^{-1}\check{\alpha}_{i_1}) = 0$, then $s_{i_1} \cdots s_{i_n}\lambda = s_{i_2} \cdots s_{i_n}\lambda$. This implies $s_{i_1} \cdots s_{i_n} W_J = s_{i_2} \cdots s_{i_n} W_J$, which yields $s_{i_1} \cdots s_{i_n} = s_{i_2} \cdots s_{i_n}$, a contradiction. Hence $(\lambda, (s_{i_2} \cdots s_{i_n})^{-1}\check{\alpha}_{i_1}) > 0$. Then $(s_{i_2} \cdots s_{i_n}\lambda, \check{\alpha}_{i_1}) = 1$ by the previous lemma. Finally, $(w\lambda, \check{\rho}) = (s_{i_2} \cdots s_{i_n}, \check{\rho}) - 1 = (\lambda, \check{\rho}) - (n-1) - 1 = (\lambda, \check{\rho}) - n$ as desired. □

Lemma 8 *Assume the hypotheses of the previous lemma, and let w_0 and w_0^J be the longest elements of W and W_J respectively. Then $2(\lambda, \check{\rho}) = l(w_0) - l(w_0^J)$.*

Proof: Let $u \in W^J$ be such that $l(u)$ is maximal. From Lemma 4, $uw_0^J = w_0$. On the one hand, $(u\lambda, \check{\rho}) = (\lambda, \check{\rho}) - l(u) = (\lambda, \check{\rho}) - l(w_0) + l(w_0^J)$. On the other hand, $(u\lambda, \check{\rho}) = (w_0\lambda, \check{\rho}) = -(\lambda, \rho)$. Thus $-(\lambda, \check{\rho}) = (\lambda, \check{\rho}) - l(w_0) + l(w_0^J)$ and the result follows. □

Proposition 2 *Let λ be minuscule and assume $W_J = W_\lambda$ is the parabolic subgroup fixing λ. Then*

$$f_\lambda(t) = W^J(t).$$

Proof: Since the dimension of each weight space is 1,

$$\begin{aligned} f_\lambda(t) &= \sum_{\mu \in \Pi(\lambda)} t^{(\lambda - w_0\mu,\, \check{\rho})} \\ &= \sum_{\mu \in \Pi(\lambda)} t^{(\lambda,\, \check{\rho}) - (\mu, w_0\check{\rho})} \\ &= \sum_{\mu \in \Pi(\lambda)} t^{(\lambda,\, \check{\rho}) - (\mu, -\check{\rho})} \\ &= \sum_{\mu \in \Pi(\lambda)} t^{(\lambda + \mu,\, \check{\rho})} \\ &= \sum_{w \in W^J} t^{(\lambda + w\lambda,\, \check{\rho})} \\ &= \sum_{w \in W^J} t^{(\lambda,\, \check{\rho}) + (\lambda,\, \check{\rho}) - l(w)} \\ &= \sum_{w \in W^J} t^{l(w_0) - l(w_0^J) - l(w)} \end{aligned}$$

where w_0 and w_0^J are the longest elements of W and W_J respectively. As a runs over W^J, $w_0 a w_0^J$ runs over W^J. Then

$$\begin{aligned} f_\lambda(t) &= \sum_{\substack{w_0 a w_0^J \\ a \in W^J}} t^{l(w_0)-l(w_0^J)-l(w_0 a w_0^J)} \\ &= \sum_{a \in W^J} t^{l(w_0)-l(w_0^J)-(l(w_0)-l(a)-l(w_0^J))} \\ &= \sum_{a \in W^J} t^{l(a)} \\ &= W^J(t). \end{aligned}$$ □

If λ is minuscule, the representations with highest weights $m\lambda$ where m is a nonnegative integer have a very nice basis labeling due to a result of Seshadri. Both Stembridge [4] and Proctor [3] study the combinatorial properties of this labeling. Let λ be minuscule and let $W_J = W_\lambda$ be the parabolic subgroup fixing it. Seshadri's monomial result states that a weight basis for $V(m\lambda)$ is indexed by weakly increasing sequences of length m in $W/W_J : \tau_1 \le \tau_2 \le \cdots \tau_m$. The weight for the vector indexed by such a sequence is simply $\tau_1\lambda + \tau_2\lambda + \cdots + \tau_m\lambda$. The main result of Stembridge [4] involves a natural involution of sequences of length m given by

$$\tau_1 \le \tau_2 \le \cdots \le \tau_m \mapsto w_0\tau_m \le \cdots \le w_0\tau_1.$$

Stembridge proves that $f_{m\lambda}(-1)$ is the number of weakly increasing sequences of length m in W/W_J fixed by this involution. Because of the above proposition, Stembridge's result can be restated in the following way in the particular case where $m = 1$.

Claim 3 Assume λ is minuscule, and let $W_J = W_\lambda$, the parabolic subgroup fixing λ. Then

$$\begin{aligned} W^J(-1) &= \text{number of left cosets } \tau \in W/W_J \text{ such that } w_0\tau = \tau \\ &= \left|\left\{ w \in W^J \;\middle|\; w_0 w w_0^J = w \right\}\right| \end{aligned}$$

where w_0 and w_0^J are the longest elements of W and W_J respectively.

4. Verification of the $t = -1$ phenomenon

Since the problem has been reduced to (W, S) irreducible and $J \subseteq S$ maximal, the result of Stembridge in Claim 3, which is a special case of Theorem 1, and some calculations of Tan [6] now suffice to prove the $t = -1$ phenomenon.

Proposition 3 *Let (W,S) be a finite irreducible reflection group with simple reflections S. Assume $J \subseteq S$ is maximal. Let W/W_J denote the left cosets for W_J in W, and set*

Table 1. Selected values of $W^J(-1)$ from Tan.

Type	$S - J$	$W^J(-1)$
B_r; D_r, r even; E_7; E_8; F_4; H_3; H_4; $I_2(m)$, m even	Any simple root	0
$I_2(m)$, m odd	Any simple root	1
D_r, r odd	s_k, $k > 1$	0
D_r, r odd	s_1	2
E_6	s_k with $k \neq 1, 6$	0

$\Omega_{SJ} = |\{\tau \in W/W_J \mid w_0\tau = \tau\}|$. *Then*

$$W^J(-1) = \Omega_{SJ}.$$

Proof: Tan [6] has already computed $W^J(-1)$, and we display some of the results in Table 1. All that is needed is to compute Ω_{SJ} and compare results.

Let s_i be the unique simple reflection in $S - J$, and α_i the corresponding simple root. Let r be the rank of (W, S), and for $j = 1, 2, \dots, r$, let H_j denote the hyperplane $\perp \alpha_j$. In other words, $H_j = \{\mu \in V \mid (\mu, \alpha_j) = 0\}$, where V is the real Euclidean space upon which W acts. The $r-1$ hyperplanes H_j where $j \neq i$ intersect in a line. As a result, we may find $\lambda \in V$ such that $(\lambda, \alpha_i) > 0$ and $(\lambda, \alpha_j) = 0$ for all $j \neq i$. For example $\lambda = \omega_i$ will work when the reflection group is a Weyl group. Evidently $W_J = \{w \in W \mid w\lambda = \lambda\}$. Then W/W_J is in one-to-one correspondence with the W-orbit of λ. Observe $\Omega_{SJ} = |\{\mu \in W\lambda \mid w_0\mu = \mu\}|$.

A_r: Every W_J where $J \subseteq S$ maximal is the fixer for some ω_i, a fundamental weight. In the A_r case, each fundamental weight is minuscule, and by Claim 3, $W^J(-1) = \Omega_{SJ}$.

B_r; D_r, r even; E_7; E_8; F_4; H_3; H_4; $I_2(m)$, m even: In these cases, (0) is the only vector which can possibly be fixed by $w_0 = -1$. However, $W\lambda$ does not contain (0) since λ has nonzero length. Hence $\Omega_{SJ} = 0$ in all these cases, matching the result of Tan for $W^J(-1)$.

D_r, r odd: Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$ be an orthonormal basis with respect to (,). Let $\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \dots, \alpha_{r-1} = \varepsilon_{r-1} - \varepsilon_r, \alpha_r = \varepsilon_{r-1} + \varepsilon_r\}$ be the simple roots. The group W may be thought of as acting on the ε's by signed permutations with an even number of sign changes. The longest element of W sends the ε_i to $-\varepsilon_i$ for $i = 1, 2, \dots, r-1$ and fixes ε_r. Each of the fundamental weights ω_i may be expressed in terms of the ε's as follows:

$$\begin{aligned}
\omega_1 &= \varepsilon_1 \\
\omega_2 &= \varepsilon_1 + \varepsilon_2 \\
&\vdots \\
\omega_{r-2} &= \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{r-2} \\
\omega_{r-1} &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{r-2} + \varepsilon_{r-1} - \varepsilon_r) \\
\omega_r &= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{r-2} + \varepsilon_{r-1} + \varepsilon_r).
\end{aligned}$$

Table 2. The E_6 case.

λ	$S-J$	$W_\lambda = W_J$	$l(w_0^J)$	$l(w_0) - l(w_0^J)$
ω_2	s_2	A_5	15	21
ω_3	s_3	$A_1 \times A_4$	11	25
ω_4	s_4	$A_2 \times A_2 \times A_1$	7	29
ω_5	s_5	$A_4 \times A_1$	11	25

Now for a fundamental weight ω_k, suppose $\mu \in W\omega_k$ is fixed by the longest element. Write $\mu = c_1\varepsilon_1 + \cdots + c_r\varepsilon_r$. Then $-c_1\varepsilon_1 - \cdots - c_{r-1}\varepsilon_{r-1} + c_r\varepsilon_r = w_0\mu = \mu = c_1\varepsilon_1 + \cdots + c_r\varepsilon_r$. Consequently $c_1 = c_2 = \cdots c_{r-1} = 0$, and $\mu = c\varepsilon_r$ for some scalar c. Hence ω_k can only be ω_1. There are precisely two elements in $W\omega_1$ fixed by w_0 namely ε_r and $-\varepsilon_r$. For $J \subseteq S$ maximal, W_J is the stabilizer for one of the fundamental weights ω_k. If $k > 1$, $\Omega_{SJ} = 0$. If $k = 1$, $\Omega_{SJ} = 2$, matching the results of Tan for $W^J(-1)$.

E_6: In the case where W_J is the stabilizer of ω_1 or ω_6, $W^J(-1) = \Omega_{SJ}$ follows from Claim 3 since both of these fundamental weights are minuscule. Consider W^J and let w_0 and w_0^J be the longest element of W and W_J respectively. Using Claim 1, $\Omega_{SJ} = |\{u \in W^J \mid w_0uw_0^J = u\}|$. Now if $w_0uw_0^J = u$, $l(w_0uw_0^J) = l(u)$. That is, $l(w_0) - l(u) - l(w_0^J) = l(u)$, or $l(u) = \frac{l(w_0) - l(w_0^J)}{2}$. In the E_6 case, $l(w_0) = \frac{|\Phi|}{2} = 36$. Consider Table 2.

Since $l(w_0) - l(w_0^J)$ is odd in these remaining cases, there is no $u \in W^J$ with $w_0uw_0^J = u$. Hence $\Omega_{SJ} = 0$ in these cases, matching Tan's computed value for $W^J(-1)$.

$I_2(m)$, m odd: If $J = s_1$, $W^J = \{1, s_2, s_1s_2, s_2s_1s_2, \ldots, (s_1s_2)^{\frac{m-1}{2}}\}$. The lengths of the m distinguished representatives are $0, 1, 2, \ldots, m-1$. Thus $W^J(-1) = 1$. Since the map $\Theta : W^J \to W^J$ sending $u \mapsto w_0uw_0^J$ sends an element of length i to an element of length $m - 1 - i$ and since m is odd, the distinguished representative of length $\frac{m-1}{2}$ will be fixed and will be the unique one fixed. As a consequence, $\Omega_{SJ} = 1 = W^J(-1)$. Similarly for $J = s_2$. □

This proposition combined with the previous work in this article, which reduces to the case (W, S) finite irreducible and $J \subseteq S$ maximal, establishes Theorem 1. The question remains at least to this writer whether a case-free proof of it can be found.

Acknowledgments

The author would like to acknowledge Professor Georgia Benkart at the University of Wisconsin–Madison for her encouragement.

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