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Completions of Goldschmidt Amalgams of Type G_4 in Dimension 3

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Abstract. The subgroups of $\mathrm{GL}_3(k)$ which are completions of the Goldschmidt G_4-amalgam are determined. We also draw attention to five related graphs which are remarkable in that they have large girth and few vertices.

Keywords: finite groups, amalgams, completions, graphs

1. Introduction

In [4] Goldschmidt determined all amalgams of finite groups $\mathcal{A}(P_1, P_2, B)$ which satisfy

(i) $P_1 \cap P_2 = B$;
(ii) $[P_i : B] = 3$ for $i = 1, 2$; and
(iii) no non-trivial subgroup of B is normal in both P_1 and P_2.

This remarkable paper marked the birth of the so-called amalgam method. The types of amalgams he found are indexed by a collection of perfect amalgams, amalgams with perfect universal completion [4]. These perfect amalgams fall into five isomorphism types, the most interesting ones being types G_3, G_4 and G_5. Among the Goldschmidt amalgams these have the most complex structure and also have P_1 and P_2 both 2-constrained. In [6] the authors addressed the problem of which classical groups in dimension 3 are quotients of the Goldschmidt G_3-amalgam, and in [5, 7] more exotic quotients of this amalgam were determined. In this note we isolate the completions of the Goldschmidt G_4-amalgam which can be found in linear groups of dimension 3. We recall from [4] that $\mathcal{A}(P_1, P_2, B)$ is a G_4-amalgam provided that

$$P_1 \cong 4^2 : \mathrm{Sym}(3),$$
$$P_2 \sim 4\mathrm{Sym}(4) \sim (4 * \mathrm{Q}_8)^{\cdot}\mathrm{Sym}(3)$$

and $B = P_1 \cap P_2$ has order 2^5. We let G^* be the universal completion of this amalgam. The objective then is to study the representation theory of G^* in dimension 3 or, equivalently, to determine which quotients of G^* can be embedded in the linear groups $\mathrm{GL}_3(k)$ where k is a field. We call the non-trivial quotients of G^* **completions** of the amalgam. The G_4-amalgam is best known through its connection with the generalized hexagon on 126 points in which case G^* maps into $\mathrm{G}_2(2)$ and the images of P_1 and P_2 are, respectively, point and line stabilizers and the image of B stabilizes an incident point line pair. Of course in this case the completion of the amalgam is $\mathrm{G}_2(2)'$ which, by chance, is isomorphic to $\mathrm{SU}_3(3)$. We shall prove

Theorem 1.1 *Suppose that G is a completion of the Goldschmidt G_4-amalgam and assume that k is a field of characteristic p. If G is isomorphic to a subgroup of* $\mathrm{GL}_3(k)$*, then p is odd, $G \cong \mathrm{SL}_3(p)$ when $p \equiv 1 \bmod 4$, and $G \cong \mathrm{SU}_3(p)$ when $p \equiv 3 \bmod 4$.*

Notice that the Goldschmidt G_5-amalgam has no completions in linear groups of dimension 3. This follows because the subgroup corresponding to B has $*$ a subgroup which is extraspecial of order 32 and as such has no matrix representations of dimension less than 4 in non-even characteristic.

We recall that given an amalgam $\mathcal{A}(P_1, P_2, B)$ we may form a graph which has vertices the left cosets of P_1 and P_2 in G two of which form an edge if and only if they intersect non-trivially. This graph is called the **coset graph** of $\mathcal{A}(P_1, P_2, B)$. We finish this introduction with a remark about the coset graphs associated with two of the amalgams appearing in Theorem 1.1.

Remark 1.2 The coset graph of the G_4-type amalgam in $\mathrm{SU}_3(3)$ is the generalized hexagon of girth 12 and the coset graph of the G_4-amalgam in $\mathrm{SL}_3(5)$ is a cubic graph of girth 20 having 7750 vertices. These graphs are respectively the smallest cubic graph of girth 12 and the smallest known cubic graph of girth 20 (at the time of writing). See [3] for more graphs of large girth.

In a similar vein we also mention

Remark 1.3 The coset graph of the G_5-type amalgam in Mat_{12} has girth 16 and the coset graph of the G_5-amalgam in $\mathrm{G}_2(3)$ is a cubic graph of girth 24 having 44226 vertices. These graphs are the smallest known cubic graphs with the given girth (at the time of writing).

We also note that after excision, [1], the $\mathrm{G}_2(3)$ graph gives the smallest known cubic graph of girth 23 with $44226 - 126 = 44100$ vertices.

2. Proof of Theorem 1.1

Let G be a completion of the G_4-amalgam, and assume that $G \leq \mathrm{GL}(V)$ where V is a 3-dimensional vector space defined over an algebraically closed field k of characteristic p. We identify P_1, P_2 and B with their images in G. We also note that, as G^* is perfect, we may assume that $G \leq \mathrm{SL}(V)$. If $p = 2$, the Sylow 2-subgroups of $\mathrm{SL}(V)$ are nilpotent of class 2, whereas, B has nilpotence class 3. Hence p is odd. We first focus on P_1. Define $Q_1 = O_2(P_1)$ and $Z_1 = \Omega_1(Q_1)$. Then Z_1 is elementary abelian of order 2^2. So, as P_1 acts transitively on the non-identity elements of Z_1, V is decomposed into the three eigenspaces of Z_1, V_1, V_2 and V_3. For $j = 1, 2, 3$ select $v_j \in V_j \backslash \{O\}$. Then $\{v_1, v_2, v_3\}$ is a basis of V. With respect to this basis we have that the three non-trivial elements of Z_1 are

$$z_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$z_3 = z_2 z_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since P_1 normalizes Z_1, it operates monomially with respect to the basis $\{v_1, v_2, v_3\}$. Now the monomial group M is isomorphic to $(k^*)^3 : \mathrm{Sym}(3)$ and the subgroup isomorphic to Q_1 is the unique normal homocyclic group of order 16 in $M \cap \mathrm{SL}(V)$. Moreover, the normalizer in $M \cap \mathrm{SL}(V)$ of a cyclic group of order 3 which permutes the basis transitively is isomorphic to Sym(3) or $3 \times \mathrm{Sym}(3)$, and thus P_1 is determined uniquely up to conjugacy in $\mathrm{GL}(V)$. Therefore, if we let i denote a square root of -1 in k, we can assume that P_1 is generated by the matrices

$$q_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$$

and

$$q_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{pmatrix}$$

which square to z_1, z_2, z_3 respectively and generate Q_1 when taken together with the monomial matrices

$$p_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$p_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

We select $B = \langle q_1, q_2, p_1\rangle$ and then $Z = \Omega_1(Z(B)) = \langle z_1\rangle$. Since P_2 centralizes Z, P_2 preserves the 2-space $\langle v_1, v_2\rangle$ which is inverted by z_1 and the 1-space V_3 which is centralized by z_1. Thus the matrices representing P_1 have shape

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$$

where $a, b, c, d, e, \in k$ and, of course, they have determinant 1. Now we locate a subgroup $W_2 \cong Q_8$, the quaternion group of order 8, which will be the subgroup $[P_2, O_2(P_2)]$. Since P_2 preserves the decomposition $\langle v_1, v_2\rangle \oplus V_3$ and W_2 is in the derived subgroup of P_2, we see that W_2 must centralize V_3 and have determinant 1 on $\langle v_1, v_2\rangle$. Therefore, $W_2 = \langle q_1, p_1\rangle$ is uniquely determined in B. We now proceed to find a further element α of P_2 which is not in B. To do this we consider all elements of $\mathrm{SL}(V)$ which conjugate q_1 to p_1 and have order 3. (We know then that B and the additional element will generate a group which together with P_1 will be a G_4-amalgam.) So as we are seeking elements of P_2, we know that it must have 'P_2 shape'. Therefore, we have an additional element

$$\alpha = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}.$$

We first ensure that

$$q_1\alpha = \alpha p_1$$

and this results in the conditions

$$ia = -b$$
$$ic = d.$$

So α must have the form

$$\alpha = \begin{pmatrix} a & -ia & 0 \\ c & ic & 0 \\ 0 & 0 & e \end{pmatrix}.$$

In particular, we note that $a \neq 0 \neq c$. Notice that the elements of order 3 in P_2 are inverted and so

$$\begin{pmatrix} a & -ia \\ c & ic \end{pmatrix}$$

has determinant 1. Hence

$$2iac = 1 \tag{1}$$

and therefore we also find that

$$e = 1.$$

Now since we require α to have order 3 we get

$$a^3 - ia^2c - ia^2c + ac^2 = 1 \tag{2}$$
$$a^2c + iac^2 - iac^2 - c^3 = 0. \tag{3}$$

Eq. (3) simplifies to

$$(a - c)(a + c)c = 0.$$

Since $c \neq 0$, we conclude that either $a = c$ or $a = -c$. Suppose first that $a = c$. Then Eq. (2) becomes

$$2a^3 - 2ia^3 = 1$$

which when combined with Eq. (1) delivers

$$a = \frac{-1}{1+i}.$$

Thus in this case we have

$$\alpha = \begin{pmatrix} \frac{-1}{1+i} & \frac{i}{1+i} & 0 \\ \frac{-1}{1+i} & \frac{-i}{1+i} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Next assume that $a = -c$. Then we substitute this into Eq. (2) to get

$$a^3 + 2ia^3 = 1$$

and this gives

$$a = \frac{-1}{1-i}.$$

Since both equations result in a unique solution and since the proposed subgroup P_2 possesses two such elements, we conclude that $P_2 = \langle B, \alpha\rangle$ is uniquely determined. In particular, G is unique up to conjugacy in $\mathrm{GL}(V)$ and $G \le \mathrm{SL}_3(k_1)$ where $k_1 = \mathrm{GF}(p)[i]$. So if $p \equiv 1 \bmod 4$, then $G \le \mathrm{SL}_3(p)$, and if $p \equiv 3 \bmod 4$, then $G \le \mathrm{SL}_3(p^2)$. If $G \le \mathrm{SL}_3(p)$, then referring to the maximal subgroups of $\mathrm{SL}_3(p)$ [2] (or see [6] for a list) and using the fact that G operates irreducibly and is perfect yields that $G = \mathrm{SL}_3(p)$ or $G \le \mathrm{SO}_3(p)$. The latter case fails, however, as $\mathrm{SO}_3(p)$ has dihedral Sylow 2-subgroups. Thus Theorem 1.1 holds in this case. Now suppose that $G \le \mathrm{SL}_3(p^2)$ and $p \equiv 3 \bmod 4$. Then $G \le \{A \in \mathrm{GL}_3(p^2) \mid A\bar{A}^T = I_3\} = \mathrm{SU}_3(p)$ (where $\bar{A}$ is the matrix aduced from A by replacing each entry $a + ib$ by its conjugate $a - ib$). Thus this time we appeal to the list of maximal subgroups of $\mathrm{SU}_3(p)$ (see [6]) to prove Theorem 1.1 in this case.

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References

1. N. Biggs, "Constructions for cubic graphs of large girth," *Electronic Journal of Combinatorics* **5** (1998) A1.
2. D. Bloom, "The Subgroups of $\mathrm{PSL}_3(q)$, for odd q," *Trans. Amer. Math. Soc.* **127** (1967), 150–178.
3. J. Bray, C. Parker, and P. Rowley, "Cayley type graphs and cubic graphs of large girth," *Discrete Mathematics* **214** (2000), 113–121.
4. D.M. Goldschmidt, "Automorphisms of trivalent graphs," *Ann. Math.* **111** (1980), 377–404.
5. C. Parker and P. Rowley, "Finite completions of the Goldschmidt G_3-amalgam and the Mathieu groups," Manchester Centre for Pure Mathematics, preprint 1997/12.
6. C. Parker and P. Rowley, "Classical groups in Dimension 3 as completions of the Goldschmidt G_3-amalgam,"*Journal of LMS*, to appear.
7. C. Parker and P. Rowley, "Sporadic simple groups and completions of the Goldschmidt G_3-amalgam," *J. Alg.* to appear.