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A Note on Maxflow-Mincut and Homomorphic Equivalence in Matroids

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Abstract. Graph homomorphisms are used to study good characterizations for coloring problems (*Trans. Amer. Math. Soc*. **384** (1996), 1281–1297; *Discrete Math.* **22** (1978), 287–300). Particularly, the following concept arises in this context: A pair of graphs (A, B) is called a *homomorphism duality* if for any graph G either there exists a homomorphism $\sigma : A \to G$ or there exists a homomorphism $\tau : G \to B$ but not both. In this paper we show that maxflow-mincut duality for matroids can be put into this framework using strong maps as homomorphisms. More precisely, we show that, if C_k denotes the circuit of length $k+1$, the pairs (C_k, C_{k+1}) are the only homomorphism dualities in the class of duals of matroids with the strong integer maxflow-mincut property (*Jour. Comb. Theor. Ser.B* **23** (1977), 189–222). Furthermore, we prove that for general matroids there is only a trivial homomorphism duality.

Keywords: matroids, strong maps, homomorphisms, duality, Menger's theorem

1. Introduction

Let e be a fixed element. Then a *matroid port* [4] is matroid M on a finite set E such that $e \in E$. Let M and M' be two matroid ports on finite ground sets $E(M)$ and $E(M')$ and $o \notin E(M')$. A map $\sigma : E(M) \to E(M') \cup \{o\}$ is called a *strong port map from M to M'* (or a homomorphism) if

SP1 $\sigma(e) = e$ and $\sigma^{-1}(\{e\}) = \{e\}$, (fixed ground point)
SP2 σ is a strong map from M to M', i.e. if $S' \subset E(M')$ is closed in M' then $\sigma^{-1}(S' \cup \{o\})$ is closed in M.
We denote the existence of a strong port map from M to M' by $M \to M'$ and by $M \not\to M'$ the non-existence of such a map. A *homomorphism duality* for a class $\mathcal{M}$ of matroid ports is a pair (A, B) such that for any matroid port $M \in \mathcal{M}$ either there exists a homomorphism $\sigma : A \to M$ or there exists a homomorphism $\tau : M \to B$ but not both. For any $k \in \mathbb{N} \cup \{0\}$ let C_k denote the $(k+1)$-*circuit* that is the matroid port consisting of the circuit $\{e, a_1, a_2, \ldots, a_k\}$. Furthermore, we define C_∞ as the free matroid on $\{e, g\}$.

We will show that the only homomorphism duality for the class of all matroids is the trivial pair (C_0, C_1). On the other hand (C_k, C_{k+1}) is a homomorphism duality for a class of matroids $\mathcal{M}$ if and only if a dual version of Menger's theorem on edge disjoint paths holds in $\mathcal{M}$. Thus, these pairs are homomorphism dualities for the class of duals of the matroids with the strong integer maxflow-mincut property [7]. We will show that they constitute the only homomorphism dualities in this class.

The paper is organized as follows. In the next section we will characterize those matroid ports that allow a homomorphism from C_k and those for which there exists a strong map to C_k. We discuss the relations to maxflow-mincut duality in Section 3. In Section 4 we put these findings into the framework of homomorphism duality and prove that, while general matroids ports have only a trivial duality, the maxflow-mincut duality is the only homomorphism duality for the duals of matroids with the strong integer maxflow-mincut property.

We assume familiarity with matroid theory, standard references are [5, 8]. All matroids will be finite and we will denote the ground set of a matroid M by $E(M)$ or sometimes just by E.

2. From and to the circuit

Let M be a matroid port and C a shortest circuit of M containing e. Then we define the *girth of M* as $\text{girth}(M) = |C| - 1$. If there is no such circuit we set the girth to infinity.

Theorem 1 *Let M be a matroid port and $k \in \mathbb{N}$. Then*

$$C_k \to M \text{ if and only if } \text{girth}(M) \leq k.$$

Proof: If $C = \{e, b_1, \ldots, b_l\}$, $l \leq k$ is a circuit in M, then clearly the port map defined by $\sigma(e) = e$, $\sigma(a_i) = b_i$ for $i \leq l$ and $\sigma(a_i) = o$ for $l < i \leq k$ is strong. Assume on the other hand that $\sigma : C_k \to M$ is a strong port map and consider $S = \sigma(C_k)\backslash\{e\}$. Then $\sigma^{-1}(S)$ is not closed, thus, as σ is strong, e must be on a circuit in $S \cup \{e\}$. □

Note that $C_\infty \to M$ for any matroid port M. This gives rise to the following:

Corollary 1 *Let $k, l \in \mathbb{N} \cup \{\infty\}$. Then*

$$C_k \to C_l \text{ if and only if } k \geq l.$$

Since the restriction of a strong map is strong we also have:

Corollary 2 *If $M \to M'$ then girth $(M) \geq$ girth (M').*

Next we study the existence of strong port maps to the $(k+1)$-circuit. We will show that this is equivalent to the existence of k "disjoint" cocircuits containing $\{e\}$. One direction of this equivalence is:

Lemma 1 *Let M be a matroid port. Assume that M has k cocircuits $C_1^*, \ldots, C_k^*$ containing e which, apart from that, are pairwise disjoint, more precisely $C_i^* \cap C_j^* = \{e\}$ for $i \neq j$.*

Let $\sigma : M \to C_k$ denote the map defined by

$$\sigma(f) = \begin{cases} e & \text{if } f = e, \\ a_i & \text{if } f \in C_i^* \backslash \{e\}, \\ o & \text{otherwise.} \end{cases}$$

Then σ is a strong port map.

Proof: Let $A \subseteq C_k$ be a closed set. We have to verify that $\sigma^{-1}(A \cup \{o\})$ is closed. If $e \notin A$, then $\sigma^{-1}(A \cup \{o\}) = \bigcap_{a_i \notin A}(E \backslash C_i^*)$ is the intersection of closed sets and thus is closed. Assume now that $e \in A$ and, for a contradiction, that $\sigma^{-1}(A \cup \{o\})$ is not closed. Hence, there exists a circuit C in M such that $C \cap (E \backslash \sigma^{-1}(A \cup \{o\})) = \{g\}$. As $E \backslash \sigma^{-1}(A \cup \{o\}) = \bigcup_{a_i \notin A}(C_i^* \backslash e)$ there exists some i_0 such that $g \in C_{i_0}^*$ and $a_{i_0} \notin A$. Since C is a circuit and $C_{i_0}^*$ a cocircuit we must have $|C \cap C_{i_0}^*| \geq 2$ and thus $e \in C$. Therefore, C has to intersect each C_i^* where $a_i \notin A$ at least twice. We conclude that $\sigma^{-1}(A \cup \{o\}) = E \backslash C_{i_0}^*$ implying $A = C_k \backslash \{a_i\}$ contradicting the fact that A is closed. □

Theorem 2 *Let M be a matroid port and $k \in \mathbb{N} \cup \{\infty\}$. Then $M \to C_k$ if and only if there exist k cocircuits $C_1^*, \ldots, C_k^*$ in M such that $C_i^* \cap C_j^* = \{e\}$ for $i \neq j$.*

Proof: Sufficiency has been proven in Lemma 1. Thus assume σ is a strong port map from $M \to C_k$. We set $\tilde{C}_i^* := \sigma^{-1}(\{e, a_i\})$ for $1 \leq i \leq k$. The claim follows if we can show that each $\tilde{C}_i^*$ contains a cocircuit C_i^* containing e. To see this note that $F := \sigma^{-1}(C_k \cup \{o\} \backslash \{e, a_i, \})$ is a closed set which is a proper subset of the ground set and does not contain e. Thus, there is a hyperplane H such that $F \subseteq H$ and $e \notin H$ and $C_i^* : = E \backslash H$ is as required. □

Note that $M \to C_\infty$ if and only if e is a cocircuit in M.

3. Maxflow-mincut and homomorphic equivalence

Given a matroid port M, a *flow* is defined to be a set of circuits $C^1, \ldots, C^k$ such that $C^i \cap C^j = \{e\}$. (We do not consider capacities here, as they can be simulated by parallels.) The value of the flow is k and a maxflow is a flow of maximum value. By the results of the last section, the existence of a flow of value k in M is equivalent to $M^* \to C_k$ where M^* denotes the matroid dual of M. We have also shown that the existence of a cocircuit C^* containing e in M^* such that $|C^* \backslash \{e\}| \leq l$ is equivalent to $C_l \to M^*$. As, obviously, there cannot be a flow of a value larger than such an l, we can formulate the well-known weak duality for matroid flows as

$$\max\{k \mid M^* \to C_k\} \leq \min\{l \mid C_l \to M^*\}.$$

If equality holds in the above inequality, we say that M has the *maxflow-mincut property*. (Note, that this is weaker than Seymour's definition of the strong integer maxflow-mincut property in [7]). Let us say that two matroid ports M, M' are *homomorphically equivalent*, denoted by $M \leftrightarrow M'$, if $M \to M'$ and $M' \to M$. With this terminology we can summarize:

Theorem 3 *Let M be a matroid port. Then there exists a unique $k \in \mathbb{N} \cup \{\infty\}$ such that $M^* \leftrightarrow C_k$ if and only if M has the maxflow-mincut property.*

This also shows that the homomorphic equivalence is a well structured property.

4. Homomorphism duality

In this section we study strong port maps from a homomorphism duality point of view as introduced in [6], [1]. Let $\mathcal{M}$ be a class of matroid ports and (A, B) denote a pair of matroid ports in $\mathcal{M}$. By $A \to$ we denote the subclass of matroid ports $M \in \mathcal{M}$ such that there exists a strong port map from $A \to M$

$$A \to := \{M \in \mathcal{M} \mid A \to M\}.$$

Similarly we set

$$\not\to B := \{M \in \mathcal{M} \mid M \not\to B\}.$$

A homomorphism duality for $\mathcal{M}$ (which is introduced in the introduction) is then the equation of classes,

$$A \to = \not\to B.$$

In other words this means that for any $M \in \mathcal{M}$ we have the homomorphic alternative: either M is the homomorphic image of A or maps onto B but not both.

We will show that for the class of all matroid ports there is no non-trivial such theorem, but restricting to the class $\mathcal{F}$ of binary matroids without F_7-minor, where F_7 denotes the Fano-matroid port containing a special element marked e, the quasiordering defined by the existence of maps has an extremely simple structure. Furthermore, we can expose Menger's theorem as the unique homomorphism duality in this class.

A trivial example of a homomorphism duality for any class of matroid ports is the observation that either e is a loop in M or it is not. With the following theorem we show that in fact this is the only homomorphism duality for the class of all matroid ports.

Theorem 4 *Assume that A, B are matroid ports such that $A \to = \not\to B$ is a homomorphism duality for the class $\mathcal{M}$ of all matroid ports. Then $A \leftrightarrow C_0$ and $B \leftrightarrow C_1$.*

We will derive Theorem 4 from the following lemma. For $n \geq k \in \mathbb{N}$ let U_k^n denote the uniform matroid port of rank k with n elements.

Lemma 2 *Let $k \in \mathbb{N}$ and B be a matroid port such that girth $(B) \geq 2$. Then there exists an n such that $U_k^n \not\to B$.*

Proof: Assume the lemma were false. Then, for n large enough and a strong map $\sigma : U_k^n \to B$, there is some element $f \in E(B)$ such that $|\sigma^{-1}(\{f, o\})| \geq k$. Let F denote the closure of f in B. By assumption $e \notin F$, thus $\sigma^{-1}(F)$ is not closed, a contradiction. □

Proof of Theorem 4: By Corollary 1 and Theorem 2 $C_0 \to M$ if and only if e is a loop in M and $M \to C_1$ if and only if e is on some cocircuit. Thus, $C_0 \to = \not\to C_1$ is a homomorphism duality theorem.

Now let $A \to = \not\to B$ be a homomorphism duality. Since all matroid ports map to the loop we must have girth $(B) \geq 1$. If girth$(B) = 1$ then $B \leftrightarrow C_1$ and, by the preceding, $A \leftrightarrow C_0$. Assume now that girth$(B) \geq 2$. Then by Lemma 2 , for all $k \in \mathbb{N}$ there is some $n(k)$ such that $U_k^{n(k)} \not\to B$. Hence, necessarily $A \to U_k^{n(k)}$. By Corollary 2 this implies girth$(A) = \infty$ and thus $A \leftrightarrow C_\infty$. Since any port maps to C_∞ we, in particular have $B \to A$, a contradiction. □

It should be clear by now that, in the following, in order to derive some interesting homomorphism duality, we restrict our considerations to a class of matroids with a strong maxflow-mincut duality. P. Seymour [7] proves that every parallel extension of a matroid M has the maxflow-mincut property if and only if M does not have an F_7^* or a $U_{2,4}$-minor that uses e. Thus, we consider the *class* $\mathcal{F}$ of *dual maxflow-mincut ports*, where $M \in \mathcal{F}$ if and only if e is neither on an F_7 nor on a $U_{2,4}$-minor.

In this class, the quasiorder defined by the existence of strong port maps on the classes of homomorphically equivalent ports has the simple structure of an infinite chain with minimum and maximum, since every port is equivalent to some C_k. We derive:

Theorem 5

1. *Let $k \in \mathbb{N}$. Then $C_k \to = \not\to C_{k+1}$ is a homomorphism duality for $\mathcal{F}$.*
2. *Let $A, B \in \mathcal{F}$ be two matroid ports and $A \to = \not\to B$ a homomorphism duality for $\mathcal{F}$. Then there exists a unique k, such that $A \leftrightarrow C_k$ and $B \leftrightarrow C_{k+1}$.*

Proof: Let $M \in \mathcal{F}$. Then e is neither on an F_7^*-minor nor on a $U_{2,4}$-minor in M^*. By Seymour's theorem ([7]) M^* has the maxflow-mincut property. Hence, by Theorem 3 there is a unique l_0 such that $M \leftrightarrow C_{l_0}$. Corollaries 1 and 2 imply that $M \to C_k \Leftrightarrow l_0 \geq k$ and $C_k \to M \Leftrightarrow k \geq l_0$. For the second statement let $A \leftrightarrow C_{l_0}$. Then $A \not\to C_{l_0+1}$ and thus $C_{l_0+1} \to B$. By Theorem 3 there exists a unique $l \leq l_0 + 1$ such that $B \leftrightarrow C_l$. Since $A \to C_l$ for $l < l_0 + 1$, the claim follows. □

We would like to point out that for graphic matroids the duality $C_k \to = \not\to C_{k+1}$ reflects the "trivial fact" that the length of a shortest *st*-path in a graph with unit weights limits the number of pairwise disjoint *st*-cuts. For cographic matroids we get the existence of either $k+1$ edge-disjoint *st*-paths or an *st*-cut of size k. Thus, the class equation "$C_k \to = \not\to C_{k+1}$" is a formulation of Menger's theorem as a theorem of the alternatives.

Remark This paper was motivated by the companion paper [2] where we considered dualities for strong maps of oriented matroids and showed that Farkas' lemma is the only instance of such a duality. In the more restrictive context of matroid ports we obtained a richer spectrum of duality results.

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