Journal of Algebraic Combinatorics **12** (2000), 85–93

On Mod-*p* Alon-Babai-Suzuki Inequality

JIN QIAN qian@math.ohio-state.edu
D.K. RAY-CHAUDHURI dijen@math.ohio-state.edu
Department of Mathematics, The Ohio State University

Received October 15, 1998; Revised April 27, 1999

Abstract. Alon, Babai and Suzuki proved the following theorem:

Let p be a prime and let K, L be two disjoint subsets of $\{0, 1, \ldots, p-1\}$. *Let* $|K| = r$, $|L| = s$, *and assume* $r(s-r+1) \le p-1$ *and* $n \ge s + k_r$ *where* k_r *is the maximal element of* K. *Let* $\mathcal{F}$ *be a family of subsets of an* n*-element set. Suppose that*

(i) $|F| \in K \pmod p$ *for each* $F \in \mathcal{F}$;
(ii) $|E \cap F| \in L \pmod p$ *for each pair of distinct sets* $E, F \in \mathcal{F}$.

Then $|\mathcal{F}| \le \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

They conjectured that the condition that $r(s-r+1) \le p-1$ in the theorem can be dropped and the same conclusion should hold. In this paper we prove that the same conclusion holds if the two conditions in the theorem, i.e. $r(s-r+1) \le p-1$ and $n \ge s + k_r$ are replaced by a single more relaxed condition $2s - r \le n$.

Keywords: combinatorial, inequality

1. Introduction

In this paper, we let n be a positive integer, $I_n = \{1, 2, \ldots, n\}$, $X = \{x_1, x_2, \ldots, x_n\}$ be an n-element set, p be a prime number and $L \subseteq I_{p-1} \cup \{0\} = \{0, 1, \ldots, p-1\}$ be an s-element set for some positive integer $s < p$. We call a family $\mathcal{F}$ of subsets of X a *mod p L-intersection family* if $|E \cap F| \in L \pmod p$, $\forall E, F \in \mathcal{F}$ with $E \ne F$. Here $n \in L (\text{mod } p)$ means there exists $l \in L$ for which $n \equiv l (\text{mod } p)$.

For any $0 \le i \le j \le n$, let $I_n(i, j)$ be the 0-1 incidence matrix of $\mathbb{P}_i(X)$ and $\mathbb{P}_j(X)$ with rows (columns) indexed by $\mathbb{P}_i(X)$ ($\mathbb{P}_j(X)$). The (A, B)-entry of $I_n(i, j)$ is 1 if $A \subseteq B$ and 0 otherwise for any $A \in \mathbb{P}_i(X)$ and $B \in \mathbb{P}_j(X)$.

Convention: Throughout the paper, unless otherwise specified, all vector spaces are assumed to be over $\mathbb{F}_p$ which we abbreviate as $\mathbb{F}$. Therefore for the sake of brevity *rank*$(I_n(i, j))$ will denote the rank of $I_n(i, j)$ considered as a matrix over $\mathbb{F}$.

Alon, Babai and Suzuki [1] proved the following inequality which generalizes the classic Frankl-Ray-Chaudhuri-Wilson Inequality [3].

Theorem *Let p be a prime and K, L be two disjoint subsets of* $\{0, 1, \ldots, p-1\}$. *Let* $|K| = r$, $|L| = s$, *and assume* $r(s-r+1) \le p-1$ *and* $n \ge s + k_r$ *where* k_r *is the maximal element of* K. *Let* $\mathcal{F}$ *be a family of subsets of an* n*-element set. Suppose that*

(i) $|F| \in K \pmod p$ *for each* $F \in \mathcal{F}$;
(ii) $|E \cap F| \in L \pmod p$ *for each pair of distinct sets* $E, F \in \mathcal{F}$.
Then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.

They went on and conjectured that the condition $r(s-r+1) \leq p-1$ in the statement of the above theorem can be dropped and the conclusion of the theorem will still hold. Snevily [7] confirmed and improved the conjecture when n is sufficiently large. He showed that when n is sufficiently large, then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-2} + \binom{n}{s-4} + \cdots + \binom{n}{s-2\lfloor s/2\rfloor}$. The main result of this paper is the following theorem which confirms the conjecture of Alon, Babai and Suzuki to a large extent.

Theorem 1 *Let p be a prime number, r, s be two positive integers with $2s-r \leq n$, L be an s-subset of $I_{p-1} \cup \{0\}$ and K be an r-subset of $I_{p-1} \cup \{0\}$ with $L \cap K = \emptyset$. If $\mathcal{F}$ is a* mod p *L-intersection family and $|E| \in K \pmod p$, $\forall E \in \mathcal{F}$, then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$.*

We note that in some instances the condition $2s - r \leq n$ holds but Alon, Babai and Suzuki's condition $n \geq s + k_r$ does not. For instance, if $n = 9$, $p = 7$, $K = \{2, 5, 6\}$ and $L = \{0, 1, 3, 4\}$, then it is clear that $2s - r = 2 \cdot 4 - 3 = 5 \leq 9 = n$, but $k_r + s = 6 + 4 > 9 = n$. In some other instances, however, the Alon, Babai and Suzuki's condition holds but the condition $2s - r \leq n$ does not. For example, $Y = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $p = 7$, $K = \{1\}$, $L = \{0, 2, 3, 4, 5, 6\}$, $\mathcal{F} = \{\{9\}, \{1, 2, 3, 4, 5, 6, 7, 8\}\}$. It is clear that $k_r + s = 7 < 9$ but $2s - r = 11 > 9$.

2. Proof of Theorem 1

For the proof of the theorem we need the following lemma which is mentioned by Frankl in [2].

Lemma 1 *If $0 \leq a \leq b < p$ and $a + b \leq n$, then $rank_p(I_n(a, b)) = \binom{n}{a}$.*

Proof: We may assume $a \neq 0$. The proof is by induction on $a + b + n$. Note that $a + b + n \geq 4$. It is clear that the lemma holds when $a + b + n = 4$.

Suppose it holds when $a + b + n < l$. Now we consider the case $a + b + n = l$. We distinguish two cases.

Case 1 $a + b = n$. In this case, it is easy to verify that $\mathbb{P}_b(X)$ is an L'-intersection family with $L' = \{n - 2a, n - 2a + 1, \ldots, n - a - 1\}$ and $b = n - a$ and $b \notin L' \pmod p$. Now we use the following result of Frankl and Wilson [3]:

If $\mathcal{G} \subseteq \mathbb{P}_k(X)$ is a mod p L-intersection family for some set L consisting of non-negative integers with $k \notin L \pmod p$ and $\binom{k-i}{l-i} \not\equiv 0 \pmod p$ for $i = 0, 1, \ldots, l$, then $|\mathcal{G}| \leq rank(I_n(l, \mathcal{G}))$, where $l = |L|$ and $I_n(l, \mathcal{G})$ is a 0-1 *incidence matrix whose rows and* *columns are indexed by* $\mathbb{P}_l(X)$ *and* $\mathcal{G}$ *respectively and the* (A, F)*-entry of* $I_n(l, \mathcal{G})$ *is* 1 *if* $A \subseteq F$ *and* 0 *otherwise for any* $A \in \mathbb{P}_l(X)$ *and* $F \in \mathcal{G}$.

Notice that if we take $\mathcal{G} = \mathbb{P}_b(X)$, then $I_n(a, \mathcal{G}) = I_n(a, b)$. So by the above result we have $\binom{n}{b} = |\mathbb{P}_b(X)| \leq rank(I_n(a, b))$. On the other hand, it is clear that $rank(I_n(a, b)) \leq \binom{n}{b}$. So $rank(I_n(a, b)) = \binom{n}{b}$, which implies $rank(I_n(a, b)) = \binom{n}{a}$ since $b = n - a$ in this case. This proves the lemma in the first case.

Case 2 $a + b < n$. In this case, we partition $\mathbb{P}_b(X)$ into two families: one consists of all those s-subsets of X not containing x_n, the other one consists of all those containing x_n. We do the same thing to $\mathbb{P}_a(X)$. It is clear that

$$I_n(a, b) = \begin{pmatrix} I_{n-1}(a, b) & B \\ 0 & I_{n-1}(a-1, b-1) \end{pmatrix} \quad \text{for some matrix } B.$$

We observe that in this case $a + b \leq n - 1$ and $a - 1 + b - 1 \leq n - 1$. By the induction hypothesis, $rank(I_{n-1}(a, b)) = \binom{n-1}{a}$ and $rank(I_{n-1}(a-1, b-1)) = \binom{n-1}{a-1}$, i.e. both the rows of $I_{n-1}(a, b)$ and the rows of $I_{n-1}(a-1, b-1)$ are linearly independent. So the rows of $I_n(a, b)$ are linearly independent, which implies that $rank(I_n(a, b)) = \binom{n}{a}$ and hence the proof of the lemma is complete. □

Remark By Lemma 1, it is clear that the row vectors of $I_n(a, b)$ can be expanded into a basis of $\mathbb{F}^{\binom{n}{b}}$ by adding some other $\binom{n}{b} - \binom{n}{a}$ vectors in $\mathbb{F}^{\binom{n}{b}}$.

Following the idea of Ramanan [6], we associate a variable x_F for each $F \in \mathcal{F}$. For $I \subseteq X$, we define the linear form L_I by

$$L_I = \sum_{F \in \mathcal{F}, I \subseteq F} x_F.$$

Now let us prove a lemma which is useful in the proof of the theorem.

Lemma 2 *For any positive integers* u, v *with* $u < v < p$ *and* $u + v \leq n$, *we have*

$$\dim\left(\frac{\langle L_J : J \in \mathbb{P}_v(X)\rangle}{\left\langle \sum_{J \in \mathbb{P}_v(X), I \subseteq J} L_J : I \in \mathbb{P}_u(X)\right\rangle}\right) \leq \binom{n}{v} - \binom{n}{u}.$$

Here $\frac{A}{B}$ *is the quotient space of two vector spaces A and B with* $B \subseteq A$ *and* $\langle L_J : J \in \mathbb{P}_v(X)\rangle$ *is the vector space spanned by* $\{L_J : J \in \mathbb{P}_v(X)\}$.

Proof: Let $V = \langle L_J : J \in \mathbb{P}_v(X)\rangle$. We define the following linear mapping $f : \mathbb{F}^{\binom{n}{v}} \rightarrow V$ as follows. We view a vector $\underline{w}$ in $\mathbb{F}^{\binom{n}{v}}$ as a mapping from $\mathbb{P}_v(X)$ to $\mathbb{F}$. For each vector $\underline{w} \in \mathbb{F}^{\binom{n}{v}}$ whose J'th component is a_J, we define $f(\underline{w}) = \sum_{J \in \mathbb{P}_v(X)} a_J L_J$.

Let W be the vector space generated by the rows of $I_n(u, v)$. It is clear that f is a surjective map that maps W to $\langle \sum_{J\in\mathbb{P}_v(X), I\subseteq J} L_J : I \in \mathbb{P}_u(X)\rangle$. By linear algebra

$$\begin{aligned}
&\dim\left(\frac{\langle L_J : J \in \mathbb{P}_v(X)\rangle}{\left\langle \sum_{J\in\mathbb{P}_v(X), I\subseteq J} L_J : I \in \mathbb{P}_u(X)\right\rangle}\right)\\
&\le \dim\left(\frac{f\left(\mathbb{F}^{\binom{n}{v}}\right)}{f(W)}\right)\\
&\le \dim\left(\frac{\mathbb{F}^{\binom{n}{v}}}{W}\right)\\
&\le \binom{n}{v} - \binom{n}{v} \quad \text{by the above remark.}
\end{aligned}$$

This proves Lemma 2. □

Consider the system of linear equations:

$$\left\{L_I = 0, \quad \text{where } I \text{ runs through } \bigcup_{i=0}^{s} \mathbb{P}_i(X)\right\}. \tag{$*$}$$

By the method employed in Qian and Ray-Chaudhuri [4] or [5], we have the following propostion.

Proposition *Assume that $L \cap K = \emptyset$. If $\mathcal{F}$ is an mod p L-intersection family with $|E| \in K$* (mod p) *for any $E \in \mathcal{F}$, then the only solution of the above system of linear equations is the trivial solution.*

Proof: Let (v_E) be a solution of (*). We need to show that (v_E) is the all-zero solution. Suppose on the contrary that not all of v_E's are 0. Let E_0 be an element in $\mathcal{F}$ with $v_{E_0} \neq 0$. Let $\mathbb{F}$ be the finite field containing p elements. Since $\binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{s}$ form a basis for the vector space spanned by all the polynomials in $\mathbb{F}(X)$ of degrees at most s, there exist $a_0, a_1, \ldots, a_s \in \mathbb{F}$ with

$$\sum_{i=0}^{s} a_i \binom{x}{i} = \prod_{j=1}^{s} (x - l_j).$$

We denote $\prod_{j=1}^{s}(x - l_j)$ by $g(x)$. Next we prove the following identity,

$$\sum_{i=0}^{s} a_i \sum_{I\in\mathbb{P}_i(X), I\subseteq E_0} L_I = \sum_{F\in\mathcal{F}} g(|F \wedge E_0|) x_F.$$

We prove it by comparing the coefficients of both sides. For any $F \in \mathcal{F}$, the coefficient of x_F in the left hand side is

$$\sum_{i=0}^{s} a_i |\{I \in \mathbb{P}_i(X) : I \subseteq E_0,\, I \subseteq F\}| = \sum_{i=0}^{s} a_i \binom{|F \wedge E_0|}{i},$$

which is equal to $g(|F \wedge E_0|)$ by the definition of $g(x)$. This proves the above identity.

Specializing $x_E = v_E$ for all $E \in \mathcal{F}$ in the above identity, we have

$$\sum_{i=0}^{s} a_i \sum_{I \in \mathbb{P}_i(X), I \subseteq E_0} L_I((v_E)) = \sum_{F \in \mathcal{F}} g(|F \wedge E_0|) v_F.$$

It is clear that left hand side is 0 since (v_E) is a solution of (*). For $F \in \mathcal{F}$ with $F \neq E_0$, $|F \wedge E_0| \in L \pmod{p}$ and so $g(|F \wedge F_0|) = 0$. So the right hand side of the above identity is equal to $g(|E_0|)v_{E_0}$. So $0 = g(|E_0|)v_{E_0}$. Since $L \cap K = \emptyset$, We have $g(|E_0|) \neq 0$ and so $v_{E_0} = 0$. This is a contradiction to the definition of E_0 and thus it proves the proposition. □

As a result of this proposition, we have:

$$|\mathcal{F}| \leq \dim\left(\left\{L_I : I \in \bigcup_{i=0}^{s} \mathbb{P}_i(X)\right\}\right). \tag{1}$$

where $\dim(\{L_I : I \in \bigcup_{i=0}^{s} \mathbb{P}_i(X)\})$ is defined to be the dimension of the space spanned by $\{L_I : I \in \bigcup_{i=0}^{s} \mathbb{P}_i(X)\}$.

The following lemma is of critical importance in the proof of the theorem.

Lemma 3 *For any $i \in \{0, 1, \ldots, s - r + 1\}$ and every $I \in \mathbb{P}_i(X)$, the linear form $\sum_{H \in \mathbb{P}_{i+r}(X), I \subseteq H} L_H$ is linearly dependent on the set of linear forms $\{L_H : i \leq |H| \leq i + r - 1, H \subseteq X\}$ over $\mathbb{F}$.*

Proof of Lemma 3: We distinguish two cases.

Case 1 $i \notin K \pmod{p}$. In this case $\forall k_j \in K$, $k_j - i \neq 0$ in $\mathbb{F}$ and so $c = (-1)^{r+1}(k_1 - i)(k_2 - i) \cdots (k_r - i) \neq 0$ in $\mathbb{F}$. It is clear that there exist $a_1, a_2, \ldots, a_{r-1} \in \mathbb{F}$, $a_r = r! \in \mathbb{F} - \{0\}$ such that

$$\begin{aligned} &a_1\binom{x}{1} + a_2\binom{x}{2} + \cdots + a_r\binom{x}{r} \\ &= (x - (k_1 - i))(x - (k_2 - i)) \cdots (x - (k_r - i)) + c, \end{aligned}$$

since the polynomial in the right hand side has constant term equal to 0.

Next we show that

$$\sum_{j=1}^{r} a_j \sum_{H \in \mathbb{P}_{i+j}(X), I \subseteq H} L_H = c \cdot L_I. \tag{2}$$

In fact both sides are linear forms in x_E's, $E \in \mathcal{F}$. The coefficient of x_E in the left hand side is $\sum_{j=1}^{r} a_j |\{H \mid I \subseteq H \subseteq E, |H| = i + j\}|$. So it is equal to 0 if $I \not\subseteq E$ and $a_1\binom{|E|-i}{1} + a_2\binom{|E|-i}{2} + \cdots + a_r\binom{|E|-i}{r}$ if $I \subseteq E$. By the above polynomial identity,

$$\begin{aligned}
&a_1\binom{|E|-i}{1} + a_2\binom{|E|-i}{2} + \cdots + a_r\binom{|E|-i}{r}\\
&\quad = (|E| - i - (k_1 - i))(|E| - i - (k_2 - i)) \cdots (|E| - i - (k_r - i)) + c\\
&\quad = c \quad \text{since } |E| \in K \pmod p.
\end{aligned}$$

The coefficient of x_E's in the right hand side is obviously the same. This proves (2).

Writing (2) in a different way, we have

$$\sum_{H \in \mathbb{P}_{i+r}(X), I \subseteq H} L_H = \frac{1}{r!}\left(cL_I - \sum_{j=1}^{r-1} a_j \sum_{H \in \mathbb{P}_{i+j}(X), I \subseteq H} L_H \right)$$

This proves the lemma in case 1.

Case 2 $i \in K \pmod p$. In this case, the constant term of $(x - (k_1 - i))(x - (k_2 - i)) \cdots (x - (k_r - i))$ is $0 \in \mathbb{F}$. So there exists $a_1, a_2, \ldots, a_{r-1} \in \mathbb{F}$, $a_r = r! \in \mathbb{F} - \{0\}$ such that

$$a_1\binom{x}{1} + a_2\binom{x}{2} + \cdots a_r\binom{x}{r} = (x - (k_1 - i))(x - (k_2 - i)) \cdots (x - (k_r - i)).$$

As a consequence we have

$$\sum_{j=1}^{r} a_j \sum_{H \in \mathbb{P}_{i+j}(X), I \subseteq H} L_H = 0 \quad \forall I \in \mathbb{P}_i(X),$$

i.e. we have

$$\sum_{H \in \mathbb{P}_{i+r}(X), I \subseteq H} L_H = -\frac{1}{r!}\left(\sum_{j=1}^{r-1} a_j \sum_{H \in \mathbb{P}_{i+j}(X), I \subseteq H} L_H \right) \quad \forall I \in \mathbb{P}_i(X).$$

This finishes the proof of this lemma. □

From the above lemma, we easily deduce the following corollary.

Corollary *With the same condition as in Lemma* 3, *we have*

$$\begin{aligned}
&\left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_H(X) \right\rangle\\
&\quad = \left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_H(X) \right\rangle + \left\langle \sum_{H \in \mathbb{P}_{i+r}(X), I \subseteq H} L_H : I \in \mathbb{P}_i(X) \right\rangle.
\end{aligned}$$

Next we prove our last lemma.

Lemma 4 *For any* $i \in \{0, 1, \ldots, s-r+1\}$,

$$\binom{n}{i} + \binom{n}{i+1} + \binom{n}{i+r-1} + \dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=i}^{s} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X)\right\rangle}\right) \leq \binom{n}{s-r+1} + \binom{n}{s-r+2} + \cdots + \binom{n}{s}.$$

Proof of Lemma 4: We induct on $s-r+1-i$. It is clearly true when $s-r+1-i=0$, i.e. $i = s-r+1$. Suppose the lemma holds for $s-r+1-i < l$ for some positive integer l. Now we want to show that it holds for $s-r+1-i=l$.

Let us recall two well-known linear algebra facts:

Fact 1. *Let A, B, C be vector spaces with $C \subseteq B$. Then* $\dim(\frac{A+B}{A+C}) \leq \dim(\frac{B}{C})$.
Fact 2. *Let $C \subseteq B \subseteq A$ be three vector spaces. Then* $\dim(\frac{A}{C}) = \dim(\frac{B}{C}) + \dim(\frac{A}{B})$.

We observe that $i+i+r \leq (s-r)+(s-r)+r \leq n$ by the condition in the theorem. By the above corollary, we have

$$\begin{aligned}
&\dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=i}^{i+r} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X)\right\rangle}\right) \\
&= \dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X)\right\rangle + \langle L_H : H \in \mathbb{P}_{i+r}(X)\rangle}{\left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X)\right\rangle + \left\langle \sum_{H\in\mathbb{P}_{i+r}(X), I\subseteq H} L_H : I \in \mathbb{P}_i(X)\right\rangle}\right) \\
&\leq \dim\left(\frac{\langle L_H : H \in \mathbb{P}_{i+r}(X)\rangle}{\left\langle \sum_{H\in\mathbb{P}_{i+r}(X), I\subseteq H} L_H : I \in \mathbb{P}_i(X)\right\rangle}\right) \quad \text{by fact 1 above} \\
&\leq \binom{n}{i+r} - \binom{n}{i} \quad \text{by Lemma 2 with } u=i \text{ and } v=i+r.
\end{aligned}$$

In summary, we have

$$\dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=i}^{i+r} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X)\right\rangle}\right) \leq \binom{n}{i+r} - \binom{n}{i}. \tag{3}$$

Now we are ready for the key part of the proof of the lemma.

$$\begin{aligned}
&\binom{n}{i} + \binom{n}{i+1} + \cdots + \binom{n}{i+r-1} + \dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=i}^{s} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X)\right\rangle}\right) \\
&= \binom{n}{i} + \binom{n}{i+1} + \cdots + \binom{n}{i+r-1} + \dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=i}^{i+r} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X)\right\rangle}\right) \\
&\quad + \dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=i}^{s} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=i}^{i+r} \mathbb{P}_j(X)\right\rangle}\right) \quad \text{by fact 2 above}
\end{aligned}$$

$$
\begin{aligned}
&= \binom{n}{i} + \binom{n}{i+1} + \cdots + \binom{n}{i+r-1} + \dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=i}^{i+r} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X)\right\rangle}\right) \\
&\quad + \dim\left(\frac{\langle L_H : H \in \mathbb{P}_i(X)\rangle + \left\langle L_H : H \in \bigcup_{j=i+1}^{s} \mathbb{P}_j(X)\right\rangle}{\langle L_H : H \in \mathbb{P}_i(X)\rangle + \left\langle L_H : H \in \bigcup_{j=i+1}^{i+r} \mathbb{P}_j(X)\right\rangle}\right) \\
&\le \binom{n}{i} + \binom{n}{i+1} + \cdots + \binom{n}{i+r-1} + \dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=i}^{i+r} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=i}^{i+r-1} \mathbb{P}_j(X)\right\rangle}\right) \\
&\quad + \dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=i+1}^{s} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=i+1}^{i+r} \mathbb{P}_j(X)\right\rangle}\right) \quad \text{by fact 1 above} \\
&\le \binom{n}{i} + \binom{n}{i+1} + \cdots + \binom{n}{i+r-1} + \binom{n}{i+r} - \binom{n}{i} \quad \text{by (3) above} \\
&\quad + \dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=i+1}^{s} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=i+1}^{i+r} \mathbb{P}_j(X)\right\rangle}\right) \\
&= \binom{n}{i+1} + \cdots + \binom{n}{i+r} + \dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=i+1}^{s} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=i+1}^{i+r} \mathbb{P}_j(X)\right\rangle}\right) \\
&\le \binom{n}{s-r+1} + \cdots + \binom{n}{s},
\end{aligned}
$$

where the last step is by the induction hypothesis since $s-r+1-(i+1)\langle s-r+1-i = l$. This completes the proof of the Lemma 4.

Now it is easy to prove Theorem 1. By (1) we have

$$
\begin{aligned}
|\mathcal{F}| &\le \dim\left(\left\langle L_H : H \in \bigcup_{i=0}^{s} \mathbb{P}_i(X)\right\rangle\right) \\
&\le \dim\left(\left\langle L_H : H \in \bigcup_{i=0}^{r-1} \mathbb{P}_i(X)\right\rangle\right) \\
&\quad + \dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=0}^{s} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=0}^{r-1} \mathbb{P}_j(X)\right\rangle}\right) \quad \text{by fact 2 above} \\
&\le \binom{n}{0} + \binom{n}{1} + \binom{n}{r-1} + \dim\left(\frac{\left\langle L_H : H \in \bigcup_{j=0}^{s} \mathbb{P}_j(X)\right\rangle}{\left\langle L_H : H \in \bigcup_{j=0}^{r-1} \mathbb{P}_j(X)\right\rangle}\right) \\
&\le \binom{n}{s-r+1} + \binom{n}{s-r+2} + \cdots + \binom{n}{s} \quad \text{by taking } i = 0 \text{ in Lemma 4,}
\end{aligned}
$$

which completes the proof of the theorem. □

References

1. N. Alon, L, Babai, and H. Suzuki, “Multilinear polynomials and Frankl-Ray-Chaudhuri-Wilson type intersection theorems,” *J. Combinatorial Theory* (A) **58** (1991), 165–180.
2. P. Frankl, “Intersection theorems and mod p rank inclusion matrices,” *J. Combinatorial Theory* (A) **54** (1990), 85–94.
3. P. Frankl and R.M. Wilson, “Intersection theorem with geometric consequences,” *Combinatorica* **1**(4) (1981), 357–368.
4. J. Qian and D.K. Ray-Chaudhuri, “Frankl-Füredi type inequalities for polynomial semi-lattices,” *Electronic Journal of Combinatorics* **4** (1997), 28.
5. J. Qian and D.K. Ray-Chaudhuri, “Extremal case of Frankl-Ray-Chaudhuri-Wilson inequality,” *J. Statist. Plann. Inference*, to be published.
6. G.V. Ramanan, “Proof of a conjecture of Frankl and Füredi,” *J. Combin. Theory Ser. A* **79**(1) (1997), 53–67.
7. H.S. Snevily, “On Generalizations of the deBruijn-Erdos Theorem,” JCT-A **68**(1) (1994), 237–238.