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General Form of Non-Symmetric Spin Models

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Abstract. A spin model (for link invariants) is a square matrix W with non-zero complex entries which satisfies certain axioms. Recently (Jaeger and Nomura, *J. Alg. Combin.* **10** (1999), 241–278) it was shown that ${}^tWW^{-1}$ is a permutation matrix (the order of this permutation matrix is called the "index" of W), and a general form was given for spin models of index 2. In the present paper, we generalize this general form to an arbitrary index m. In particular, we give a simple form of W when m is a prime number.

Keywords: spin model, association scheme, Bose-Mesner algebra

1. Introduction

Spin models were introduced by Vaughan Jones [7] to construct invariants of knots and links. A spin model is essentially a square matrix W with nonzero entries which satisfies two conditions (type II and type III conditions). In his definition of a spin model, Jones considered only symmetric matrices. It was generalized to non-symmetric case by Kawagoe-Munemasa-Watatani [8].

Recently, François Jaeger and the second author [6] introduced the notion of "index" of a spin model. For every spin model W, the transpose ${}^tW$ is obtained from W by a permutation of rows. Let σ denote the corresponding permutation of $X = \{1, \ldots, n\}$ (n is the size of W). Then the index m is the order of σ. In [6], it was shown that X is partitioned into m subsets $X_0, X_1, \ldots, X_{m-1}$ such that $W(x, y) = \eta^{i-j} W(y, x)$ holds for all $x \in X_i$, $y \in X_j$. Moreover, the case of $m = 2$ was deeply investigated, and a general form of spin models of index 2 was given.

In the present paper, we investigate the structure of spin models of an arbitrary index m. In Section 4, we show that W is decomposed into blocks W_{ij}, and W_{ij} splits into Kronecker product of two matrices S_{ij} and T_{ij} (Proposition 4.3). In Section 5, we give conditions on T_{ij} (Propositions 5.1 and 5.5). In Section 6, we apply this general form to some special cases (Propositions 6.1 and 6.2). In particular, we give a simple form of W when the index m is a prime number (Corollary 6.3).

2. Preliminaries

In this section, we give some basic materials concerning spin models and association schemes. For more details the reader can refer to [4–7].

Let X be a finite non-empty set with n elements. We denote by $Mat_X(\mathbf{C})$ the set of square matrices with complex entries whose rows and columns are indexed by X. For $W \in Mat_X(\mathbf{C})$ and $x, y \in X$, the (x, y)-entry of W is denoted by $W(x, y)$.

A *type II matrix* on X is a matrix $W \in Mat_X(\mathbf{C})$ with nonzero entries which satisfies the *type II condition*:

$$\sum_{x \in X} \frac{W(a, x)}{W(b, x)} = n\delta_{a,b} \quad (\text{for all } a, b \in X). \tag{1}$$

Let $W^- \in Mat_X(\mathbf{C})$ be defined by $W^-(x, y) = W(y, x)^{-1}$. Then type II condition is written as $WW^- = nI$ (I denotes the identity matrix). Hence, if W is a type II matrix, then W is non-singular with $W^{-1} = n^{-1}W^-$. It is clear that W^{-1} and ${}^tW$ are also type II matrices.

A type II matrix W is called a *spin model* on X if W satisfies *type III condition*:

$$\sum_{x \in X} \frac{W(a, x)W(b, x)}{W(c, x)} = D\frac{W(a, b)}{W(a, c)W(c, b)} \quad (\text{for all } a, b, c \in X) \tag{2}$$

for some nonzero complex number D. The number D is called the *loop variable* of W. Setting $b = c$ in (2), $\sum_{x \in X} W(a, x) = DW(b, b)^{-1}$ holds, so that the diagonal entries $W(b, b)$ is a constant, which is called the *modulus* of W.

For a spin model W with loop variable D, any nonzero scalar multiple λW is a spin model with loop variable $\lambda^2 D$. Usually W is normalized so that $D^2 = n$, but we allow any nonzero value of D in this paper to simplify our arguments.

Observe that, for any spin models W_i on X_i with loop variable D_i ($i = 1, 2$), their tensor (Kronecker) product $W_1 \otimes W_2$ is a spin model with loop variable $D = D_1 D_2$. Conversely, it is not difficult to show that, if $W_1 \otimes W_2$ and W_1 are spin models, then W_2 must be a spin model.

A (*class* d) *association scheme* on X is a partition of $X \times X$ with nonempty relations $R_0, R_1, \ldots, R_d$, where $R_0 = \{(x, x) \mid x \in X\}$ which satisfy the following conditions:

- (i) For every i in $\{0, 1, \ldots, d\}$, there exists i' in $\{0, 1, \ldots, d\}$ such that $R_{i'} = \{(y, x) \mid (x, y) \in R_i\}$.
- (ii) There exist integers p_{ij}^k ($i, j, k \in \{0, 1, \ldots, d\}$) such that for every $(x, y) \in R_k$, there are precisely p_{ij}^k elements z such that $(x, z) \in R_i$ and $(z, y) \in R_j$.
- (iii) $p_{ij}^k = p_{ji}^k$ for every i, j in $\{0, 1, \ldots, d\}$.

Let A_i denote the adjacency matrix of the relation R_i, so $A_i \in Mat_X(\mathbf{C})$ is a $\{0, 1\}$-matrix whose (x, y)-entry is equal to 1 if and only if $(x, y) \in R_i$. Clearly $A_0 = I$, $A_i \circ A_j = \delta_{i,j} A_i$

(entry-wise product), $\sum_{i=0}^{d} A_i = J$ (all 1's matrix), and $A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$ hold. The linear span $\mathcal{A}$ of $\{A_0, A_1, \ldots, A_d\}$ becomes a subalgebra of $Mat_X(\mathbf{C})$, called the *Bose-Mesner algebra* of the association scheme. Observe that $\mathcal{A}$ is closed under entry-wise product, $\mathcal{A}$ is closed under transposition $A \mapsto {}^tA$, and $\mathcal{A}$ contains I, J.

3. Associated permutation

Let W be a spin model on X. Then there exists an association scheme $R_0, \ldots, R_d$ on X such that the corresponding Bose-Mesner algebra $\mathcal{A}$ contains W ([5] Theorem 11). In [6], it was shown that ${}^tWW^{-1} = A_s$ (the adjacency matrix of R_s) for some $s \in \{0, 1, \ldots, d\}$, and moreover A_s is a permutation matrix ([6] Proposition 2). Let σ denote the corresponding permutation on X, so that $A_s(x, y) = 1$ if $y = \sigma(x)$ and $A_s(x, y) = 0$ otherwise. The order m of σ is called the *index* of W.

Observe that $m = 1$ if and only if W is symmetric. Also observe that, for two spin models W_i of index m_i $(i = 1, 2)$, the index of $W_1 \otimes W_2$ is equal to the least common multiple of m_1 and m_2. In particular, tensor product of a spin model of index m with any symmetric spin model has index m.

Lemma 3.1

(i) $W(x, \sigma(x)) = W(y, \sigma(y))$ $(x, y \in X)$.
(ii) $W(y, x) = W(\sigma(x), y)$ $(x, y \in X)$.
(iii) *Every orbit of σ has length m.*

Proof:

(i) Observe that, since $W \in \mathcal{A}$, W is written as a linear combination $W = \sum_{i=0}^{d} t_i A_i$, so $W(x, y) = t_i$ for $(x, y) \in R_i$. Since $(x, \sigma(x)) \in R_s$ (for every $x \in X$), it holds that $W(x, \sigma(x)) = t_s = W(y, \sigma(y))$.
(ii) $W(y, x) = {}^tW(x, y) = (A_s W)(x, y) = W(\sigma(x), y)$.
(iii) Pick any i $(0 < i < m)$. Since A_s^i is a linear combination of $A_0, \ldots, A_d$ and since A_s^i is a permutation matrix, we get $A_s^i = A_j$ for some $j \neq 0$. Observe that the diagonal entries of A_j are all zero since $j \neq 0$. This means that σ^i (which corresponds the permutation matrix A_j) has no fixed point on X. We have shown that σ^i fixes no point $(0 < i < m)$. Thus every orbit of σ must have length m. □

Lemma 3.2 *There is a partition $X = X_0 \cup \cdots \cup X_{m-1}$ such that (for all $i, j \in \{0, \ldots, m-1\}$)*

$$W(x, y) = \eta^{i-j} W(y, x) \quad (\text{for all } x \in X_i,\ y \in X_j), \tag{3}$$

where η denotes a primitive m-root of unity. Moreover, for every i, $\sigma(X_i) = X_j$ holds for some j.

Proof: The existence of such a partition follows from [6] Proposition 3. As in the proof of Lemma 3.1(i), we have $(x, \sigma(x)) \in R_s$ and $W(x, \sigma(x)) = t_s$ for all $x \in X$. Then there exists s' such that $(\sigma(x), x) \in R_{s'}$, so that $W(\sigma(x), x) = t_{s'}$. Now pick any $x \in X_i$. Then $\sigma(x) \in X_j$ for some j. On the other hand, $W(x, \sigma(x)) = \eta^{i-j} W(\sigma(x), x)$. These imply $\eta^{i-j} = t_s t_{s'}^{-1}$. This means that j is independent of the choice of $x \in X_i$, so that $\sigma(X_i) = X_j$. □

We fix a primitive m-root of unity η, and let $X_0, \ldots, X_{m-1}$ be the partition of X given in Lemma 3.2. We identify the index set $\{0, 1, \ldots, m-1\}$ with $\mathbf{Z}_m = \mathbf{Z}/m\mathbf{Z}$. By Lemma 3.2, there is a permutation π on $\mathbf{Z}_m$ such that $\sigma(X_i) = X_{\pi(i)}$ $(i \in \mathbf{Z}_m)$. Let t denote the order of π, and set $k = m/t$.

Lemma 3.3 $\pi(i) - i = \pi(j) - j$ *for all* $i, j \in \mathbf{Z}_m$.

Proof: Pick any $x \in X_i$, $y \in X_j$. We have $\sigma(x) \in X_{\pi(i)}$, $\sigma(y) \in X_{\pi(j)}$. By Lemma 3.2, $W(x, \sigma(x)) = \eta^{i-\pi(i)} W(\sigma(x), x)$ and $W(y, \sigma(y)) = \eta^{i-\pi(j)} W(\sigma(y), y)$. On the other hand, $W(x, \sigma(x)) = W(y, \sigma(y))$ by Lemma 3.1(i), and also $W(\sigma(x), x) = W(x, x) =$ (the modulus of W) $= W(y, y) = W(\sigma(y), y)$ by Lemma 3.1(ii). These imply $\eta^{i-\pi(i)} = \eta^{j-\pi(j)}$. □

Lemma 3.4 *There exists an automorphism* φ *of the additive group* $\mathbf{Z}_m$ *such that* $\pi(\varphi(i)) = \varphi(i + k)$ *for all* $i \in \mathbf{Z}_m$. *Moreover*, $W(x, y) = (\eta^{\varphi(1)})^{i-j} W(y, x)$ *for every* $x \in X_{\varphi(i)}$, $y \in X_{\varphi(j)}$.

Proof: Set $k' = \pi(0)$. Then $\pi(i) = i + k'$ $(i \in \mathbf{Z}_m)$ by Lemma 3.3. Thus $k'\mathbf{Z}_m = \{0, k', 2k', \ldots, (t-1)k'\}$ is an orbit of π. Note that every orbit of π has length t, and hence the number of orbits of π is equal to $k = m/t$ (in particular, k must be an integer). Clearly $k'\mathbf{Z}_m$ is the unique subgroup of $\mathbf{Z}_m$ of order t, so $k'\mathbf{Z}_m = k\mathbf{Z}_m$. Hence there is an automorphism φ of the additive group $k\mathbf{Z}_m$ such that $\varphi(k) = k'$.

We claim that φ can be extended to an automorphism of $\mathbf{Z}_m$. In fact, for any cyclic group G and for any subgroup H of G, any automorphism of H can be extended to an automorphism of G. This fact can be easily shown when G is a cyclic p-group. For general case, decompose G into the Sylow subgroups.

Now we have an automorphism φ of $\mathbf{Z}_m$ such that $\varphi(k) = k'$. Since $\pi(i) = i + k'$ for all $i \in \mathbf{Z}_m$, we get $\pi(\varphi(i)) = \varphi(i) + k' = \varphi(i) + \varphi(k) = \varphi(i + k)$.

Let $x \in X_{\varphi(i)}$, $y \in X_{\varphi(j)}$. Then, by Lemma 3.2, $W(x, y) = \eta^{\varphi(i)-\varphi(j)} W(y, x)$ holds for all $x \in X_{\varphi(i)}$, $y \in X_{\varphi(j)}$. Here $\varphi(i) - \varphi(j) = \varphi(i \cdot 1) - \varphi(j \cdot 1) = i\varphi(1) - j\varphi(1) = \varphi(1)(i - j)$. Hence $W(x, y) = (\eta^{\varphi(1)})^{i-j} W(y, x)$. □

Thus, by reordering the indices $\{0, 1, \ldots, m-1\}$ by φ, and by replacing η with $\eta^{\varphi(1)}$, we may assume that

$$\pi(i) = i + k \quad (i \in \mathbf{Z}_m). \tag{4}$$

4. General form of W

We use the notation of the previous section. We also use the notation:

$$\gamma_k(\ell, i) = \eta^{-\ell i-(k/2)\ell(\ell-1)}. \tag{5}$$

Proposition 4.1 *Let $i, j \in \mathbf{Z}_m$ and $x \in X_i$, $y \in X_j$. Then for $\ell, \ell' \in \mathbf{Z}$,*

$$W(\sigma^{\ell}(x),\ \sigma^{\ell'}(y)) = \gamma_k(\ell-\ell',\ i-j)\, W(x,y). \tag{6}$$

Proof: Assume $\ell \geq 0$ and $\ell' \geq 0$. First we consider the case of $\ell' = 0$. We proceed by induction on ℓ. Obviously (6) holds for $\ell = 0$. By Lemma 3.1(ii) and Lemma 3.2, $W(y,x) = W(\sigma(x), y)$ and $W(y,x) = \eta^{j-i}W(x,y)$. Hence $W(\sigma(x), y) = \eta^{j-i}W(x,y)$, so (6) holds for $\ell = 1$. Now assume $\ell > 1$. Noting $\sigma(x) \in X_{\pi(i)} = X_{i+k}$ and using induction,

$$\begin{aligned} W(\sigma^{\ell}(x), y) &= W(\sigma^{\ell-1}(\sigma(x)), y) \\ &= \gamma_k(\ell-1, (i+k)-j)W(\sigma(x), y) \\ &= \gamma_k(\ell-1, (i+k)-j)\eta^{j-i}W(x,y) \\ &= \gamma_k(\ell, i-j)W(x,y). \end{aligned}$$

Hence (6) holds for $\ell' = 0$. Now suppose $\ell' > 0$. Noting $\sigma^{\ell'}(y) \in X_{j+\ell' k}$ and using Lemma 3.2,

$$\begin{aligned} W(\sigma^{\ell}(x), \sigma^{\ell'}(y)) &= \gamma_k(\ell, i-(j+\ell' k))W(x, \sigma^{\ell'}(y)) \\ &= \gamma_k(\ell, i-(j+\ell' k))\eta^{i-(j+\ell' k)}W(\sigma^{\ell'}(y), x) \\ &= \gamma_k(\ell, i-(j+\ell' k))\eta^{i-(j+\ell' k)}\gamma_k(\ell', j-i)W(y,x) \\ &= \gamma_k(\ell, i-(j+\ell' k))\eta^{i-(j+\ell' k)}\gamma_k(\ell', j-i)\eta^{j-i}W(x,y) \\ &= \gamma_k(\ell-\ell', i-j)W(x,y). \end{aligned}$$

Thus (6) holds for non-negative integers ℓ, ℓ'.

Since $\sigma^{-\ell}(x) \in X_{i-\ell k}$,

$$W(x,y) = W(\sigma^{\ell}(\sigma^{-\ell}(x)), y) = \gamma_k(\ell, (i-\ell k)-j)W(\sigma^{-\ell}(x), y).$$

Hence

$$\begin{aligned} W(\sigma^{-\ell}(x), y) &= \gamma_k(\ell, i-\ell k-j)^{-1}W(x,y) \\ &= \eta^{\ell(i-\ell k-j)+(k/2)\ell(\ell-1)}W(x,y) \\ &= \eta^{\ell(i-j)+(k/2)\ell(\ell+1)}W(x,y) \\ &= \gamma_k(-\ell, i-j)W(x,y). \end{aligned}$$

Since $\sigma^{-\ell'}(y) \in X_{j-\ell' k}$,

$$W(x,y) = W(x, \sigma^{\ell'}(\sigma^{-\ell'}(y))) = \gamma_k(-\ell', i-(j-\ell' k))W(x, \sigma^{-\ell'}(y)).$$

Hence

$$W(x,\sigma^{-\ell'}(y)) = \gamma_k(-\ell', i-j+\ell' k)^{-1} W(x,y)$$
$$= \gamma_k(\ell', i-j) W(x,y).$$

Since $\sigma^{\ell'}(y) \in X_{j+\ell' k}$,

$$W(\sigma^{-\ell}(x),\sigma^{\ell'}(y)) = \gamma_k(-\ell, i-(j+\ell' k)) W(x,\sigma^{\ell'}(y))$$
$$= \gamma_k(-\ell, i-j-\ell' k)\gamma_k(-\ell', i-j) W(x,y)$$
$$= \gamma_k(-\ell-\ell', i-j) W(x,y).$$

Similarly, we can show that

$$W(\sigma^{\ell}(x),\sigma^{-\ell'}(y)) = \gamma_k(\ell+\ell', i-j) W(x,y),$$

and

$$W(\sigma^{-\ell}(x),\sigma^{-\ell'}(y)) = \gamma_k(-\ell+\ell', i-j) W(x,y).$$

This completes the proof of (6). □

Lemma 4.2 *If m is even, then k is even.*

Proof: We apply Proposition 4.1 for $\ell = m$, $\ell' = 0$ and $i = j$. Then (6) implies $\gamma_k(m,0) = 1$, and this becomes $(\eta^{-m/2})^{k(m-1)} = 1$. Observe that $\eta^{-m/2} = -1$, since η is a primitive m-root of unity and m is even. Hence $(-1)^{k(m-1)} = 1$, so that k must be even. □

For $i \in \mathbf{Z}_m$, set

$$\Delta_i = \bigcup_{h=0}^{t-1} X_{i+hk}.$$

Observe that $|\Delta_i| = t(n/m) = tn/(kt) = n/k$, and that

$$X = \bigcup_{i=0}^{k-1} \Delta_i,$$

Since $\sigma(\Delta_i) = \Delta_i$, Δ_i is partitioned into σ-orbits Y_α^i:

$$\Delta_i = \bigcup_{\alpha=1}^{r} Y_\alpha^i \quad (i = 0, \ldots, k-1),$$

where $r = |\Delta_i|/m = n/(mk)$. Observe that $|Y^i_\alpha| = m$ and $|Y^i_\alpha \cap X_i| = k$. We choose representative elements

$$y^i_\alpha \in Y^i_\alpha \cap X_i \quad (i = 0, \ldots, k-1, \alpha = 1, \ldots, r).$$

Then

$$X = \left\{\sigma^\ell\left(y^i_\alpha\right) \,\middle|\, i = 0, \ldots, k-1, \alpha = 1, \ldots, r, \ell = 0, \ldots, m-1\right\}, \tag{7}$$

and

$$W\left(\sigma^\ell\left(y^i_\alpha\right), \sigma^{\ell'}\left(y^j_\beta\right)\right) = \gamma_k(\ell - \ell', i - j)\, W\left(y^i_\alpha, y^j_\beta\right) \tag{8}$$

for $\ell, \ell' \in \mathbf{Z}_m$, $i, j = 0, \ldots, k-1$ and $\alpha, \beta = 1, \ldots, r$.

We define square matrices T_{ij} of size r and S_{ij} of size m $(i, j = 0, \ldots, k-1)$ by

$$T_{ij}(\alpha, \beta) = W\left(y^i_\alpha, y^j_\beta\right) \quad (\alpha, \beta = 1, \ldots, r),$$
$$S_{ij}(\ell, \ell') = \gamma_k(\ell - \ell', i - j) \quad (\ell, \ell' = 0, \ldots, m-1).$$

For subsets A, B of X, let $W|_{A\times B}$ denote the restriction (submatrix) of W on $A \times B$. For two matrices S, T, we denote the Kronecker product by $S \otimes T$.

Proposition 4.3 *For* $i, j = 0, \ldots, k-1$,

$$W|_{Y^i_\alpha \times Y^j_\beta} = T_{ij}(\alpha, \beta)\, S_{ij} \quad (\alpha, \beta = 1, \ldots, r),$$

and

$$W|_{\Delta_i \times \Delta_j} = S_{ij} \otimes T_{ij}. \tag{9}$$

Proof: Clear. □

Thus W decomposes into blocks $W_{ij} = W|_{\Delta_i \times \Delta_j}$ $(i, j = 0, \ldots, k-1)$, and each block has the form $W_{ij} = S_{ij} \otimes T_{ij}$ $(i, j = 0, \ldots, k-1)$.

5. Type II and Type III conditions

Let m, k, t, r be positive integers with $m = kt$.

Let T_{ij} $(i, j = 0, \ldots, k-1)$ be any matrices of size r with nonzero entries, and let S_{ij} $(i, j = 0, \ldots, k-1)$ be the matrix of size m defined by

$$S_{ij}(\ell, \ell') = \gamma_k(\ell - \ell', i - j) \quad (\ell, \ell' = 0, \ldots, m-1),$$

where γ_k is defined by (5) for a primitive m-root of unity η. Now set

$$W_{ij} = S_{ij} \otimes T_{ij} \quad (i, j = 0, \dots, k-1),$$

and let W be the matrix of size $n = kmr$ whose (i, j) block is W_{ij} $(i, j = 0, \dots, k-1)$. We index the rows and the columns of W by the set:

$$X = \{[i, \ell, \alpha] \mid 0 \le i \le k-1,\ 0 \le \ell \le m-1,\ 1 \le \alpha \le r\},$$

so that

$$W([i, \ell, \alpha],\ [j, \ell', \beta]) = S_{ij}(\ell, \ell')T_{ij}(\alpha, \beta). \tag{10}$$

Proposition 5.1 *W is a type II matrix if and only if T_{ij} is a type II matrix for all i, $j \in \{0, \dots, k-1\}$.*

Proof: The type II condition (1) for $a = [i_1, \ell_1, \alpha_1]$, $b = [i_2, \ell_2, \alpha_2]$ becomes

$$\sum_{i=0}^{k-1}\sum_{\ell=0}^{m-1}\sum_{\alpha=1}^{r} \frac{W([i_1, \ell_1, \alpha_1],\ [i, \ell, \alpha])}{W([i_2, \ell_2, \alpha_2],\ [i, \ell, \alpha])} = n\delta_{i_1,i_2}\delta_{\ell_1,\ell_2}\delta_{\alpha_1,\alpha_2}. \tag{11}$$

Using (10), we rewrite the left-hand-side as follows:

$$\begin{aligned}\text{l.h.s.} &= \sum_{i=0}^{k-1}\sum_{\ell=0}^{m-1}\sum_{\alpha=1}^{r} \frac{\gamma_k(\ell_1 - \ell, i_1 - i)T_{i_1 i}(\alpha_1, \alpha)}{\gamma_k(\ell_2 - \ell, i_2 - i)T_{i_2 i}(\alpha_2, \alpha)}\\ &= \eta^{-\ell_1 i_i + \ell_2 i_2 - (k/2)(\ell_1-\ell_2)(\ell_1+\ell_2-1)} \sum_{i=0}^{k-1} \eta^{(\ell_1-\ell_2)i} \sum_{\alpha=1}^{r} \frac{T_{i_1 i}(\alpha_1, \alpha)}{T_{i_2 i}(\alpha_2, \alpha)} \sum_{\ell=0}^{m-1} \eta^{(i_1 - i_2 + k(\ell_1-\ell_2))\ell}.\end{aligned}$$

Observe that, since η is a primitive m-root of unity,

$$\sum_{\ell=0}^{m-1} \eta^{((i_1-i_2)+k(\ell_1-\ell_2))\ell} = \begin{cases} m & \text{if } (i_1 - i_2) + k(\ell_1 - \ell_2) \equiv 0 \pmod{m},\\ 0 & \text{otherwise.}\end{cases}$$

Observe that $(i_1 - i_2) + k(\ell_1 - \ell_2) \equiv 0 \pmod{m}$ if and only if $i_1 = i_2$ and $\ell_1 \equiv \ell_2 \pmod{t}$, since $0 \le i_1,\ i_2 \le k-1$ and $m = kt$.

Now suppose that T_{ij} are type II $(i, j = 0, \dots, k-1)$. We must show that the l.h.s. of (11) becomes zero for $[i_1, \ell_1, \alpha_1] \ne [i_2, \ell_2, \alpha_2]$. By the above observation, we may assume that $i_1 = i_2$ and $\ell_1 \equiv \ell_2 \pmod{t}$. We set $\ell_1 - \ell_2 = ts$. If $\alpha_1 \ne \alpha_2$, then l.h.s. of (11) vanishes by type II condition for $T_{i_1 i}$. Hence we may assume $\alpha_1 = \alpha_2$. Thus we have $i_1 = i_2$, $\alpha_1 = \alpha_2$, $\ell_1 - \ell_2 \equiv 0 \pmod{t}$ and $\ell_1 \ne \ell_2$. Hence

$$\text{l.h.s.} = mr\eta^{-\ell_1 i_1 + \ell_2 i_2 - (k/2)(\ell_1-\ell_2)(\ell_1+\ell_2-1)} \sum_{i=0}^{k-1} \eta^{tsi}.$$

Observe that η^t is a primitive k-root of unity. So, $\sum_{i=0}^{k-1}(\eta^t)^{si}=0$, since $s \not\equiv 0 \pmod{k}$. We have shown that W is type II.

Next suppose that W is type II. Pick any distinct $\alpha_1, \alpha_2 \in \{1, \dots, r\}$. From (11) at $i_1 = i_2$ and $\ell_1 \equiv \ell_2 \pmod{t}$, we obtain

$$\sum_{i=0}^{k-1} \eta^{(\ell_1-\ell_2)i} \sum_{\alpha=1}^{r} \frac{T_{i_i i}(\alpha_1, \alpha)}{T_{i_2 i}(\alpha_2, \alpha)} = 0.$$

Setting

$$K_i = \sum_{\alpha=1}^{r} \frac{T_{i_1 i}(\alpha_1, \alpha)}{T_{i_1 i}(\alpha_2, \alpha)}$$

and considering the case $\ell_2 = 0$, the above equation implies

$$\sum_{i=0}^{k-1} \left(\eta^{\ell_1}\right)^i K_i = 0 \quad (\ell_1 = 0, t, 2t, \dots, (k-1)t),$$

or equivalently

$$\sum_{i=0}^{k-1} (\eta^t)^{ei} K_i = 0 \quad (e = 0, 1, \dots, k-1).$$

Observe that $(\eta^t)^e$ $(e = 0, 1, \dots, k-1)$ are distinct, since η^t is a primitive k-root of unity. Hence $K_i = 0$ $(i = 0, 1, \dots, k-1)$ by Vandermonde determinant. Thus

$$\sum_{\alpha=1}^{r} \frac{T_{i_1 i}(\alpha_1, \alpha)}{T_{i_1 i}(\alpha_2, \alpha)} = 0 \quad (i = 0, \dots, k-1),$$

so that $T_{i_1 i}$ is type II. □

Lemma 5.2 *Assume k is even when m is even. Then the matrix W satisfies the type III condition* (2) *if and only if the following equation holds for all $i_1, i_2, i_3 \in \{0, \dots, k-1\}$ and for all $\alpha_1, \alpha_2, \alpha_3 \in \{1, \dots, r\}$:*

$$\sum_{i=0}^{k-1} \left(\sum_{\ell=0}^{m-1} \eta^{-k\ell} \gamma_k(\ell, i - i_1 - i_2 + i_3) \right) \left(\sum_{\alpha=1}^{r} \frac{T_{i_1,i}(\alpha_1, \alpha) T_{i_2,i}(\alpha_2, \alpha)}{T_{i_3,i}(\alpha_3, \alpha)} \right)$$
$$= D \frac{T_{i_1,i_2}(\alpha_1, \alpha_2)}{T_{i_1,i_3}(\alpha_1, \alpha_3) T_{i_3,i_2}(\alpha_3, \alpha_2)}.$$

Proof: The type III condition (2) for $a = [i_1, \ell_1, \alpha_1]$, $b = [i_2, \ell_2, \alpha_2]$, $c = [i_3, \ell_3, \alpha_3]$ becomes

$$\sum_{i=0}^{k-1}\sum_{\ell=0}^{m-1}\sum_{\alpha=1}^{r} \frac{\gamma_k(\ell_1-\ell, i_1-i)\gamma_k(\ell_2-\ell, i_2-i)}{\gamma_k(\ell_3-\ell, i_3-i)} \cdot \frac{T_{i_1,i}(\alpha_1,\alpha)T_{i_2,i}(\alpha_2,\alpha)}{T_{i_3,i}(\alpha_3,\alpha)}$$
$$= D\frac{\gamma_k(\ell_1-\ell_2, i_1-i_2)}{\gamma_k(\ell_1-\ell_3, i_1-i_3)\gamma_k(\ell_3-\ell_2, i_3-i_2)} \cdot \frac{T_{i_1,i_2}(\alpha_1,\alpha_2)}{T_{i_1,i_3}(\alpha_1,\alpha_3)T_{i_3,i_2}(\alpha_3,\alpha_2)}.$$

By a direct (but somewhat long) computation, we obtain

$$\frac{\gamma_k(\ell_1-\ell, i_1-i)\gamma_k(\ell_2-\ell, i_2-i)}{\gamma_k(\ell_3-\ell, i_3-i)} \cdot \frac{\gamma_k(\ell_1-\ell_3, i_1-i_3)\gamma_k(\ell_3-\ell_2, i_3-i_2)}{\gamma_k(\ell_1-\ell_2, i_1-i_2)}$$
$$= \eta^{-k(\ell-\hat{\ell})}\gamma_k(\ell-\hat{\ell}, i-\hat{i}),$$

where $\hat{\ell} = \ell_1+\ell_2-\ell_3$, $\hat{i} = i_1+i_2-i_3$. So the type III condition becomes

$$\sum_{i=0}^{k-1}\left(\sum_{\ell=0}^{m-1}\eta^{-k(\ell-\hat{\ell})}\gamma_k(\ell-\hat{\ell}, i-\hat{i})\right)\left(\sum_{\alpha=1}^{r}\frac{T_{i_1,i}(\alpha_1,\alpha)T_{i_2,i}(\alpha_2,\alpha)}{T_{i_3,i}(\alpha_3,\alpha)}\right)$$
$$= D\frac{T_{i_1,i_2}(\alpha_1,\alpha_2)}{T_{i_1,i_3}(\alpha_1,\alpha_3)T_{i_3,i_2}(\alpha_3,\alpha_2)}.$$

To complete our proof, we must show that

$$\sum_{\ell=0}^{m-1}\eta^{-k(\ell-\hat{\ell})}\gamma_k(\ell-\hat{\ell}, i-\hat{i}) = \sum_{\ell=0}^{m-1}\eta^{-k\ell}\gamma_k(\ell, i-\hat{i}).$$

To show this, it is enough to show that $\gamma_k(\ell+m, j) = \gamma_k(\ell, j)$ holds for all j, ℓ.

$$\begin{aligned}\gamma_k(\ell+m, j) &= \eta^{-(\ell+m)j-(k/2)(\ell+m)(\ell+m-1)}\\ &= \gamma_k(\ell, j)\eta^{-mj-km\ell}\eta^{-(k/2)m(m-1)}\\ &= \gamma_k(\ell, j)\eta^{-km(m-1)/2}.\end{aligned}$$

When m is odd, $(m-1)/2$ is an integer. When m is even, k is even by our assumption, and so $k/2$ is an integer. Thus $\eta^{-km(m-1)/2} = 1$. □

Lemma 5.3 *For all* u, s $(0 \le u \le t-1, 0 \le s \le k-1)$,

$$\gamma_k(u+st, j) = ((-1)^{t-1}\eta^{-tj})^s\gamma_k(u, j).$$

Proof: We compute $\gamma_k(u+st, j)$ as follows.

$$\begin{aligned}\gamma_k(u+st, j) &= \eta^{-(u+st)j-(k/2)(u+st)(u+st-1)}\\ &= \eta^{-uj-(k/2)u(u-1)}\eta^{-stj}\eta^{-(kt)su}\eta^{-(k/2)st(st-1)}\\ &= \gamma_k(u, j)\eta^{-stj}\eta^{-(k/2)st(st-1)}.\end{aligned}$$

So it is enough to show that

$$\eta^{-(k/2)st(st-1)} = (-1)^{(t-1)s}. \tag{12}$$

If t is even, then m is even and so $\eta^{(m/2)} = -1$. Hence, noting $st-1$ is odd,

$$\eta^{-(k/2)st(st-1)} = \left(\eta^{-(m/2)}\right)^{(st-1)s} = \left((-1)^{(st-1)}\right)^{s} = (-1)^{s},$$

so (12) holds. Next assume t is odd. If s is even, then

$$\eta^{-(k/2)st(st-1)} = \eta^{-(kt)(s/2)(st-1)} = \eta^{-m(s/2)(st-1)} = 1.$$

If s is odd, then $st-1$ is even. Hence

$$\eta^{-(k/2)st(st-1)} = \eta^{-(kt)s(st-1)/2} = (\eta^{-m})^{s(st-1)/2} = 1.$$

Therefore (12) holds in each case. □

Lemma 5.4
(i) *If t is odd, then*

$$\sum_{\ell=0}^{m-1} \eta^{-k\ell}\gamma_k(\ell, j) = \begin{cases} k\displaystyle\sum_{u=0}^{t-1}\eta^{-uj-ku(u+1)/2} & \text{if } j \equiv 0 \pmod{k}, \\ 0 & \text{otherwise}. \end{cases}$$

(ii) *If t and k are even, then*

$$\sum_{\ell=0}^{m-1} \eta^{-k\ell}\gamma_k(\ell, j) = \begin{cases} k\displaystyle\sum_{u=0}^{t-1}\eta^{-uj-ku(u+1)/2} & \text{if } j \equiv \dfrac{k}{2} \pmod{k}, \\ 0 & \text{otherwise}. \end{cases}$$

Proof: Using Lemma 5.3, we proceed as follows.

$$\begin{aligned}
\sum_{\ell=0}^{m-1} \eta^{-k\ell}\gamma_k(\ell, j) &= \sum_{u=0}^{t-1}\sum_{s=0}^{k-1}\eta^{-k(u+st)}\gamma_k(u+st, j) \\
&= \sum_{u=0}^{t-1}\sum_{s=0}^{k-1}\eta^{-(ku+ms)}((-1)^{t-1}\eta^{-tj})^{s}\gamma_k(u, j) \\
&= \left(\sum_{s=0}^{k-1}((-1)^{t-1}\eta^{-tj})^{s}\right)\left(\sum_{u=0}^{t-1}\eta^{-ku}\gamma_k(u, j)\right).
\end{aligned}$$

If t is odd, then the first factor becomes

$$\sum_{s=0}^{k-1}(\eta^{-tj})^{s} = \sum_{s=0}^{k-1}(\eta^{-t})^{js} = \begin{cases} k & \text{if } j \equiv 0 \pmod{k}, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose t and k are even. In this case, m is also even, so that $\eta^{(kt/2)} = \eta^{(m/2)} = -1$. Hence the first factor becomes

$$\begin{aligned}\sum_{s=0}^{k-1}((-1)\eta^{-tj})^s &= \sum_{s=0}^{k-1}\left(\eta^{(kt/2)}\eta^{-tj}\right)^s \\ &= \sum_{s=0}^{k-1}(\eta^t)^{((k/2)-j)s} \\ &= \begin{cases} k & \text{if } \dfrac{k}{2} - j \equiv 0 \pmod{k}, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Now the result follows by

$$\eta^{-ku}\gamma_k(u, j) = \eta^{-uj-ku(u+1)/2}. \qquad \square$$

Proposition 5.5 *Assume k is even when m is even. Then the matrix W satisfies the type III condition* (2) *if and only if the following equation holds for all $i_1, i_2, i_3 \in \{0, \dots, k-1\}$ and for all $\alpha_1, \alpha_2, \alpha_3 \in \{1, \dots, r\}$*:

$$\left(\sum_{u=0}^{t-1}\eta^{-u(i-\hat{i})-ku(u+1)/2}\right)\left(\sum_{\alpha=1}^{r}\frac{T_{i_1,i}(\alpha_1,\alpha)T_{i_2,i}(\alpha_2,\alpha)}{T_{i_3,i}(\alpha_3,\alpha)}\right) = (D/k)\frac{T_{i_1,i_2}(\alpha_1,\alpha_2)}{T_{i_1,i_3}(\alpha_1,\alpha_3)T_{i_3,i_2}(\alpha_3,\alpha_2)},$$

where $\hat{i} = i_1 + i_2 - i_3$, and i denotes the integer in $\{0, \dots, k-1\}$ such that

$$i \equiv \begin{cases}\hat{i} \pmod{k} & \text{if } t \text{ is odd},\\ \hat{i} + \dfrac{k}{2} \pmod{k} & \text{if } t \text{ is even}.\end{cases}$$

Proof: This is a direct consequence of Lemmas 5.2 and 5.4. □

6. Some special cases

We use the notation in Section 4.

Proposition 6.1 *Suppose $k = 1$. Then m is odd, and*

$$W = S \otimes T,$$

where S is a spin model of size m and index m which is given by

$$S(\ell, \ell') = \eta^{-(1/2)(\ell-\ell')(\ell-\ell'-1)} \quad (\ell, \ell' = 0, 1, \dots, m-1),$$

and T is a symmetric spin model of size n/m.

Proof: When $k = 1$, we have $X = \Delta_0$ and $r = n/m$. Setting $S = S_{00}$ and $T = T_{00}$, $W = W|_{\Delta_0} = S \otimes T$ by Proposition 4.3. Obviously $S(\ell, \ell') = S_{00}(\ell, \ell') = \gamma_1(\ell - \ell', 0)$. For $\alpha, \beta \in \{1, \ldots, r\}$, $T(\alpha, \beta) = W(y^0_\alpha, y^0_\beta) = \eta^{0-0} W(y^0_\beta, y^0_\alpha) = T(\beta, \alpha)$, so that T is symmetric.

Since m is odd by Lemma 4.2, S is a spin model on the cyclic group of order m, which was constructed in [2] (see also [1, 3]). Since $W = S \otimes T$ with W, S are spin models, T must be a spin model. □

Proposition 6.2 *Suppose $k = m$. Then*

$$W|_{X_i \times X_j} = S_{ij} \otimes T_{ij} \quad (i, j = 0, 1, \ldots, m-1),$$

and

$$S_{ij}(\ell, \ell') = \eta^{-(\ell-\ell')(i-j)} \quad (\ell, \ell' = 0, \ldots, m-1).$$

The matrices T_{ij} are type II matrices of size $r = n/m^2$. Moreover the following equation holds for all $i_1, i_2, i_3 \in \{0, \ldots, m-1\}$ and for all $\alpha_1, \alpha_2, \alpha_3 \in \{1, \ldots, r\}$:

$$\sum_{\alpha=1}^{r} \frac{T_{i_1,i}(\alpha_1, \alpha) T_{i_2,i}(\alpha_2, \alpha)}{T_{i_3,i}(\alpha_3, \alpha)} = (D/m) \frac{T_{i_1,i_2}(\alpha_1, \alpha_2)}{T_{i_1,i_3}(\alpha_1, \alpha_3) T_{i_3,i_2}(\alpha_3, \alpha_2)},$$

where i denotes the integer in $\{0, \ldots, m-1\}$ such that $i \equiv i_1 + i_2 - i_3 \pmod m$.

Proof: We have $t = 1$, $r = n/(mk) = n/m^2$ and $X_i = \Delta_i$ $(i = 0, \ldots, m-1)$. Hence $W|_{X_i \times X_j} = S_{ij} \otimes T_{ij}$ by Proposition 4.3. Since $\eta^{(m/2)\ell(\ell-1)} = (\eta^m)^{\ell(\ell-1)/2} = 1$, $\gamma_m(\ell, i) = \eta^{-\ell i}$. So $S_{ij}(\ell, \ell') = \eta^{-(\ell-\ell')(i-j)}$. By Proposition 5.1, T_{ij} is a type II matrix. Now the result follows from Proposition 5.1 and 5.5. □

Corollary 6.3 *Let W be a spin model on X of prime index m. Then one of the following holds, where η denotes a primitive m-root of unity.*

(i) *$W = S \otimes T$, where S is a spin model of size m with*

$$S(\ell, \ell') = \eta^{-(1/2)(\ell-\ell')(\ell-\ell'-1)} \quad (\ell, \ell' = 0, 1, \ldots, m-1),$$

and T is a symmetric spin model of size $|X|/m$.

(ii) *W decomposes into m^2 blocks W_{ij} $(i, j = 0, \ldots, m-1)$ with $W_{ij} = S_{ij} \otimes T_{ij}$, where S_{ij} are matrices of size m defined by*

$$S_{ij}(\ell, \ell') = \eta^{-(\ell-\ell')(i-j)} \quad (\ell, \ell' = 0, 1, \ldots, m-1),$$

and T_{ij} are type II matrices of size $r = n/m^2$ which satisfy the following equation for all $i_1, i_2, i_3 \in \{0, \ldots, m-1\}$ and for all $\alpha_1, \alpha_2, \alpha_3 \in \{1, \ldots, r\}$:

$$\sum_{\alpha=1}^{r} \frac{T_{i_1,i}(\alpha_1, \alpha) T_{i_2,i}(\alpha_2, \alpha)}{T_{i_3,i}(\alpha_3, \alpha)} = (D/m) \frac{T_{i_1,i_2}(\alpha_1, \alpha_2)}{T_{i_1,i_3}(\alpha_1, \alpha_3) T_{i_3,i_2}(\alpha_3, \alpha_2)},$$

where i denotes the integer in $\{0, \ldots, m-1\}$ such that $i \equiv i_1 + i_2 - i_3 \pmod m$.

References

1. Et. Bannai, “Modular invariance property and spin models attached to cyclic group association schemes,” *J. Stat. Plann. and Inference* **51** (1996), 107–124.
2. E. Bannai and Et. Bannai, “Spin models on finite cyclic groups,” *J. Alg. Combin.* **3** (1994), 243–259.
3. E. Bannai, Et. Bannai, and F. Jaeger, “On spin models, modular invariance, and duality,” *J. Alg. Combin.* **6** (1997), 203–228.
4. E. Bannai and T. Ito, *Algebraic Combinatorics I*, Benjamin/Cummings, Menlo Park, 1984.
5. F. Jaeger, M. Matsumoto, and K. Nomura, “Bose-Mesner algebras related to type II matrices and spin models,” *J. Alg. Combin.* **8** (1998), 39–72.
6. F. Jaeger and K. Nomura, “Symmetric versus non-symmetric spin models for link invariants,” *J. Alg. Combin.* **10** (1999), 241–278.
7. V.F.R. Jones, “On knot invariants related to some statistical mechanical models,” *Pac. J. Math.* **137** (1989), 311–336.
8. K. Kawagoe, A. Munemasa, and Y. Watatani, “Generalized spin models,” *J. of Knot Theory and its Ramifications* **3** (1994), 465–475.