Journal of Algebraic Combinatorics **12** (2000), 25–36

Distance-Regular Graphs Related to the Quantum Enveloping Algebra of *sl*(2)

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Received May 5, 1998; Revised April 21, 1999

Abstract. We investigate a connection between distance-regular graphs and $U_q(sl(2))$, the quantum universal enveloping algebra of the Lie algebra $sl(2)$. Let Γ be a distance-regular graph with diameter $d \geq 3$ and valency $k \geq 3$, and assume Γ is not isomorphic to the d-cube. Fix a vertex x of Γ, and let $\mathcal{T} = \mathcal{T}(x)$ denote the Terwilliger algebra of Γ with respect to x. Fix any complex number $q \notin \{0, 1, -1\}$. Then $\mathcal{T}$ is generated by certain matrices satisfying the defining relations of $U_q(sl(2))$ if and only if Γ is bipartite and 2-homogeneous.

Keywords: distance-regular graph, Terwilliger algebra, quantum group

1. Introduction

We investigate a connection between distance-regular graphs and $U_q(sl(2))$, the quantum universal enveloping algebra of the Lie algebra $sl(2)$. It is well-known that there is a "natural" $sl(2)$ action on the d-cubes (see Proctor [9] or Go [4]). Here we describe the distance-regular graphs with a similar natural $U_q(sl(2))$ action. We show that these graphs are precisely the bipartite distance-regular graphs which are 2-homogeneous in the sense of [7, 8], excluding the d-cubes. To state this precisely, we recall some definitions.

Let $U(sl(2))$ denote the unital associative $\mathbf{C}$-algebra generated by X^-, X^+, and Z subject to the relations

$$ZX^- - X^-Z = 2X^-, \quad ZX^+ - X^+Z = -2X^+, \quad X^-X^+ - X^+X^- = Z. \tag{1}$$

$U(sl(2))$ is called the *universal enveloping algebra of* $sl(2)$. For any complex number q satisfying

$$q \neq 1, \quad q \neq 0, \quad q \neq -1, \tag{2}$$

let $U_q(sl(2))$ denote the unital associative $\mathbf{C}$-algebra generated by X^-, X^+, Y, and Y^{-1} subject to the relations

$$YY^{-1} = Y^{-1}Y = 1, \tag{3}$$

$$YX^- = q^2X^-Y, \quad YX^+ = q^{-2}X^+Y, \quad X^-X^+ - X^+X^- = \frac{Y - Y^{-1}}{q - q^{-1}}. \tag{4}$$

$U_q(sl(2))$ is called the *quantum universal enveloping algebra of* $sl(2)$. For more on $U_q(sl(2))$ and its relation to $U(sl(2))$ see [5, 6].

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges and having vertex set X, edge set R, distance function ∂, and diameter d. Γ is said to be *distance-regular* whenever for all integers ℓ, i, j $(0 \le \ell, i, j \le d)$ there exists a scalar p^{ℓ}_{ij} such that for all $x, y \in X$ with $\partial(x, y) = \ell$, $|\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}| = p^{\ell}_{ij}$. Assume that Γ is distance-regular. Set $c_0 = 0$, $c_i = p^i_{1i-1}$ $(1 \le i \le d)$, $a_i = p^i_{1i}$ $(0 \le i \le d)$, $b_i = p^i_{1i+1}$ $(0 \le i \le d-1)$, and $b_d = 0$. Γ is regular with valency $k = b_0 = p^0_{11}$, and $c_i + a_i + b_i = k$ $(0 \le i \le d)$. Γ is bipartite precisely when $a_i = 0$ $(0 \le i \le d)$.

Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph. Γ is said to be 2-*homogeneous* whenever for all integers i $(1 \le i \le d)$ there exists a scalar γ_i such that for all $x, y, z \in X$ with $\partial(x, y) = i$, $\partial(x, z) = i$, $\partial(y, z) = 2$, $|\{w \in X \mid \partial(x, w) = i-1, \partial(y, w) = 1, \partial(z, w) = 1\}| = \gamma_i$. Γ may be 2-homogeneous despite the fact that some structure constant γ_i is not uniquely determined: This occurs when there are no $x, y, z \in X$ with $\partial(x, y) = i$, $\partial(x, z) = i$, $\partial(y, z) = 2$. It is known that γ_d is not uniquely determined when Γ is 2-homogeneous [8]. The d-*cube* is the graph with vertex set $X = \{0, 1\}^d$ (the d-tuples with entries in $\{0, 1\}$) such that two vertices are adjacent if and only if they differ in precisely one coordinate. The d-cube is a 2-homogeneous bipartite distance-regular graph with $\gamma_i = 1$ $(1 \le i \le d-1)$. The 2-homogeneous bipartite distance-regular graphs have been studied in [3, 8, 11].

Let $Mat_X(\mathbf{C})$ denote the $\mathbf{C}$-algebra of matrices with rows and columns indexed by X. Let $A \in Mat_X(\mathbf{C})$ denote the adjacency matrix of Γ. For the rest of this section fix $x \in X$. For all i $(0 \le i \le d)$, define $E^*_i = E^*_i(x)$ to be the diagonal matrix in $Mat_X(\mathbf{C})$ such that for all $y \in X$, E^*_i has (y, y)-entry equal to 1 if $\partial(x, y) = i$, and 0 otherwise. Let $\mathcal{T} = \mathcal{T}(x)$ denote the subalgebra of $Mat_X(\mathbf{C})$ generated by $A, E^*_0, E^*_1, \ldots, E^*_d$.

Set $L = \sum_{i=0}^{d-1} E^*_i A E^*_{i+1}$ and $R = \sum_{i=1}^{d} E^*_i A E^*_{i-1}$. Proctor [9] showed that if Γ is isomorphic to the d-cube, then the matrices $X^- = L$, $X^+ = R$, and $Z = \sum_{i=0}^{d} (d-2i) E^*_i$ satisfy the relations of (1) (see also Go [4]). We must slightly relax the form of these matrices to admit a $U_q(sl(2))$ structure. Specifically, we consider matrices of the form:

$$X^- = \sum_{i=0}^{d-1} x^-_i E^*_i A E^*_{i+1}, \quad X^+ = \sum_{i=1}^{d} x^+_i E^*_i A E^*_{i-1}, \quad Y = \sum_{i=0}^{d} y_i E^*_i, \tag{5}$$

where x^-_i $(0 \le i \le d-1)$, x^+_i $(1 \le i \le d)$, and y_i $(0 \le i \le d)$ are arbitrary complex scalars. Y is invertible if and only if $y_i \ne 0$ $(0 \le i \le d)$, in which case $Y^{-1} = \sum_{i=0}^{d} y_i^{-1} E^*_i$.

Theorem 1.1 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \ge 3$ and valency $k \ge 3$. Assume that Γ is not isomorphic to the d-cube. Fix $x \in X$, and write $E^*_i = E^*_i(x)$ $(0 \le i \le d)$ and $\mathcal{T} = \mathcal{T}(x)$. Let X^-, X^+, and Y be any matrices of the form* (5), *and let q be any nonzero complex number. Then the following are equivalent.*

(i) *Y is invertible, X^-, X^+, Y, Y^{-1} generate $\mathcal{T}$, and* (2)–(4) *hold.*

(ii) *Γ is bipartite and 2-homogeneous, $(q + q^{-1})^2 = c_2^2 b_2^{-1} (k-2)(c_2 - 1)^{-1}$, and there* *exists* $\epsilon \in \{1, -1\}$ *such that*

$$y_i = \epsilon q^{d-2i} \quad (0 \le i \le d),$$
$$x_i^- x_{i+1}^+ = \epsilon q^{-2i+1}(q^d + q^{2i})(q^d + q^{2i+2})(q^d + q^2)^{-2} \quad (0 \le i \le d-1).$$

The condition (i) of Theorem 1.1 means that the Terwilliger algebra $\mathcal{T}$ is a homomorphic image of $U_q(sl(2))$. The factor of ϵ appears in (ii) because the defining relations of $U_q(sl(2))$ are invariant under changing the signs of any two of X^-, X^+, and Y.

2. Background

Throughout this section, let $\Gamma = (X, R)$ denote a distance-regular graph with diameter d. Let $Mat_X(\mathbf{C})$ denote the $\mathbf{C}$-algebra of matrices with rows and columns indexed by X. For all i $(0 \le i \le d)$, define A_i to be the matrix in $Mat_X(\mathbf{C})$ such that for all $y, z \in X$ the (y, z)-entry of A_i is 1 if $\partial(y, z) = i$ and 0 otherwise. Observe that $A_0 = I$ (the identity matrix), $A := A_1$ is the adjacency matrix of Γ, and $\sum_{i=0}^d A_i = J$ (the all 1's matrix). Observe that $A_i A_j = A_j A_i = \sum_{\ell=0}^d p_{ij}^\ell A_\ell$ $(0 \le i, j \le d)$. It follows that the linear span $\mathcal{M}$ of $A_0, A_1, \ldots, A_d$ is a commutative subalgebra of $Mat_X(\mathbf{C})$. The algebra $\mathcal{M}$ is called the *Bose-Mesner algebra* of Γ. It is known that $\mathcal{M}$ is generated by A. See [1, 2] for more on distance-regular graphs and their Bose-Mesner algebras.

For the rest of this section fix $x \in X$. For all i $(0 \le i \le d)$, define $E_i^* = E_i^*(x)$ to be the diagonal matrix in $Mat_X(\mathbf{C})$ such that for all $y \in X$, the (y, y)-entry of E_i^* is $E_i^*(y, y) = A_i(x, y)$. Observe that $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \le i, j \le d)$ and $\sum_{i=0}^d E_i^* = I$. It follows that the linear span $\mathcal{M}^* = \mathcal{M}^*(x)$ of $E_0^*, E_1^*, \ldots, E_d^*$ is a commutative subalgebra of $Mat_X(\mathbf{C})$. The algebra $\mathcal{M}^*$ is called the *dual Bose-Mesner algebra* of Γ with respect to x. Let $\mathcal{T} = \mathcal{T}(x)$ denote the subalgebra of $Mat_X(\mathbf{C})$ generated by $\mathcal{M} \cup \mathcal{M}^*$. The algebra $\mathcal{T}$ is called the *Terwilliger algebra* of Γ with respect to x. See [10] for more on Terwilliger algebras.

Fix ℓ, i, j $(0 \le \ell, i, j \le d)$. Observe that for all $y, z \in X$, the (y, z)-entry of $E_i^* A_\ell E_j^*$ is 0 or 1, and it is equal to 1 if and only if $\partial(x, y) = i$, $\partial(y, z) = \ell$ and $\partial(x, z) = j$. Thus, considering the positions of the nonzero entries,

$$\{E_i^* A_\ell E_j^* \ne 0 \mid 0 \le \ell, i, j \le d\} \text{ is linearly independent,} \tag{6}$$
$$E_i^* A_\ell E_j^* \ne 0 \text{ if and only if } p_{ij}^\ell \ne 0. \tag{7}$$

Observe that $p_{ij}^\ell = 0$ if one of ℓ, i, j is greater than the sum of the other two, and $p_{ij}^\ell \ne 0$ if one of ℓ, i, j is equal to the sum of the other two. It follows that $E_i^* A E_j^* = E_j^* A E_i^* = 0$ whenever $|i - j| > 1$. Hence $A = \sum_{i=0}^d \sum_{j=0}^d E_i^* A E_j^* = L + F + R$, where

$$L = \sum_{i=0}^{d-1} E_i^* A E_{i+1}^*, \quad F = \sum_{i=0}^{d} E_i^* A E_i^*, \quad R = \sum_{i=1}^{d} E_i^* A E_{i-1}^*.$$

Observe that $E_i^* A E_i^* = 0$ if and only if $a_i = 0$, so Γ is bipartite if and only if $F = 0$.

We wish to emphasize the following combinatorial interpretation of L and R. For all i $(0 \le i \le d)$ and for all $y \in X$, let $\Gamma_i(y) = \{z \in X \mid \partial(y, z) = i\}$. Identify each vertex with its characteristic column vector, and note that $Mat_X(\mathbf{C})$ acts on the vertices by left multiplication. For all i $(0 \le i \le d)$ and all $y \in \Gamma_i(x)$, $Ly = \sum_{w \in \Gamma_1(y) \cap \Gamma_{i-1}(x)} w$, $Ry = \sum_{w \in \Gamma_1(y) \cap \Gamma_{i+1}(x)} w$, and $E_j^* y = \delta_{ij} y$ $(0 \le j \le d)$. Fix i $(0 \le i \le d)$. For all y, $z \in \Gamma_i(x)$, set

$$\beta(y,z) = |\Gamma_1(y) \cap \Gamma_1(z) \cap \Gamma_{i+1}(x)|, \quad \gamma(y,z) = |\Gamma_1(y) \cap \Gamma_1(z) \cap \Gamma_{i-1}(x)|. \tag{8}$$

Observe that for all $y, z \in \Gamma_i(x)$,

$$(LRE_i^*)(y,z) = \beta(y,z), \quad (RLE_i^*)(y,z) = \gamma(y,z). \tag{9}$$

In particular, $(LRE_i^*)(y,y) = b_i$, $(RLE_i^*)(y,y) = c_i$, and when $\partial(y,z) > 2$, $(LRE_i^*)(y,z) = (RLE_i^*)(y,z) = 0$.

3. Construction of *U*(*sl*(2)) and *U*$_q$(*sl*(2)) structures

In this section, we construct a $U(sl(2))$ structure on the d-cubes and a $U_q(sl(2))$ structure on the remaining 2-homogeneous bipartite distance-regular graphs. Throughout this section, let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \ge 3$ and valency $k \ge 3$. Fix $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \le i \le d)$, $\mathcal{M}^* = \mathcal{M}^*(x)$, $\mathcal{T} = \mathcal{T}(x)$.

Lemma 3.1 *Let $z_0, z_1, \ldots, z_d$ denote distinct complex scalars. Then $Z = \sum_{i=0}^{d} z_i E_i^*$ generates $\mathcal{M}^*$.*

Proof: Observe that $Z^j = \sum_{i=0}^{d} z_i^j E_i^*$ $(0 \le j \le d)$, where the $j = 0$ equation is interpreted as $I = \sum_{i=0}^{d} E_i^*$. Viewing $E_0^*, E_1^*, \ldots, E_d^*$ as unknowns, this is a system of linear equations with Vandermonde (hence invertible) coefficient matrix. Thus $E_i^* \in \operatorname{span}\{Z^j \mid 0 \le j \le d\}$ $(0 \le i \le d)$, so Z generates $\mathcal{M}^*$. □

Lemma 3.2 [4, 9] *Suppose Γ is isomorphic to the d-cube. Then $X^- = L$, $X^+ = R$ and $Z = \sum_{i=0}^{d} (d-2i) E_i^*$ generate $\mathcal{T}$ and satisfy* (1).

Proof: Observe that Z generates $\mathcal{M}^*$ by Lemma 3.1. Observe that $F = 0$ since Γ is bipartite, so $A = L + R$. A generates $\mathcal{M}$, so L, R, and Z generate $\mathcal{T}$.

The relations $ZL - LZ = 2L$ and $ZR - RZ = -2R$ are easily verified using the definitions of L, R, and Z and the fact that $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \le i, j \le d)$. It remains to verify $LR - RL = Z$. Since $\sum_{i=0}^{d} E_i^* = I$, it is enough to show that for all i $(0 \le i \le d)$

$$LRE_i^* - RLE_i^* = (d-2i)E_i^*. \tag{10}$$

Fix i $(0 \le i \le d)$, and pick $y, z \in \Gamma_i(x)$. Let r, s, t denote the (y,z)-entries of LRE_i^*, RLE_i^*, and E_i^*, respectively. From (8), (9) we find the following. Suppose $\partial(y,z) > 2$. Then $r = s = t = 0$. Suppose $\partial(y,z) = 2$. Then $r = c_2 - \gamma_i = 1$, $s = \gamma_i = 1$, and $t = 0$. The case $\partial(y,z) = 1$ does not occur since $a_i = 0$. Finally suppose $y = z$. Then $r = b_i = d - i$, $s = c_i = i$, and $t = 1$. In all cases $r - s = (d-2i)t$, so (10) holds. □

Theorem 3.3 ([3, Theorem 35]) *Suppose Γ is not isomorphic to the d-cube. Then Γ is bipartite and 2-homogeneous if and only if there exists a complex scalar $q \notin \{0, 1, -1\}$ such that*

$$c_i = e_i [\, i\,], \quad b_i = e_i [\, d - i\,] \quad (0 \le i \le d), \tag{11}$$

where

$$e_i = q^{i-1}(q^d + q^2)(q^d + q^{2i})^{-1}, \quad [\, i\,] = (q^i - q^{-i})(q - q^{-1})^{-1} \tag{12}$$

for all integers i. Suppose the above equivalent conditions hold. Then

$$\gamma_i = e_2 e_i e_{i+1}^{-1} \quad (1 \le i \le d-1). \tag{13}$$

Corollary 3.4 ([3, Corollary 36]) *Suppose Γ is bipartite and 2-homogeneous, but not isomorphic to the d-cube. Then any complex scalar $q \notin \{0, 1, -1\}$ satisfying (11) and (12) is real and*

$$(q + q^{-1})^2 = c_2^2 b_2^{-1}(b_0 - 2)(c_2 - 1)^{-1}. \tag{14}$$

The set of q satisfying (14) is of the form $\{\lambda, \lambda^{-1}, -\lambda, -\lambda^{-1}\}$ for some real number $\lambda > 1$. When d is even, all such q satisfy (11). When d is odd, only $q \in \{\lambda, \lambda^{-1}\}$ satisfy (11) since $q + q^{-1} = c_2 \gamma_r^{-1} > 0$, where $r = (d-1)/2$ (see [3, Corollary 36]).

Lemma 3.5 *Suppose Γ is bipartite and 2-homogeneous but not isomorphic to the d-cube. Let $q \notin \{0, 1, -1\}$ be any complex scalar such that (11), (12) hold, and let e_i $(0 \le i \le d)$ be as in (12). Then the matrices*

$$X^- = \sum_{j=0}^{d-1} e_j^{-1} E_j^* A E_{j+1}^*, \quad X^+ = \sum_{j=1}^{d} e_j^{-1} E_j^* A E_{j-1}^*, \quad Y = \sum_{j=0}^{d} q^{d-2j} E_j^*$$

generate $\mathcal{T}$ and satisfy (4).

Proof: Observe that Y generates $\mathcal{M}^*$ by Lemma 3.1. Now $L = (\sum_{i=0}^{d-1} e_i E_i^*) X^-$ and $R = (\sum_{i=1}^{d} e_i E_i^*) X^+$ are in the algebra generated by Y, X^- and X^+. Observe that $F = 0$ since Γ is bipartite, so $A = L + R$. A generates $\mathcal{M}$, so X^-, X^+, and Y generate $\mathcal{T}$.

The relations $YX^- = q^2 X^- Y$ and $YX^+ = q^{-2} X^+ Y$ are easily verified using the definitions of X^-, X^+, and Y and the fact that $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \le i,\ j \le d)$. It remains to verify $X^- X^+ - X^+ X^- = (Y - Y^{-1})/(q - q^{-1})$. Observe that for all i $(0 \le i \le d)$, $X^- X^+ E_i^* = e_i^{-1} e_{i+1}^{-1} L R E_i^*$, $X^+ X^- E_i^* = e_{i-1}^{-1} e_i^{-1} R L E_i^*$, and $(Y - Y^{-1})/(q - q^{-1}) E_i^* = [\, d - 2i\,] E_i^*$. Thus, since $I = \sum_{i=0}^{d} E_i^*$, it is enough to show that for all i $(0 \le i \le d)$

$$e_i^{-1} e_{i+1}^{-1} L R E_i^* - e_{i-1}^{-1} e_i^{-1} R L E_i^* = [\, d - 2i\,] E_i^*. \tag{15}$$

Fix i $(0 \leq i \leq d)$, and pick $y, z \in \Gamma_i(x)$. Let r, s, and t denote the (y, z)-entries of LRE^*_i, RLE^*_i, and E^*_i, respectively. From (8), (9) we find the following. Suppose $\partial(y, z) > 2$. Then $r = s = t = 0$. Suppose $\partial(y, z) = 2$. Then by the definition of 2-homogeneous, $r = c_2 - \gamma_i$, $s = \gamma_i$, and $t = 0$. It can be verified by a direct computation using (11)–(13) that $e_i^{-1} e_{i+1}^{-1}(c_2 - \gamma_i) - e_{i-1}^{-1} e_i^{-1} \gamma_i = 0$. The case $\partial(y, z) = 1$ does not occur since $a_i = 0$. Finally suppose $y = z$. Then $r = b_i$, $s = c_i$, and $t = 1$. It can be verified by a direct (albeit long) computation using (11), (12) that $e_i^{-1} e_{i+1}^{-1} b_i - e_{i-1}^{-1} e_i^{-1} c_i = [\, d - 2i\,]$. In all cases $e_i^{-1} e_{i+1}^{-1} r - e_{i-1}^{-1} e_i^{-1} s = [\, d - 2i\,] t$, so (15) holds. □

The $U(sl(2))$ structure on the d-cube is very similar to the $U_q(sl(2))$ structure on the remaining 2-homogeneous bipartite distance-regular graphs. In the sequel, we exploit this similarity to prove the following result and Theorem 1.1 simultaneously.

Theorem 3.6 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$ and valency $k \geq 3$. Fix $x \in X$, and write $E^*_i = E^*_i(x)$ $(0 \leq i \leq d)$, $\mathcal{T} = \mathcal{T}(x)$. Let X^-, X^+, and Z be of the form $X^- = \sum_{i=0}^{d-1} x_i^- E^*_i A E^*_{i+1}$, $X^+ = \sum_{i=1}^{d} x_i^+ E^*_i A E^*_{i-1}$, $Z = \sum_{i=0}^{d} z_i E^*_i$ for some complex scalars x_i^- $(0 \leq i \leq d-1)$, x_i^+ $(1 \leq i \leq d)$, z_i $(0 \leq i \leq d)$. Then the following are equivalent.*
(i) *X^-, X^+, and Z generate $\mathcal{T}$ and satisfy* (1).
(ii) *Γ is isomorphic to the d-cube, and*

$$x_i^- x_{i+1}^+ = 1 \quad (0 \leq i \leq d-1),$$
$$z_i = d - 2i \quad (0 \leq i \leq d).$$

As in Theorem 1.1, The condition (i) of Theorem 3.6 means that the Terwilliger algebra $\mathcal{T}$ is a homomorphic image of $U(sl(2))$.

4. Combinatorial structure

We show that the $U(sl(2))$ and $U_q(sl(2))$ structures of Lemmas 3.2 and 3.5 can only occur on a 2-homogeneous bipartite distance-regular graph. Specifically, we show the following.

Theorem 4.1 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$ and valency $k \geq 3$. Fix $x \in X$, and write $E^*_i = E^*_i(x)$ $(0 \leq i \leq d)$, $\mathcal{T} = \mathcal{T}(x)$. Suppose that $\mathcal{T}$ is generated by $\{X^-, X^+\} \cup \mathcal{M}^*$ and that $X^- X^+ - X^+ X^- = Z$, where X^- and X^+ are of the form* (5) *and Z is of the form $Z = \sum_{i=0}^{d} z_i E^*_i$ for some complex scalars z_i $(0 \leq i \leq d)$. Then Γ is bipartite and* 2*-homogeneous.*

The hypotheses of this result are met by both Theorems 1.1(i) and 3.6(i). Throughout this section, we adopt the notation and assumptions of Theorem 4.1 as we prove this result in a series of lemmas. The first step in our proof of Theorem 4.1 is to show that $a_i = 0$ $(1 \leq i \leq d-1)$. To do so, we consider certain matrices in the left ideal $\mathcal{T} E^*_1$ of $\mathcal{T}$:

$$K_i = E^*_i J E^*_1 \quad (0 \leq i \leq d),$$
$$N_0 = 0, \qquad N_i = E^*_i A_{i-1} E^*_1 \quad (1 \leq i \leq d).$$

Lemma 4.2 *$LK_0 = 0$, $LK_i = b_{i-1}K_{i-1}$ $(1 \le i \le d)$, $RK_i = c_{i+1}K_{i+1}$ $(0 \le i \le d-1)$, $RK_d = 0$, and $X^-K_0 = 0$, $X^-K_i = x^-_{i-1}b_{i-1}K_{i-1}$ $(1 \le i \le d)$, $X^+K_i = x^+_{i+1}c_{i+1}K_{i+1}$ $(0 \le i \le d-1)$, $X^+K_d = 0$.*

Proof: Clearly $LK_0 = LE^*_0K_0 = 0$. Fix i $(1 \le i \le d)$. Fix $y, z \in X$, and let r and s denote the (y, z)-entries of LK_i and K_{i-1}, respectively. Observe that $r = s = 0$ unless $y \in \Gamma_{i-1}(x)$ and $z \in \Gamma_1(x)$, so suppose $y \in \Gamma_{i-1}(x)$ and $z \in \Gamma_1(x)$. Then

$$\begin{aligned} r &= (E^*_{i-1}AE^*_iJE^*_1)(y, z) = \sum_{p\in X} E^*_{i-1}(y, y)A(y, p)E^*_i(p, p)J(p, z)E^*_1(z, z) \\ &= \sum_{p\in X} A(y, p)E^*_i(p, p) = |\Gamma_1(y) \cap \Gamma_i(x)| = b_{i-1}, \\ s &= (E^*_{i-1}JE^*_1)(y, z) = E^*_{i-1}(y, y)J(y, z)E^*_1(z, z) = 1. \end{aligned}$$

In all cases $r = b_{i-1}s$, so $LK_i = b_{i-1}K_{i-1}$. The equations for RK_i are proved similarly. The equations involving X^- and X^+ follow since $X^-E^*_i = x^-_{i-1}LE^*_i$ $(1 \le i \le d)$ and $X^+E^*_i = x^+_{i+1}RE^*_i$ $(0 \le i \le d-1)$. □

Lemma 4.3 *$x^-_i \ne 0$ and $x^+_{i+1} \ne 0$ $(0 \le i \le d-1)$. In particular, $s_i := x^-_ix^+_{i+1} \ne 0$ $(0 \le i \le d-1)$.*

Proof: Suppose $x^-_i = 0$ for some i $(0 \le i \le d-1)$, and set $\mathcal{U} = \text{span}\{K_h \mid i+1 \le h \le d\}$. Then $\mathcal{U}$ is closed under left multiplication by the generators X^-, X^+, and $\mathcal{M}^*$ of $\mathcal{T}$ by Lemma 4.2 and construction. Hence $\mathcal{U}$ is a left ideal of $\mathcal{T}$. However, $LK_{i+1} = b_iK_i \ne 0$ and $K_i \notin \mathcal{U}$, a contradiction. Hence $x^-_i \ne 0$ $(0 \le i \le d-1)$. A similar argument shows that $x^+_{i+1} \ne 0$ $(0 \le i \le d-1)$. □

Lemma 4.4 *$X^+N_i = x^+_{i+1}c_iN_{i+1}$ $(1 \le i \le d-1)$ and $X^+N_d = 0$.*

Proof: Fix i $(1 \le i \le d-1)$. Pick $y, z \in X$, and let r and s denote the (y, z)-entries of X^+N_i and N_{i+1}, respectively. Observe that $r = s = 0$ unless $y \in \Gamma_{i+1}(x)$ and $z \in \Gamma_1(x)$, so suppose $y \in \Gamma_{i+1}(x)$ and $z \in \Gamma_1(x)$. Then

$$\begin{aligned} r &= x^+_{i+1}(E^*_{i+1}AE^*_iA_{i-1}E^*_1)(y, z) \\ &= x^+_{i+1}\sum_{p\in X} E^*_{i+1}(y, y)A(y, p)E^*_i(p, p)A_{i-1}(p, z)E^*_1(z, z) \\ &= x^+_{i+1}|\Gamma_1(y) \cap \Gamma_i(x) \cap \Gamma_{i-1}(z)|, \\ s &= (E^*_{i+1}A_iE^*_1)(y, z) = A_i(y, z). \end{aligned}$$

Observe that $r = s = 0$ when $\partial(y, z) \ne i$, and $r = x^+_{i+1}c_i$, $s = 1$ when $\partial(y, z) = i$. In all cases $r = x^+_{i+1}c_is$, so $X^+N_i = x^+_{i+1}c_iN_{i+1}$. Clearly $X^+N_d = X^+E^*_dN_d = 0$. □

Lemma 4.5 *$X^-N_i \in \text{span}\{N_{i-1},\ K_{i-1}\}$ $(1 \le i \le d)$.*

Proof: It is easy to show that $X^- N_1 = x_0^- K_0$ by entry-wise computation. We proceed by induction: Fix i $(2 \leq i \leq d)$, and assume $X^- N_{i-1} = gN_{i-2} + hK_{i-2}$ for some scalars g, h. We compute

$$
\begin{aligned}
X^- X^+ N_{i-1} &= X^-(x_i^+ c_{i-1} N_i) = x_i^+ c_{i-1}(X^- N_i),\\
X^+ X^- N_{i-1} &= X^+(gN_{i-2} + hK_{i-2}) = g x_{i-1}^+ c_{i-2} N_{i-1} + h x_{i-1}^+ c_{i-1} K_{i-1},\\
Z N_{i-1} &= z_{i-1} N_{i-1}.
\end{aligned}
$$

Now we may apply the relation $X^- X^+ - X^+ X^- = Z$ to N_{i-1} and solve to find $X^- N_i \in \operatorname{span}\{N_{i-1}, K_{i-1}\}$ since $x_i^+ c_{i-1} \neq 0$. The result follows by induction. □

Lemma 4.6 $a_i = 0$ $(1 \leq i \leq d-1)$.

Proof: By Lemmas 4.2–4.5 and construction, $\mathcal{U} = \operatorname{span}\{K_i \mid 0 \leq i \leq d\} + \operatorname{span}\{N_i \mid 1 \leq i \leq d\}$ is a left ideal of $\mathcal{T}$. In fact, $\mathcal{U} = \mathcal{T}E_1^*$ since $E_1^* = N_1$. Now fix i $(1 \leq i \leq d-1)$. Then $E_i^*\mathcal{T}E_1^* = E_i^*\mathcal{U} = \operatorname{span}\{E_i^* K_i, E_i^* N_i\}$, so $\dim_{\mathbf{C}} E_i^*\mathcal{T}E_1^* \leq 2$. Observe that the subspace $E_i^*\mathcal{T}E_1^*$ contains $E_i^* A_j E_1^*$ $(j = i-1, i, i+1)$, and $E_i^* A_{i-1} E_1^* \neq 0$, $E_i^* A_{i+1} E_1^* \neq 0$ by (7). If $E_i^* A_i E_1^* \neq 0$, then these three matrices are linearly independent by (6), contradicting $\dim_{\mathbf{C}} E_i^*\mathcal{T}E_1^* \leq 2$. Thus $E_i^* A_i E_1^* = 0$, so $a_i = 0$ by (7). □

We show that $a_d = 0$ by showing that there is a unique vertex at distance d from x.

Lemma 4.7 *Set* $s_i = x_i^- x_{i+1}^+$ $(0 \leq i \leq d-1)$ *and* $s_{-1} = s_d = 0$. *Then for all* i $(0 \leq i \leq d)$,

$$s_i LRE_i^* - s_{i-1} RLE_i^* = z_i E_i^*, \tag{16}$$

$$s_i \beta(y,z) - s_{i-1}\gamma(y,z) = \delta_{yz} z_i \quad (y, z \in \Gamma_i(x)), \tag{17}$$

where $\beta(y,z)$ *and* $\gamma(y,z)$ *are as in* (8). *In particular*, $\beta(y,z) = 0$ *if and only if* $\gamma(y,z) = 0$ *for any distinct* $y, z \in \Gamma_i(x)$.

Proof: Fix i $(0 \leq i \leq d)$. Apply the relation $X^- X^+ - X^+ X^- = Z$ to E_i^* to get (16). Fix $y, z \in \Gamma_i(x)$. Computing the (y,z)-entry of (16) gives (17) by (9). It is clear from (17) and Lemma 4.3 that $\beta(y,z) = 0$ if and only if $\gamma(y,z) = 0$ when y, z are distinct. □

Lemma 4.8 $|\Gamma_d(x)| = 1$ *and* Γ *is bipartite.*

Proof: By a *down-up walk* of length 2ℓ $(1 \leq \ell \leq d)$, we mean a sequence of vertices $v_0, v_1, \ldots, v_{2\ell}$ such that v_i and v_{i+1} are adjacent $(0 \leq i \leq 2\ell - 1)$, $v_i, v_{2\ell-i} \in \Gamma_{d-i}(x)$ $(0 \leq i \leq \ell)$, and $v_0 \neq v_{2\ell}$. Assume $|\Gamma_d(x)| \geq 2$. For all distinct $y, z \in \Gamma_d(x)$ there exists a down-up walk of length $2d$ (taking $v_0 = y$, $v_d = x$, $v_{2d} = z$), but there is no down-up walk of length 2 since $|\Gamma_{d-1}(x) \cap \Gamma_1(y) \cap \Gamma_1(z)| = 0$ by Lemma 4.7.

Fix a down-up walk $v_0, v_1, \ldots, v_{2\ell}$ of minimal length 2ℓ. By minimality of the length of this down-up walk, $v_{\ell-1}$ and $v_{\ell+1} \in \Gamma_{d-\ell+1}(x)$ are distinct. Let $\gamma(v_{\ell-1}, v_{\ell+1})$, $\beta(v_{\ell-1}, v_{\ell+1})$

be as in (8). Observe that $\gamma(v_{\ell-1}, v_{\ell+1}) > 0$, so $\beta(v_{\ell-1}, v_{\ell+1}) > 0$ by Lemma 4.7. Fix $w \in \Gamma_{d-\ell+2}(x) \cap \Gamma_1(v_{\ell-1}) \cap \Gamma_1(v_{\ell+1})$. Fix a path $w_{d-\ell+2} = w, w_{d-\ell+3}, \ldots, w_d$ such that $w_i \in \Gamma_i(x)$ (such a path exists since $b_i > 0$ $(0 \leq i \leq d-1)$). Suppose $w_d \neq v_0$. Then $v_0, \ldots, v_{\ell-1}, w_{d-\ell+2}, \ldots, w_d$ is a down-up path of length $2\ell - 2$, contradicting the minimality of length 2ℓ. Thus $w_d = v_0$. Similarly, $v_{2\ell} = w_d$, contradicting $v_0 \neq v_{2\ell}$. It follows that $|\Gamma_d(x)| = 1$, so $a_d = 0$. Hence Γ is bipartite in light of Lemma 4.6. □

Lemma 4.9 Γ *is* 2*-homogeneous.*

Proof: By [3, Theorem 16] it is enough to show that for all i $(1 \leq i \leq d)$ and for all y, $z \in \Gamma_i(x)$ with $\partial(y, z) = 2$, the number $\gamma(y, z)$ of (8) is independent of the choice of y, z.

Fix i $(1 \leq i \leq d-1)$, and pick any $y, z \in \Gamma_i(x)$ with $\partial(y, z) = 2$. By Lemma 4.8, Γ is bipartite, so $\beta(y, z) + \gamma(y, z) = c_2$. By (17), $s_i\beta(y, z) - s_{i-1}\gamma(y, z) = 0$. Thus $(s_i + s_{i-1})\gamma(y, z) = c_2 s_i$. Since $s_i \neq 0$ by Lemma 4.3, the right side is nonzero and hence the left side is also nonzero. Thus we may solve this equation for $\gamma(y, z)$ independent of y and z. Observe that when $i = d$ there is nothing to show by Lemma 4.8. □

5. Proof of Theorem 1.1

In this section we prove Theorems 1.1 and 3.6. We continue with the notation and assumptions of Theorem 4.1 throughout this section. We begin by considering the uniqueness of the $U(sl(2))$ and $U_q(sl(2))$ structures.

Lemma 5.1 *Set* $s_i = x_i^- x_{i+1}^+$ $(0 \leq i \leq d-1)$. *Then the scalars* s_i $(0 \leq i \leq d-1)$ *and* z_i $(0 \leq i \leq d)$ *are uniquely determined up to the same scalar multiple.*

Proof: Observe that for all i $(0 \leq i \leq d)$ and for all $y \in \Gamma_i(x)$, $\beta(y, y) = b_i$ and $\gamma(y, y) = c_i$, where $\beta(y, y)$ and $\gamma(y, y)$ are as in (8). Thus applying (17) with $y = z$ gives

$$s_0 = z_0 b_0^{-1}, \quad s_i b_i - s_{i-1} c_i = z_i \ (1 \leq i \leq d-1), \quad s_{d-1} c_d = -z_d. \tag{18}$$

Applying the relation $X^- X^+ - X^+ X^- = Z$ to K_i $(1 \leq i \leq d-1)$ and simplifying with Lemma 4.2 gives

$$s_i b_i c_{i+1} - s_{i-1} b_{i-1} c_i = z_i \quad (1 \leq i \leq d-1). \tag{19}$$

Fix i $(1 \leq i \leq d-1)$. Subtracting (18) from (19) gives $s_i b_i (c_{i+1} - 1) = s_{i-1} c_i (b_{i-1} - 1)$. Since s_i, s_{i-1} are nonzero by Lemma 4.3, $b_{i-1} = 1$ if and only if $c_{i+1} = 1$. Suppose $b_{i-1} = c_{i+1} = 1$. Then $1 \leq b_i \leq b_{i-1} = 1$ and $1 \leq c_i \leq c_{i+1} = 1$ since the c_i form a nondecreasing sequence and the b_i form a nonincreasing sequence by [2, Proposition 4.1.6]. Thus $k = c_i + b_i = 2$, a contradiction. Thus we may solve for s_i as $s_i = c_i(b_{i-1} - 1)s_{i-1}/(b_i(c_{i+1} - 1))$. In particular, since $s_0 = z_0 b_0^{-1}$, the numbers s_j $(0 \leq j \leq d-1)$ are determined by the intersection numbers and z_0. The numbers z_j $(1 \leq j \leq d)$ are determined by (18). In these formulas z_0 is a factor of s_j $(0 \leq j \leq d-1)$ and z_j $(1 \leq j \leq d)$, so the result follows. □

Lemma 5.2 *Suppose that* Γ *is isomorphic to the* d*-cube. Then*, *after multiplying* X^+ *and* Z *by some same scalar*, $Z = \sum_{i=0}^{d}(d-2i)E_i^*$ *and* X^-, X^+, Z *satisfy* (1).

Proof: By (16), $s_i LRE_i^* - s_{i-1}RLE_i^* = z_i E_i^*$ $(0 \le i \le d)$, and by (10), $LRE_i^* - RLE_i^* = (d-2i)E_i^*$ $(0 \le i \le d)$. One possibility is $s_i = 1$ $(0 \le i \le d-1)$, and in this case $z_i = d - 2i$ $(0 \le i \le d)$. Thus by Lemma 5.1, there exists a scalar α such that $\alpha s_i = 1$ $(0 \le i \le d-1)$ and $\alpha z_i = d - 2i$ $(0 \le i \le d)$. Hence, after replacing X^+ with αX^+ and Z with αZ, we find that $Z = \sum_{i=0}^{d}(d-2i)E_i^*$ and X^-, X^+, and Z satisfy (1). □

Lemma 5.3 *Suppose* Γ *is not isomorphic to the* d*-cube. Then*, *after multiplying* X^+ *and* Z *by some same scalar*, $Z = \sum_{i=0}^{d} [\, d-2i\,] E_i^*$ *and* X^-, X^+, $Y = \sum_{i=0}^{d} q^{d-2i} E_i^*$ *satisfy* (4) *for some real number* $q \notin \{0, 1, -1\}$.

Proof: By (16), $s_i LRE_i^* - s_{i-1}RLE_i^* = z_i E_i^*$ $(0 \le i \le d)$, and by (15), $e_i^{-1}e_{i+1}^{-1}LRE_i^* - e_{i-1}^{-1}e_i^{-1}RLE_i^* = [\, d-2i\,]E_i^*$ $(0 \le i \le d)$, where e_j and $[\, j\,]$ are as in (12) for all integers j. One possibility is $s_i = e_i^{-1}e_{i+1}^{-1}$ $(0 \le i \le d-1)$, and in this case $z_i = [\, d-2i\,]$ $(0 \le i \le d)$. Thus by Lemma 5.1, there exists a scalar α such that $\alpha s_i = e_i^{-1}e_{i+1}^{-1}$ $(0 \le i \le d-1)$ and $\alpha z_i = [\, d-2i\,]$ $(0 \le i \le d)$. Hence $\alpha Z = \sum_{i=0}^{d} [\, d-2i\,]E_i^*$, and, after replacing X^+ with αX^+, we find that X^-, X^+, and $Y = \sum_{i=0}^{d} q^{d-2i}E_i^*$ satisfy (4). □

Lemma 5.4 *The conclusions of Lemma* 5.3 *do not hold when* Γ *is isomorphic to the* d*-cube*, *and the conclusions of Lemma* 5.2 *do not hold when* Γ *is not isomorphic to the* d*-cube.*

Proof: If this is not the case, then arguing as in Lemmas 5.2 and 5.3, we find that there is a scalar α such that $\alpha(d-2i) = [\, d-2i\,]$ $(0 \le i \le d)$, where $[\, d-2i\,]$ is as in (12) for some real number $q \notin \{0, 1, -1\}$. When d is odd, this equation at $i = (d-1)/2$ and $i = (d-3)/2$ routinely implies $q \in \{1, -1\}$, and when d is even, this equation at $i = d/2 - 1$ and $i = d/2 - 2$ routinely implies $q \in \{1, -1\}$, a contradiction. □

We are ready to prove Theorems 1.1 and 3.6.

Proof of Theorem 3.6:

(i)⇒(ii): Observe that Γ is isomorphic to the d-cube by Theorem 4.1 and Lemma 5.4.
Applying the relation $ZX - X^-Z = 2X^-$ to K_i $(1 \le i \le d)$ and simplifying with Lemma 4.2, we find that $z_{i-1}x_{i-1}^- b_{i-1}K_{i-1} - z_i x_{i-1}^- b_{i-1}K_{i-1} = 2x_{i-1}^- b_{i-1}K_{i-1}$ $(1 \le i \le d)$. Thus $z_i = z_{i-1} - 2$ $(1 \le i \le d)$, so $z_i = \beta + d - 2i$ $(0 \le i \le d)$, where $\beta = z_0 - d$. By Lemma 5.2, there exists a scalar α such that $\alpha z_i = d - 2i$ $(0 \le i \le d)$. Comparing these formulas for z_i, we find that $\alpha = 1$ and $\beta = 0$. It follows from Lemma 5.2 that $z_i = d - 2i$ $(0 \le i \le d)$ and $s_i = 1$ $(0 \le i \le d-1)$.

(ii)⇒(i): The relations are verified exactly as in Lemma 3.2. We may argue as in Lemma 3.5 to show that these matrices generate $\mathcal{T}$. □

Proof of Theorem 1.1:

(i)⇒(ii): Γ is bipartite and 2-homogeneous by Theorem 4.1. Note that it is not isomorphic to the d-cube by assumption. We apply our results to $U_p(sl(2))$ and use q to denote the parameter of Theorem 3.3 while showing that the formulas for $(p+p^{-1})^2$ and $x_i^- x_{i+1}^+$ hold.

Applying the relation $YX^- = p^2 X^- Y$ to K_i $(1 \le i \le d)$ and simplifying with Lemma 4.2, we find that $y_{i-1}x_{i-1}^- b_{i-1}K_{i-1} = p^2 y_i x_{i-1}^- b_{i-1} K_{i-1}$ $(1 \le i \le d)$. Thus $y_i = y_{i-1}p^{-2}$ $(1 \le i \le d)$, so $y_i = \beta p^{d-2i}$ $(0 \le i \le d)$, where $\beta = y_0 p^{-d}$. By Lemma 5.3, there exists a scalar α such that $\alpha(y_i - y_i^{-1})(p-p^{-1})^{-1} = (q^{d-2i} - q^{-d+2i})(q-q^{-1})^{-1}$ $(0 \le i \le d)$. Combining these formulas,

$$\begin{aligned}&\alpha(\beta p^{d-2i} - \beta^{-1}p^{-d+2i})(p-p^{-1})^{-1}\\ &\quad = (q^{d-2i} - q^{-d+2i})(q-q^{-1})^{-1} \quad (0 \le i \le d).\end{aligned} \tag{20}$$

Suppose d is odd. Then (20) at $i = (d-1)/2$ and $i = (d+1)/2$ routinely implies that $\alpha = \beta \in \{1, -1\}$. Now (20) at $i = (d-3)/2$ gives $p^2 + p^{-2} = q^2 + q^{-2}$. Suppose d is even. Then (20) at $i = d/2$ routinely implies that $\beta \in \{1, -1\}$. Now (20) at $i = d/2 - 1$ and $i = d/2 - 2$ routinely implies that $\alpha = \beta$ and $p^2 + p^{-2} = q^2 + q^{-2}$. In both cases $(p+p^{-1})^2 = (q+q^{-1})^2$, so the formula for $(p+p^{-1})^2$ follows from Corollary 3.4. The formula for $x_i^- x_{i+1}^+$ follows from Lemma 5.3 (with $\epsilon = \alpha$).

(ii)⇒(i): Identical to Lemma 3.5 since the expression for $x_i^- x_{i+1}^+$ in (ii) equals $\epsilon e_i^{-1} e_{i+1}^{-1}$ $(0 \le i \le d-1)$. □

6. Remarks

The 2-homogeneous bipartite distance-regular graphs are essentially known.

Theorem 6.1 [8, 11] *Let $\Gamma = (X, R)$ denote distance-regular graph with diameter $d \ge 3$ and valency $k \ge 3$, and assume that Γ is not isomorphic to the d-cube. Then Γ is bipartite and 2-homogeneous if and only if it is one of the following*:

(i) *the complement of the $2 \times (k+1)$-grid*;

(ii) *a Hadamard graph of order 4γ for some positive integer γ*;

(iii) *a bipartite distance-regular graph with diameter 5 and intersection array*

$$\{b_0, b_1, \ldots, b_4;\ c_1, c_2, \ldots, c_4\} = \{k, k-1, k-\mu, \mu, 1; 1, \mu, k-\mu, k-1, k\},$$

where $k = \gamma(\gamma^2 + 3\gamma + 1)$, $\mu = \gamma(\gamma+1)$ for some integer $\gamma \ge 2$.

When $\gamma = 2$, (iii) is uniquely realized by the antipodal 2-cover of the Higman-Sims graph. No examples of (iii) with $\gamma \ge 3$ are known.

We present some examples of distance-regular graphs related to $U(sl(2))$ and $U_q(sl(2))$ which do not satisfy hypotheses of Theorem 1.1.

Let $\Gamma = (X, R)$ denote the $2d$-cycle $(d \ge 2)$. Fix $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \le i \le d)$ and $\mathcal{T} = \mathcal{T}(x)$. Observe that Γ is vacuously 2-homogeneous. Let q be a primitive

$2d$th root of unity, and set $X^- = \sum_{i=0}^{d-1}[d-i]E_i^* A E_{i+1}^*$, $X^+ = \sum_{i=1}^{d}[i]E_i^* A E_{i-1}^*$, and $Y = \sum_{i=0}^{d} q^{d-2i} E_i^*$. Then X^-, X^+, and Y satisfy (4). However, these matrices do not generate $\mathcal{T}$. The 4-cycle is exceptional. In addition to the $U(sl(2))$ structure of Theorem 3.6, the 4-cycle has the $U_q(sl(2))$ structure of Theorem 1.1 for any non-zero complex number q such that $q^4 \neq 1$.

Let $\Gamma = (X, R)$ denote the Hamming graph $H(d, n)$, $n \geq 3$. Fix $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \leq i \leq d)$ and $\mathcal{T} = \mathcal{T}(x)$. By [10, p. 202], $X^- = L$, $X^+ = R$, and $Z = LR - RL$ satisfy (1). However, these matrices do not generate $\mathcal{T}$ and $Z \notin \mathcal{M}^*$.

It is hoped that some further light will be shed upon the Q-polynomial distance-regular graphs through our work on the 2-homogeneous bipartite distance-regular graphs. Thus in a future paper we will relate the algebraic properties of $\mathcal{T}$ to those of $U_q(sl(2))$.

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