# Distance-Regular Graphs of Valency 6 and $a_1 = 1$

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**Abstract.** We give a complete classification of distance-regular graphs of valency 6 and $a_1 = 1$.

## 1. Introduction

In this paper we only consider undirected finite graphs without loops or multiple edges. Let $\Gamma$ be a connected graph. We identify $\Gamma$ with the set of vertices. For two vertices $\alpha, \beta$, let $\partial(\alpha, \beta)$ denote the usual distance between $\alpha$ and $\beta$ in $\Gamma$. Let

$$\Gamma_i(\alpha) = \{\beta \in \Gamma \mid \partial(\alpha, \beta) = i\} \quad \text{and} \quad \Gamma(\alpha) = \Gamma_1(\alpha).$$

$\Gamma$ is said to be distance-regular if the cardinality of the set

$$D_j^i(\alpha, \beta) = \Gamma_i(\alpha) \cap \Gamma_j(\beta) \quad \text{for every} \quad i, j$$

depends only on the distance between $\alpha$ and $\beta$. In this case we write

$$p_{ij}^m = \left| D_j^i(\alpha, \beta) \right|,$$

where $m = \partial(\alpha, \beta)$. Let $d = d(\Gamma)$ denote the diameter, i.e., the maximal distance of $\Gamma$, and

$$k_i = p_{ii}^0 = |\Gamma_i(\alpha)|.$$

In particular $k = k_1$ is the valency of $\Gamma$. Let $c_i = p_{1\,i-1}^i$, $a_i = p_{1\,i}^i$ and $b_i = p_{1\,i+1}^i$, $0 \le i \le d$.

$$\iota(\Gamma) = \begin{Bmatrix} * & c_1 & \cdots & c_i & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_i & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_i & \cdots & b_{d-1} & * \end{Bmatrix}$$

is called the intersection array of $\Gamma$. Note that $c_i + a_i + b_i = k$, $b_0 = k$ and $c_1 = 1$.

Let $l(c, a, b)$ denote the number of columns ${}^t(c, a, b)$ in $\iota(\Gamma)$ and

$$r(\Gamma) = l(c_1, a_1, b_1).$$

The girth of $\Gamma$, denoted by $g$, is the length of a shortest circuit. In particular, the girth $g$ equals 3 if and only if $a_1 \ne 0$ for a distance-regular graph $\Gamma$.

Information about general theory of distance-regular graphs is given in [1, 5, 8].

In this paper we prove the following theorem.

**Theorem 1.1** *Let $\Gamma$ be a distance-regular graph of valency 6 and $a_1 = 1$. Then one of the following holds.*
(1) *$\Gamma$ is isomorphic to the collinearity graph of the generalized quadrangle of order $(2, 2)$.*
(2) *$\Gamma$ is isomorphic to the collinearity graph of one of the two generalized hexagons of order $(2, 2)$.*
(3) *$\Gamma \simeq H(3, 3)$, the Hamming graph $3^3$.*
(4) *$\Gamma$ is isomorphic to the 3-cover of the collinearity graph of a generalized quadrangle of order $(2, 2)$, the halved Foster graph.*

In [12] A.A. Ivanov proved that the diameter $d(\Gamma)$ of a distance-regular graph $\Gamma$ is bounded by a function of the valency $k$ and $r(\Gamma)$. So in order to classify distance-regular graphs of fixed valency $k$, the major part of work is to give an upper bound of $r(\Gamma)$. On the other hand if $r(\Gamma) \ge 2$, it is easy to see that every maximal clique has size $s + 1 = a_1 + 2$. In particular, $t + 1 = k/(a_1 + 1)$ is an integer. So we define the following.

A distance-regular graph $\Gamma$ is said to be of order $(s, t)$ if $\Gamma(\alpha) \simeq (t + 1) \cdot K_s$ for every vertex $\alpha$. If $\Gamma$ does not have an induced subgraph isomorphic to $K_{2,1,1}$, then a distance-regular graph $\Gamma$ is of order $(s, t)$ for some $s$ and $t$. In particular, this is the case if $r(\Gamma) > 1$ or $a_1 \le 1$. In this terminology this paper is concerned with a classification of distance-regular graphs of order $(2, 2)$.

Let $\Gamma$ be a distance-regular graph of order $(s, t)$.

If $t = 0$, it is clear that $\Gamma$ is a complete graph.

If $t = 1$, $\Gamma$ is a line graph and we have a classification of such graphs. See [8, 13] and Proposition 6.2.

We are interested in the next case, $t = 2$. If $s = 1$, i.e., $a_1 = 0$ then $k = 3$. A classification of distance-regular graphs of valency 3 is completed by Ito [11], Biggs–Boshier–Shawe-Taylor [6] and Bannai–Ito [3]. In this paper we treat the case $s = t = 2$. It seems that the situation is a little different in each of the following cases.

$$t < s, \quad t = s \quad \text{and} \quad t > s.$$

Actually, stimulated by our result and the techniques developed in this paper, N. Yamazaki proved the following [16].

**Theorem 1.2** *Let $\Gamma$ be a distance-regular graph of order $(s, 2)$ with $s > 2$. Then one of the following holds.*
(1) $d(\Gamma) \leq r(\Gamma) + 2$.
(2) $\Gamma$ is a bipartite half of a distance-biregular graph with vertices of valency three.

We note here that the condition $s > 2$ is essential in his proof. We believe that our case is one of the key parts of the classification of distance-regular graphs with $t = 2$.

We also note that $k = 6$, $a_1 = 1$ is the smallest unsettled case with girth equals 3.

For the convenience of the reader, we also give a classification of distance-regular graphs of valency $k \leq 7$, girth 3 in the last section. Except the case $k = 6$, $a_1 = 1$, the results may be known to some specialists.

In [15], the third author called a distance-regular graph *extra thin* if $b_1 = c_{d-1}$. If an extra thin distance-regular graph satisfies $a_d \neq 0$, then $k = a_d(a_d + 1)$ and $a_1 = a_d - 1$. So Theorem 1.1 includes the nonexistence of extra thin distance-regular graphs with $a_d = 2$.

Our proof is divided into two parts.

In the first part we apply combinatorial arguments to show that either $d(\Gamma) \leq r(\Gamma) + 2$ or $\Gamma$ is a bipartite half of a bipartite distance-regular graph of valency 3. We use intersection diagrams and investigate the clique patterns on the diagram of rank 1. (See the last part of Section 2.) After determining the clique patterns, we apply circuit chasing techniques. See Sections 3 and 4.

Since bipartite distance-regular graphs of valency 3 are completely classified [11], in the second part we assume $d(\Gamma) \leq r(\Gamma) + 2$. We use eigenvalue techniques. We follow mainly the techniques developed by E. Bannai and T. Ito [2–4]. Using additional information in our case and refinement in computation, we could obtain a bound $r(\Gamma) \leq 17$. Now it is not hard to determine the feasible arrays either by computer testing the integrality condition of multiplicities of eigenvalues, or by hand checking the divisibility condition coming from the number of circuits of certain types.

We also note here the importance of these two parts. Under our assumption, it is not hard to show that if $l(c, a, b) \geq 2$, then $(c, a, b) = (1, 1, 4), (2, 2, 2)$ or $(4, 1, 1)$. By a result of E. Bannai and T. Ito in [2], $l(2, 2, 2) \leq 10 \cdot 6 \cdot 2^6$. Now we can apply the main theorem in [4] and $d(\Gamma)$ is bounded. So in this sense, $d(\Gamma)$ is theoretically bounded. In order to get a complete classification, however, we need to obtain a reasonable upper bound of $d(\Gamma)$. For that reason, it was essential to show that $d(\Gamma) \leq r(\Gamma) + 2$.

Our notation and terminologies are standard except the following.

Let $e(A, B)$ denote the number of edges between subsets A, B of $\Gamma$. Instead of $e(\{x\}, B)$, we will write $e(x, B)$.

For an edge $\alpha \sim \beta$,

$$D(\alpha, \beta) = D_1^1(\alpha, \beta) = \Gamma(\alpha) \cap \Gamma(\beta).$$

Let $u, v$ be vertices at distance $i$. We call the induced subgraph on $\Gamma_{i-1}(u) \cap \Gamma(v)$, the $c_i$-graph or the $c_i$-graph $\Gamma_{i-1}(u) \cap \Gamma(v)$. The $b_i$-graph and the $a_i$-graph are defined similarly.

## 2.  Preliminaries

In this section we collect several results we apply in the following sections.

**Proposition 2.1** [10, *Lemma* 3.2]   *Let* $\Gamma$ *be a distance-regular graph. Let* $r = r(\Gamma)$. *Suppose* $c_{r+1} = 1$ *and for all vertices* $\alpha, \beta \in \Gamma$ *with* $\partial(\alpha, \beta) = r + 2$, *the* $c_{r+2}$-*graph* $\Gamma_{r+1}(\alpha) \cap \Gamma(\beta)$ *is a coclique. Then there exists an integer* $n \geq a_1 + 1$ *such that* $a_{r+1} - a_1 = n(a_1 + 1)$.

**Lemma 2.2** [8, *Lemma* 5.5.2]   *Let* $\Gamma$ *be a distance-regular graph and let* $2 \leq i \leq d - 1$. *Suppose that for all vertices* $\alpha, \beta$ *with* $\partial(\alpha, \beta) = i + 1$, *the* $c_{i+1}$-*graph* $\Gamma_i(\alpha) \cap \Gamma(\beta)$ *is a clique. Then* $c_{i+1} = 1$.

**Lemma 2.3**   *Let* $\Gamma$ *be a distance-regular graph with* $k = 6, a_1 = 1$. *If* $b_{d-1} = 1$, *then one of the following holds.*
(1) $c_d = 6$ *and* $k_{d-1} \equiv 0 \pmod 3$.
(2) $c_d = 4$ *and* $k_d \equiv 0 \pmod 3$.

**Proof:**   If $c_d = 6$, the assertion is obvious.

Suppose $c_d \neq 6$, i.e., $a_d \neq 0$. Let $\partial(u, v) = d$ and $D_j^i = \Gamma_i(u) \cap \Gamma_j(v)$. Since $b_{d-1} = 1$, $e(D_1^{d-1}, D_1^d \cup D_2^d) = 0$. See figure 1. So a subgraph $D_1^{d-1}$ is 1-regular and $c_d \equiv 0 \pmod 2$. If $c_d = 2$, the $c_d$-graph $D_1^{d-1}$ is a clique. This contradicts Lemma 2.2. Hence $c_d = 4 = b_1$. So $e(D_1^d, D_2^d) = 0$. This implies that every connected component of $\Gamma_d(\alpha)$ is a clique of size 3. Thus $k_d \equiv 0 \pmod 3$.                                                      $\square$

**Proposition 2.4** [9]   *Let* $\Gamma$ *be a distance-regular graph of valency* $k$, $a = a_1 \neq 0$. *Let* $r = r(\Gamma)$. *Suppose*

$$(c_{r+1}, a_{r+1}, b_{r+1}) = \cdots = (c_{r+t}, a_{r+t}, b_{r+t}) = (2, 2a, b).$$
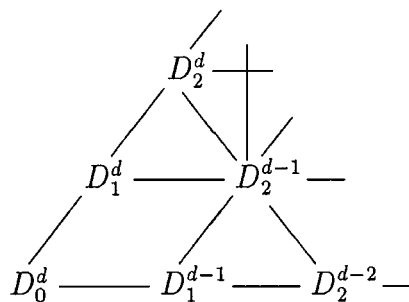
*Then* $t \leq 1$.



*Figure 1.*   Rank $d$ diagram with $b_{d-1} = 1$.

In this and the following two sections we will use intersection diagrams of rank 1. For intersection diagrams, see for example [7, 9].

**Lemma 2.5** *Let $\Gamma$ be a distance-regular graph of order $(s, t)$ and $r = r(\Gamma)$. For every pair of vertices $\alpha$ and $x$ with $\partial(\alpha, x) = r + 1$, the $c_{r+1}$-graph $\Gamma_r(\alpha) \cap \Gamma(x)$ is a coclique.*

**Proof:** We may assume that $c_{r+1} \geq 2$. Take any $y_1, y_2 \in \Gamma_r(\alpha) \cap \Gamma(x)$. Let $\{\beta\} = \Gamma(\alpha) \cap \Gamma_{r-1}(y_1)$.

The intersection diagram with respect to $(\alpha, \beta)$ has the following shape. (See [9].) In particular, we have $e(D_r^{r+1}, D_r^r) = 0$.

Note that $y_1 \in D_{r-1}^r$ and $x \in D_r^{r+1}$. Since

$$1 = c_r = e(x, D_{r-1}^r) = |\{y_1\}|,$$

$y_2$ must be in $D_{r+1}^r$. Hence we have $y_1 \not\sim y_2$. □

Let $\Gamma$ be a distance-regular graph with $k = 6$, $a_1 = 1$, and $r = r(\Gamma)$. Then for each vertex $x \in \Gamma$, $\Gamma(x) = 3 \cdot K_2$, i.e., a disjoint union of three $K_2$'s. We fix the following notation in this and the following two sections.

$$\partial(\alpha, \beta) = 1, \quad D_j^i = \Gamma_i(\alpha) \cap \Gamma_j(\beta), \quad D_1^1 = \{\gamma\}.$$

Note that $|D_1^1| = 1$ as $a_1 = 1$.

We introduce three terms which play key role in the following sections. They are '*clique type*', '*vertex type*' and '*clique pattern*'. Let $x = \{\zeta, \eta, \xi\}$ be a clique. By the clique type (with respect to a vertex $\pi$) of $x$, we mean

$$(\partial(\pi, \zeta) - r, \partial(\pi, \eta) - r, \partial(\pi, \xi) - r)$$

the list of distances from $\pi$ minus $r$ of vertices in a clique.

We call

$$\vec{\delta} = (\partial(\alpha, \delta) - r, \partial(\gamma, \delta) - r, \partial(\beta, \delta) - r)$$

the type of a vertex $\delta$ (with respect to $(\alpha, \beta)$). We also use column vectors in the figures. The clique pattern at $\delta$ is the collection of types of vertices in $\Gamma(\delta)$ with edges among them.
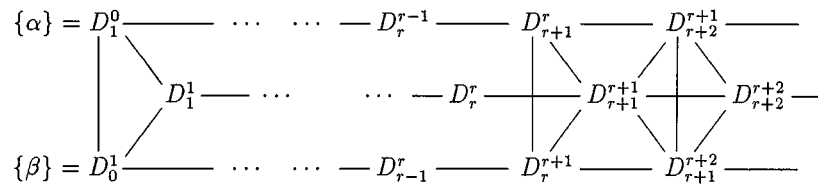


*Figure 2.* Rank 1 diagram with $c_{r+1} > 1$.

The first part of our proof is to determine these things. For example, if $(c_{r+1}, a_{r+1}, b_{r+1}) = (2, 2, 2)$, Lemma 2.5 tells us that there are no cliques of type $(0, 0, 1)$, $(1, 1, 1)$, $(1, 1, 2)$. This is equivalent to say that there are no vertices of the same types. Next we determine the possibilities of the clique pattern at a vertex which is of certain type. By this argument and circuit chasing techniques we will show that either $d(\Gamma) \leq r + 2$ or $\Gamma$ is a halved graph of a bipartite graph of valency 3.

## 3.    The case $c_{r+1} \geq 2$

In this section we prove the following.

**Theorem 3.1**    *Let $\Gamma$ be a distance-regular graph with $k = 6, a_1 = 1$. Let $r = r(\Gamma)$. If $c_{r+1} \geq 2$, then $d \leq r + 2$ and one of the following holds.*
*(1)  $d = r + 1$ and $c_{r+1} \leq 3$.*
*(2)  $(c_{r+1}, a_{r+1}, b_{r+1}) = (2, 2, 2)$ and $c_{r+2} = 3$.*
*(3)  $(c_{r+1}, a_{r+1}, b_{r+1}) = (2, 3, 1)$ and $c_{r+2} = 6$.*

**Proof:**    As we remarked in the previous section, for each vertex $x \in \Gamma$, $\Gamma(x) \simeq 3 \cdot K_2$, i.e., a disjoint union of three $K_2$'s.

Since every $c_{r+1}$-graph is a coclique by Lemma 2.5, $c_{r+1} \leq 3$. Moreover if $c_{r+1} = 3$, then $b_{r+1} = 0$, i.e., $d = r + 1$. Hence we may assume that $c_{r+1} = 2$ and $b_{r+1} \geq 1$ to prove our theorem.

Take any edge $x \sim y$ with $x \in \Gamma_{r+1}(\alpha)$ and $y \in \Gamma_{r+2}(\alpha)$. Since $c_{r+1} = 2$, we have $\{z_1, z_2\} = \Gamma_r(\alpha) \cap \Gamma(x)$. We have $z_1 \not\sim z_2$ by Lemma 2.5. Hence $\{w_j\} = D(x, z_j) \subset \Gamma_{r+1}(\alpha)$ for $j = 1, 2$. This means $b_{r+1} \leq 2$.

**Case 1.**    $b_{r+1} = 2$, i.e., $\{y'\} = D(x, y) \subset \Gamma_{r+2}(\alpha)$.

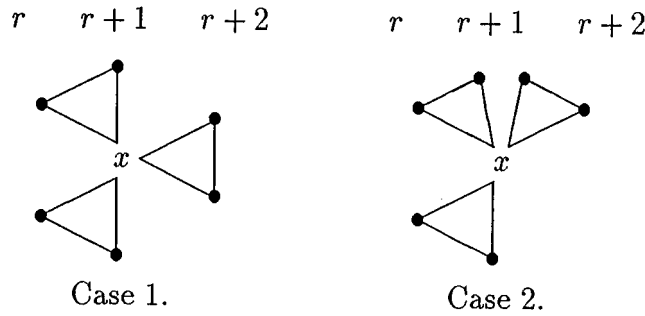The types of cliques in $\Gamma(x) \cup \{x\}$ for each $x \in \Gamma_{r+1}(\alpha)$ are as depicted in figure 3.



*Figure 3.*    Clique types of $x \in \Gamma_{r+1}(\alpha)$ with $c_{r+1} > 1$.

This implies that every $c_{r+2}$-graph is a coclique. Hence we have $c_{r+2} \leq 3$. Moreover, if $c_{r+2} = 3$, then $d = r + 2$, we have (2).

Suppose $c_{r+2} = 2$. Then $e(D_{r+1}^{r+2}, D_{r+2}^{r+1}) = 0$ and by Proposition 2.4, $b_{r+2} \neq 2$. So we have an edge $y \sim u$ with $y \in D_{r+1}^{r+2}$ and $u \in D_{r+2}^{r+2}$.

Suppose there exists a vertex $v \in \Gamma(u) \cap D_{r+1}^{r+1}$. Since $\alpha, \beta \in \Gamma_{r+1}(v)$, we have $\gamma \in \Gamma_r(v)$ from figure 3. This means that $\partial(u, \gamma) = r + 1$. Moreover, $\alpha \in \Gamma_{r+2}(y)$ and $\beta \in \Gamma_{r+1}(y)$ implies $\gamma \in \Gamma_{r+2}(y)$. Hence

$$D_{r+2}^{r+1}(\gamma, \beta) \ni u \sim y \in D_{r+1}^{r+2}(\gamma, \beta).$$

This is a contradiction.

On the other hand suppose $e(u, D_{r+1}^{r+1}) = 0$. Then $\Gamma(u) \cap (D_{r+2}^{r+1} \cup D_{r+1}^{r+2})$ is a union of two cocliques of size 2 without edges in between, because it contains two $c_{r+2}$-graphs $\Gamma_{r+1}(\alpha) \cap \Gamma(u)$ and $\Gamma_{r+1}(\beta) \cap \Gamma(u)$, and $e(D_{r+1}^{r+2}, D_{r+2}^{r+1}) = 0$. Hence $\Gamma(u)$ contains a coclique of size 4, a contradiction. Thus we have (2) in this case.

**Case 2.**     $b_{r+1} = 1$, i.e., $\{y'\} = D(x, y) \subset \Gamma_{r+1}(\alpha)$.

The types of cliques in $\Gamma(x) \cup \{x\}$ for each $x \in \Gamma_{r+1}(\alpha)$ are as depicted in figure 3.

This implies that every $c_{r+1}$-graph is a coclique of size 2 and every $c_{r+2}$-graph is a union of $K_2$'s. If $c_{r+2} = 6$, we have (3). Hence we may assume that $c_{r+2} = 4$, as every $c_{r+2}$-graph always contains a $c_{r+1}$-graph as a subgraph.

Let $\delta \in D_{r+1}^{r+1}$. Since $\alpha, \beta \in \Gamma_{r+1}(\delta)$, we have $\gamma \in \Gamma_r(\delta) \cup \Gamma_{r+2}(\delta)$ from figure 3. Moreover, $D_{r+1}^{r+1} \cap \Gamma_{r+2}(\gamma) \neq \emptyset$, for example every vertex in $\Gamma_{r+1}(\alpha) \cap \Gamma_{r+2}(\gamma)$ satisfies this condition. Note that $\delta \in D_{r+1}^{r+1} \cap \Gamma_{r+2}(\gamma)$, i.e., $\delta$ is of type $(1, 2, 1)$, if and only if $e(\delta, D_r^r) = 0$.

Take a vertex $x$ of type $(1, 2, 1)$. Let $\{z_1, z_2\} = \Gamma(x) \cap D_{r+1}^r$, $\{z_1', z_2'\} = \Gamma(x) \cap D_r^{r+1}$ with $z_1 \sim z_1'$, $z_2 \sim z_2'$.

Now we apply a circuit chasing technique.

Take a circuit $x_0 \sim x_1 \sim \cdots \sim x_{2r+2} \sim x_0$ of length $2r + 3$ such that

$$x_0 \in D_1^0, \qquad x_i \in D_{i-1}^i, \quad i = 1, \ldots, r+1, \quad x_{r+1} = z_2'$$

$$x = x_{r+2} \in D_{r+1}^{r+1}, \qquad x_{r+j} \in D_{r+4-j}^{r+3-j}, \quad j = 3, \ldots, r+2, \quad x_{r+3} = z_1.$$

Let $\{y_i\} = D(x_i, x_{i+1})$. We have $y_0 = \gamma$, $y_{r+1} = z_2$, $y_{r+2} = z_1'$, and this circuit does not contain a triangle, i.e., $x_i \not\sim x_{i+2}$. See figure 4.

Changing the base points to $x_1, x_2$, we have easily that

$$x_{r+2} \in D_r^{r+1}, \qquad x_{r+3} \in D_{r+1}^{r+1}, \qquad x_{r+4} \in D_{r+1}^r.$$

Since $\partial(x_1, y_{r+2}) = r$, $y_{r+2} \in D_{r+1}^r$. Since $\{x_{r+4}, y_{r+2}\} \subset D_{r+1}^r \cap \Gamma(x_{r+3})$, $e(x_{r+3}, D_r^r) = 0$. This implies that $x_{r+3} \in \Gamma_{r+2}(y_1)$ and it is of same type as $x$. In particular, $\partial(x_2, y_{r+3}) = r$. By induction we have that
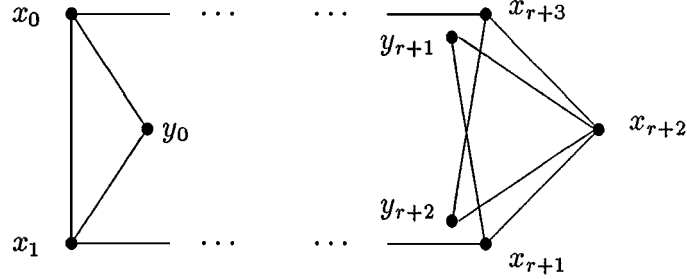
$$r = \partial(x_{r+2}, y_0) = \partial(x, y_0) = r + 2.$$

*Figure 4.*   Circuit of length $2r + 3$.

This is a contradiction.

This completes the proof of Theorem 3.1.                                              □

## Remark

1. For Case 2, our original proof was different. We showed that if $d \geq r+2$, $\Gamma$ is a bipartite half of a bipartite distance-regular graph of valency 3. (See $(c_{r+1}, a_{r+1}, b_{r+1}) = (1, 3, 2)$ case in the next section.) The above proof was suggested by N. Yamazaki.
2. Case (3) in Theorem 3.1 does not occur. Actually, we could eliminate this case. However we decided to eliminate this case after bounding the diameter in Section 5 to avoid lengthy arguments.

## 4.   The case $c_{r+1} = 1$

In this case we prove the following.

**Theorem 4.1**   *Let $\Gamma$ be a distance-regular graph with $k = 6, a_1 = 1$. Let $r = r(\Gamma)$. If $c_{r+1} = 1$, then one of the following holds.*
(1) $d = r + 1$.
(2) $d = r + 2$,    $a_{r+1} = 4$,    $c_{r+2} = 6$.
(3) $d = r + 2$,    $a_{r+1} = 3$,    $c_{r+2} = 3$ *or* 4.
(4) $d = r + 2$,    $a_{r+1} = 2$,    $c_{r+2} = 2, 3$ *or* 4.
(5) $\Gamma$ *is a bipartite half of a bipartite distance-regular graph of valency* 3.

**Proof:**   Throughout this proof we assume that $d \geq r+2$. First we note that $e(D_{r+1}^r, D_r^{r+1}) = 0$ as $c_{r+1} = 1$. By Proposition 2.1, there is a $c_{r+2}$-graph, which is not a coclique. In particular $c_{r+2} \geq 2$. We argue three cases separately depending on the values of $a_{r+1}$.

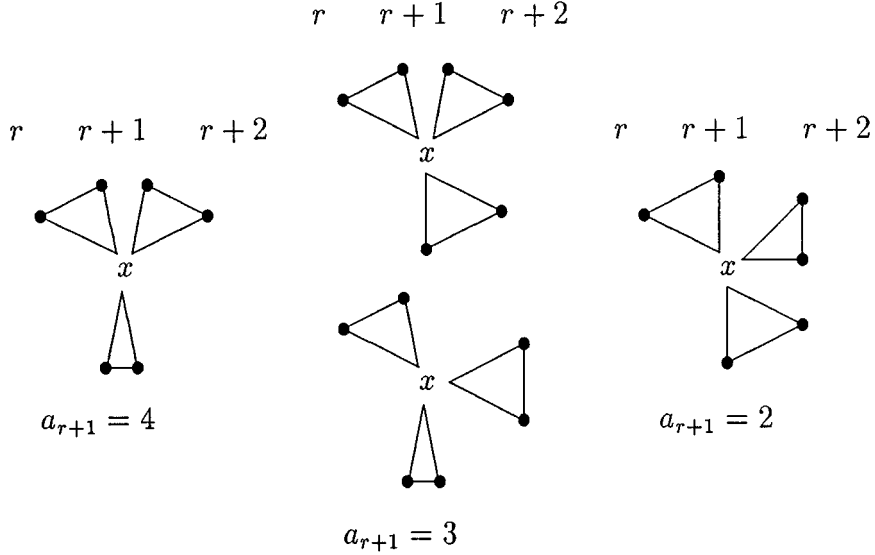The following are the clique types of vertices in $\Gamma(x) \cup \{x\}$ for each $x \in \Gamma_{r+1}(\alpha)$.

*Figure 5.* Clique types of $x \in \Gamma_{r+1}(\alpha)$ with $c_{r+1} = 1$.

**Case 1.** $a_{r+1} = 4$.

By figure 5, it is easy to see that every $c_{r+2}$-graph is a union of $K_2$'s, and therefore $c_{r+2} \in \{2, 4, 6\}$. $c_{r+2} = 2$ is impossible by Lemma 2.2. We want to show that $c_{r+2} = 6$. So we will assume $c_{r+2} = 4$ to derive a contradiction.

*Step 1.* $D_{r+1}^{r+2} \cup D_r^{r+1} \cup D_{r+1}^r \cup D_{r+2}^{r+1} \subset \Gamma_{r+1}(\gamma)$.
    Since $\{\alpha, \beta, \gamma\}$ is a clique, the assertion follows easily from figure 5.
*Step 2.* $e(D_{r+1}^{r+2}, D_{r+2}^{r+1}) = 0$.
    We may assume $d = r + 2$, as otherwise $b_{r+1} = b_{r+2} = 1$ and the assertion is obvious. Suppose there exists an edge $z \sim z'$ such that $z \in D_{r+1}^{r+2}$ and $z' \in D_{r+2}^{r+1}$. Let $\{y\} = D_r^{r+1} \cap \Gamma(z)$. Since $c_{r+1} = b_{r+1} = 1$, $\{y'\} = D(z, y) \subset D_{r+1}^{r+1}$ and $\{x\} = D(z, z') \subset D_{r+1}^{r+1}$. Hence the clique pattern at $z$ is as in figure 6. Consider the clique pattern at $x$. Then there are adjacent vertices $w, w'$ in $\Gamma(x)$ both of type $(1, 1, 1)$. Hence the clique pattern at $w$ is as in figure 6. Then $\Gamma(w) \cap \Gamma_{r+2}(\gamma) = \emptyset$ by Step 1, a contradiction.
*Step 3.* Let $u \in D_{r+1}^{r+1}$ such that $\partial(u, \gamma) \geq r + 1$. Then there are three possible clique patterns. See figure 7.

$$\text{Let}\{v\} = D_r^{r+1} \cap \Gamma(u), \qquad \{v'\} = D_{r+1}^r \cap \Gamma(u).$$

Suppose $\{w\} = D(u, v) \subset D_{r+1}^{r+2}$. Then there exists a vertex $w' \in \Gamma(u) \cap D_{r+2}^{r+1}$. Since $b_{r+1} = c_{r+1} = 1$, the other two vertices $x, y$ in $\Gamma(u)$ are in $D_{r+1}^{r+1}$. Since $w$ is of type
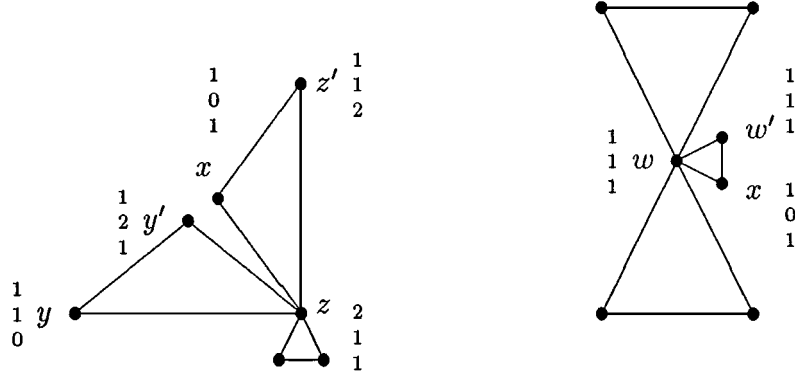
*Figure 6.*



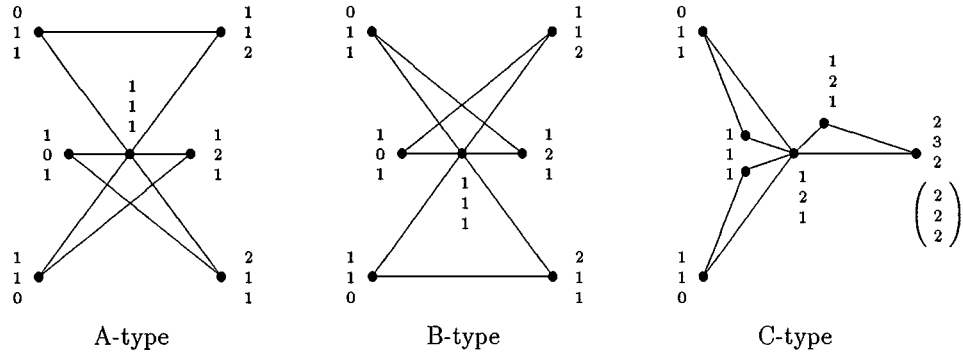A-type                              B-type                              C-type

*Figure 7.*   Clique patterns of the case $a_{r+1} = 4$.

$(2, 1, 1)$, $u$ cannot be of type $(1, 2, 1)$ by Step 2, as otherwise

$$D_{r+1}^{r+2}(\alpha, \gamma) \ni w \sim u \in D_{r+2}^{r+1}(\alpha, \gamma).$$

Hence $u$ is of type $(1, 1, 1)$, and we may assume that

$$\vec{x} = (1, 0, 1), \qquad \vec{y} = (1, 2, 1).$$

Since $y \not\sim x$ and $y \not\sim w'$, $x \sim v'$. We have B-type. By symmetry, we have A-type if $D(u, v') \subset D_{r+2}^{r+1}$.

Now we may assume that

$$D(u, v) \cup D(u, v') \subset D_{r+1}^{r+1}.$$

Let $\{x\} = D(u, v)$, $\{x'\} = D(u, v')$. The other two vertices $y$, $w$ in $\Gamma(u)$ must lie in $D_{r+1}^{r+1}$ and $D_{r+2}^{r+2}$ by Step 2. Since $b_{r+1} = 1$, $\partial(w, \gamma) \geq r + 2$, as otherwise $\partial(w, \gamma) = r + 1$ and

$$2 = |\{\alpha, \beta\}| \leq |\Gamma_{r+2}(w) \cap \Gamma(\gamma)| = b_{r+1}.$$

We have $\partial(y, \gamma) \geq r + 1$. Hence $\vec{u} = (1, 2, 1)$, as otherwise $u$ must be adjacent to a vertex in $\Gamma_r(\gamma)$, which is impossible. Therefore we also have $\vec{x} = \vec{x}' = (1, 1, 1)$.

If $d \geq r + 3$, then $\vec{w} = (2, 3, 2)$, $\quad \vec{y} = (1, 2, 1)$, and we have C-type.

If $d = r + 2$, then $\vec{w} = (2, 2, 2)$, $\quad \vec{y} = (1, 2, 1)$, as $\vec{y} = (1, 1, 1)$ implies $b_{r+1} \geq 2$.

*Step 4.* $r \equiv 0 \pmod 3$.

We apply a circuit chasing technique.

Take a circuit $x_0 \sim x_1 \sim \cdots \sim x_{2r+2} \sim x_0$ of length $2r + 3$ such that

$$x_0 \in D_1^0, \qquad x_i \in D_{i-1}^i, \quad i = 1, \ldots, r+1,$$
$$x_{r+2} \in D_{r+1}^{r+1}, \qquad x_{r+j} \in D_{r+4-j}^{r+3-j}, \quad j = 3, \ldots, r + 2.$$

It is easy to see that with respect to the base points $x_i$, $x_{i+1}$,

$$x_{i+j} \in D_{j-1}^j, \quad j = 1, \ldots, r+1, \qquad x_{r+2+i} \in D_{r+1}^{r+1},$$
$$x_{r+i+j} \in D_{r+4-j}^{r+3-j}, \quad j = 3, \ldots, r + 3,$$

where the indices of $x_i$'s are taken modulo $2r + 3$.

Assume that $x_{r+2}$ is of A-type. Let $\{y_i\} = D(x_i, x_{i+1})$. In particular

$$y_{r+2} \in \Gamma_{r+1}(x_0) \cap \Gamma_{r+1}(y_0) \cap \Gamma_{r+2}(x_1).$$

Changing the base points to $x_1$, $x_2$, we have $y_{r+2} \in D_{r+1}^{r+2}$, as

$$x_{r+2} \in D_r^{r+1}, \qquad x_{r+3} \in D_{r+1}^{r+1}, \quad \text{and} \quad y_{r+2} \in \Gamma_{r+2}(x_1).$$

Hence $x_{r+3}$ is of B-type and $y_{r+3} \in \Gamma_{r+2}(y_1) \cap D_{r+1}^{r+1}$.

Changing the base points to $x_2$, $x_3$, we have $y_{r+3} \in D_{r+1}^{r+1}$ as before.

We claim that $y_{r+3} \in \Gamma_{r+1}(y_2)$. Since $y_{r+3} \sim x_{r+3} \in D_r^{r+1}$, $\partial(y_{r+3}, y_2) \geq r + 1$. If $y_{r+3} \in \Gamma_{r+2}(y_2)$, then the $b_{r+1}$-graph $\Gamma(x_2) \cap \Gamma_{r+2}(y_{r+3})$ contains $y_1$, $y_2$. This contradicts $b_{r+1} = 1$. Thus $y_{r+3} \in \Gamma_{r+1}(y_2)$ and $y_{r+3}$ is of A-type. So $x_{r+4}$ must be of C-type and $y_{r+4} \in \Gamma_{r+1}(y_2) \cap D_{r+1}^{r+1}$.

Changing again the base points to $x_3$, $x_4$, we have $y_{r+4} \in D_{r+1}^{r+1}$.

We claim that $y_{r+4} \in \Gamma_{r+2}(y_3)$. If $y_{r+4} \in \Gamma_{r+1}(y_3)$, then

$$x_2, y_2, x_4, y_3 \in \Gamma_{r+1}(y_{r+4}) \cap \Gamma(x_3)$$
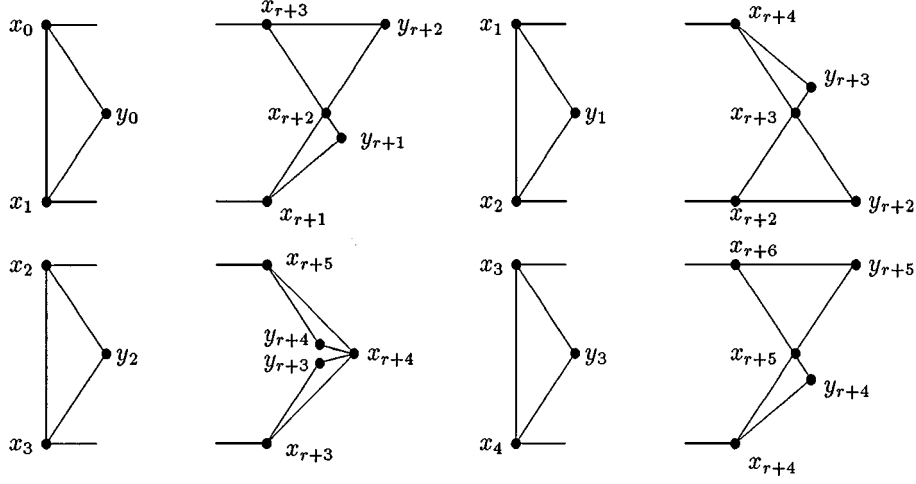
*Figure 8.*   Circuit of length $2r + 3$.

form $2 \cdot K_2$. This is impossible by figure 5. Hence $y_{r+4} \in \Gamma_{r+2}(y_3)$ and $x_{r+5}$ is of A-type. We have proved that the type changes

$$A \to B \to C \to A$$

with period 3, as we change the base points successively in this circuit. Thus we can conclude that $2r + 3 \equiv 0 \pmod 3$.

*Step 5.*   The case $c_{r+2} = 4$ is not possible.

In the following, we show that $r \equiv 1 \pmod 3$ to derive a contradiction.

Take a circuit $x_0 \sim x_1 \sim \cdots \sim x_{2r+3} \sim x_0$ of length $2r + 4$ such that

$$x_0 \in D_1^0, \qquad x_i \in D_{i-1}^i, \quad i = 1, \ldots, r + 2,$$

$$x_{r+3} \in D_{r+1}^{r+1}, \qquad x_{r+j} \in D_{r+5-j}^{r+4-j}, \quad j = 4, \ldots, r + 3.$$

Assume that $x_{r+3}$ is of A-type. We note that this circuit does not contain triangles. Let $\{y_i\} = D(x_i, x_{i+1})$.

Changing the base points to $x_1, x_2$, we have $x_{r+3} \in D_{r+1}^{r+1}$, and $\partial(x_{r+3}, y_1) \geq r + 1$.

If $\partial(x_{r+3}, y_1) = r+1$, then we can argue as before and conclude that $\Gamma_{r+1}(x_{r+3}) \cap \Gamma(x_1) \ni x_0, y_0, x_2, y_1$ is an $a_{r+1}$-graph containing $2 \cdot K_2$, which is a contradiction. Hence $x_{r+3}$ is of C-type and $y_{r+2} \in \Gamma_{r+1}(y_1)$. Moreover $y_{r+3} \in D_{r+2}^{r+2}$, as $y_{r+3} \in \Gamma_{r+2}(x_1) \cap \Gamma(x_{r+3})$, and we have $x_{r+4} \in \Gamma_{r+2}(y_1) \cap D_{r+1}^{r+1}$.

Changing the base points to $x_2, x_3$, we have $x_{r+4} \in D_{r+1}^{r+1}$, and $y_{r+3} \in D_{r+1}^{r+2}$. Hence $x_{r+4}$ is of B-type. Since $x_{r+5} \in D_1^1(x_{r+4}, x_{r+6})$, and $x_{r+4} \not\sim x_{r+6}$, we have $x_{r+5} \in D_{r+2}^{r+1}$.

Changing again the base points to $x_3, x_4$, we have $x_{r+5} \in D_{r+1}^{r+2}, x_{r+6} \in D_{r+1}^{r+1}$ by Step 2. Since $x_{r+4} \not\sim x_{r+6}$, $x_{r+6}$ is not of B-type. Thus $x_{r+6}$ is of A-type.
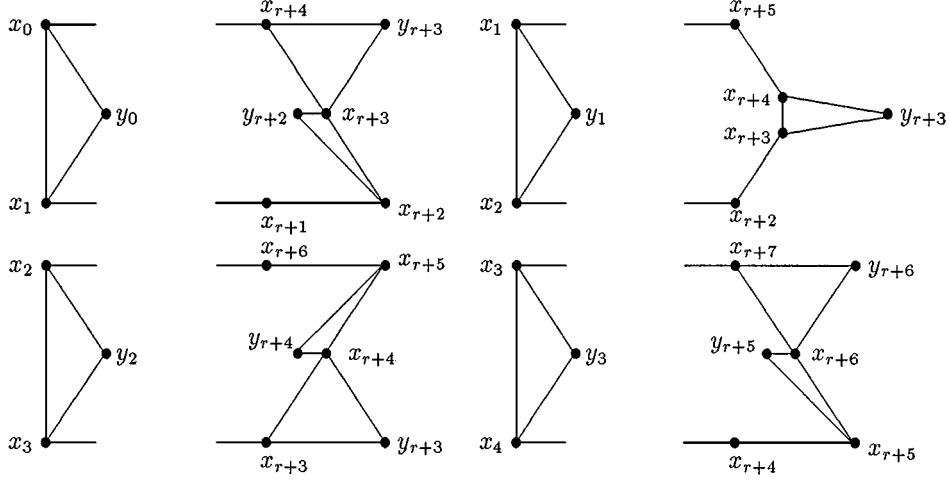
*Figure 9.* Circuit of length $2r + 4$.

Hence we have the same profile with respect to $x_0, x_1$.

Therefore we conclude that $2r + 4 \equiv 0 \pmod 3$. This contradicts Step 4.

Therefore we have (2) in this case.

**Case 2.** $a_{r+1} = 3$.

In this case, $c_{r+2} = 2, 3, 4,$ or $6$.

Suppose $c_{r+2} = 3$, and $d \geq r + 3$. If $b_{r+2} = 2$, then every $b_{r+2}$-graph is a clique, hence so is every $b_{r+1}$-graph. This is impossible as $c_{r+2} = 3$.

If $b_{r+2} = 1$, then we have a contradiction by Lemma 2.3. Hence $d \leq r + 2$, in this case. Next we treat the case $c_{r+2} = 2$.

**Lemma 4.2** *If* $(c_{r+1}, a_{r+1}, b_{r+1}) = (1, 3, 2)$, *with* $c_{r+2} = 2$, *then* $e(D_{r+2}^{r+1}, D_{r+1}^{r+2}) \neq 0$.

**Proof:** Suppose $e(D_{r+2}^{r+1}, D_{r+1}^{r+2}) = 0$.

*Step 1.* Let $u \in D_{r+1}^{r+1}$, $v \in D_r^{r+1}$, and $w \in D_{r+1}^{r+2}$ be a triangle. Then the clique pattern at the vertex $u$ is as in figure 10.

Since $c_{r+2} = 2$, $e(w, D_{r+1}^{r+1}) = 1$ and $w \in \Gamma_{r+2}(\gamma)$, as otherwise $w \in \Gamma_{r+1}(\gamma)$ and there would exist a vertex $u'$ such that

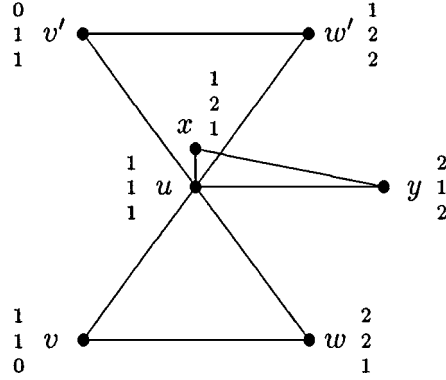$$u \neq u' \in \Gamma_r(\gamma) \cap \Gamma(w) \subset D_{r+1}^{r+1}.$$

*Figure 10.*   Clique pattern of the case $a_{r+1} = 3$ (1).

Let $w'' \in \Gamma(v) \cap D_{r+1}^{r+2}$ with $w'' \neq w$. Then $w'' \in \Gamma_{r+2}(\gamma)$ as above. Since $b_{r+1} = 2$ and $v \in \Gamma_{r+1}(\gamma)$, $u \in \Gamma_{r+1}(\gamma)$. Hence there is a vertex $x \in D_{r+1}^{r+1} \cap \Gamma(u) \cap \Gamma_r(\gamma)$. Now Step 1 follows immediately.

*Step 2.*   There is no triangle $u \in D_{r+1}^{r+1}, v \in D_r^{r+1}, w \in D_{r+1}^{r+2}$.

Suppose there exists a triangle $\{u, v, w\}$ with $u \in D_{r+1}^{r+1}$, $v \in D_r^{r+1}$ and $w \in D_{r+1}^{r+2}$. Take a circuit $x_0 \sim x_1 \sim \cdots \sim x_{2r+2} \sim x_0$ of length $2r + 3$ such that

$$x_0 \in D_1^0, \qquad x_i \in D_{i-1}^i, \quad i = 1, \ldots, r+1,$$

$$x_{r+2} \in D_{r+1}^{r+1}, \qquad x_{r+j} \in D_{r+4-j}^{r+3-j}, \quad j = 3, \ldots, r+2.$$

Let $\{y_i\} = D(x_i, x_{i+1})$. Suppose $u = x_{r+2}$, $v = x_{r+1}$, $w = y_{r+1}$. So $x_0 \in \Gamma_{r+2}(y_{r+1})$. Then by Step 1, $y_{r+2} \in D_{r+2}^{r+1}$. See figure 11.

Changing the base points to $x_1, x_2$, we have that $x_{r+2} \in D_r^{r+1}, x_{r+3} \in D_{r+1}^{r+1}$, and $x_{r+4} \in D_{r+1}^r$. Since $y_{r+2} \in \Gamma_{r+2}(x_1)$, $y_{r+2} \in D_{r+1}^{r+2}$. Thus again by Step 1, $y_{r+3} \in D_{r+2}^{r+1}$ and $x_{r+3} \in \Gamma_{r+1}(y_1)$.
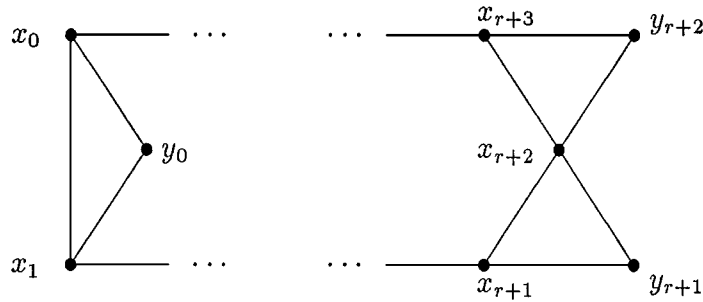


*Figure 11.*   Circuit of length $2r + 3$.

By induction we can conclude that

$$x_0 \in D_{r+1}^{r+1}(x_{r+1}, x_{r+2}) \quad \text{and therefore} \quad x_0 \in \Gamma_{r+1}(y_{r+1}).$$

Since $x_0 \in \Gamma_{r+2}(y_{r+1})$, this is a contradiction.

By Step 2, we have that the $c_{r+2}$-graph is always a coclique, if we assume $e(D_{r+2}^{r+1}, D_{r+1}^{r+2}) = 0$. This contradicts Proposition 2.1.

This proves Lemma 4.2. □

Suppose $c_{r+2} = 2$. By the previous lemma, we may assume that there are vertices, $w$, $w'$ such that

$$D_{r+1}^{r+2} \ni w \sim w' \in D_{r+2}^{r+1}.$$

Let $v \in \Gamma(w) \cap D_r^{r+1}$, $w'' \in D(w, v)$. Since $c_{r+2} = 2$, $w'' \in D_{r+1}^{r+2}$. Let $\{u, u'\} = \Gamma(v) \cap D_{r+1}^{r+1}$. Then $u \sim u'$.

Suppose $u$ is of type (1, 1, 1) and $v' \in \Gamma(u) \cap D_{r+1}^r$. Then there is a vertex $x$ in $\Gamma_r(\gamma) \cap \Gamma(u) \subset D_{r+1}^{r+1}$, and $x \not\sim v'$. So it is impossible to have $b_{r+1} = 2$. So $u$ and $u'$ are of type (1, 2, 1) and $w \in \Gamma_{r+1}(\gamma)$, as $\partial(\gamma, v) = r + 1$ and $\Gamma_{r+2}(\gamma) \cap \Gamma(v) = \{u, u'\}$. There must be a vertex in $\Gamma(w) \cap \Gamma_r(\gamma) \subset D_{r+1}^{r+1}$. This is impossible as $c_{r+2} = 2$. Thus $c_{r+2} \neq 2$.

Suppose $c_{r+2} = 4$. If $d \geq r + 3$, it is clear that $(c_j, a_j, b_j) = (4, 1, 1)$ for $r + 2 \leq j \leq d - 1$. So $c_d = 4$ or 6 by Lemma 2.3. We treat three cases together, namely,

(1) $d \geq r + 3, c_{r+2} = c_d = 4$,
(2) $d \geq r + 3, c_{r+2} = 4, c_d = 6$; and
(3) $d = r + 2, c_{r+2} = 6$.

Our goal is to show (5) in the theorem.

Firstly, in all these three cases every $b_{r+1}$-graph is a coclique, as every $c_{r+2}$-graph is a union of $K_2$'s. So there is no triangle in $\Gamma_{r+1}(x)$ for $x \in \Gamma$. This implies $D_{r+1}^{r+1} \cap \Gamma_{r+1}(\gamma) = \emptyset$. Hence

$$\Gamma_{r+1}(\gamma) = D_{r+1}^{r+2} \cup D_{r+2}^{r+1} \cup D_{r+1}^r \cup D_r^{r+1}.$$

For example, $D_{r+1}^{r+2} \subset \Gamma_{r+1}(\gamma)$ as otherwise, there is a vertex $x \in D_{r+1}^{r+2} \cap \Gamma_{r+2}(\gamma)$ and the $b_{r+1}$-graph $\Gamma(\beta) \cap \Gamma_{r+2}(x)$ is a clique. Now it is easy to determine clique patterns.

Let $\Delta$ be the set of all maximal cliques, i.e., $K_3$'s in $\Gamma$. Let $\tilde{\Gamma} = \Gamma \cup \Delta$ be the incidence graph, i.e., a bipartite graph defined by the following adjacency.

$$\alpha \sim x \text{ in } \tilde{\Gamma} \quad \text{if and only if} \quad \alpha \in x, \text{ for } \quad \alpha \in \Gamma, \ x \in \Delta.$$

We use~notation for the graph $\tilde{\Gamma}$.

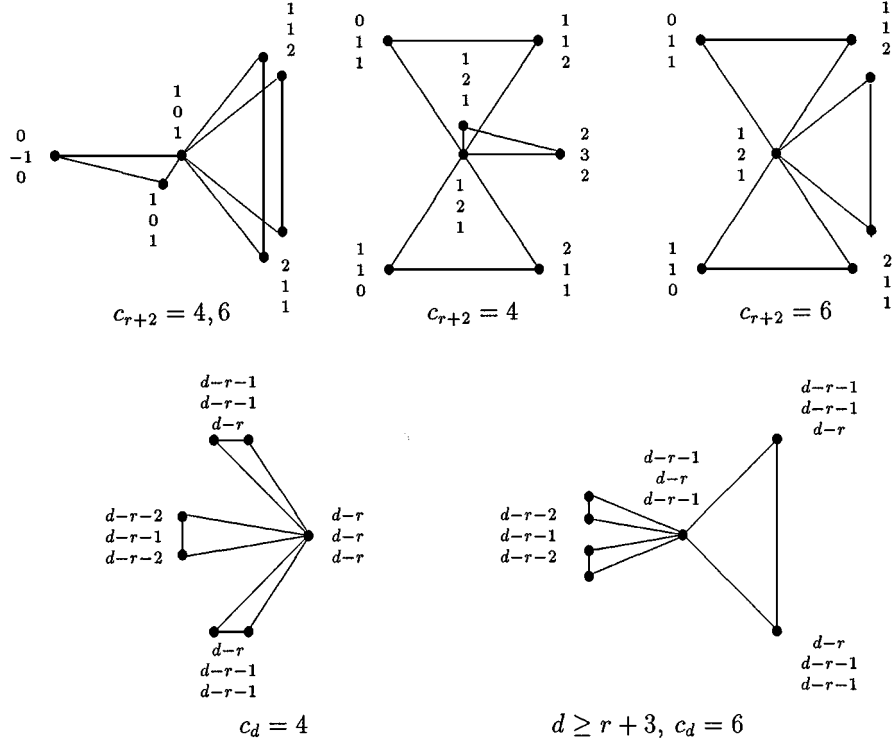It is straightforward to show the distance-regularity of $\tilde{\Gamma}$ by the clique patterns described above.

$$c_{r+2} = 4, 6 \qquad\qquad c_{r+2} = 4 \qquad\qquad c_{r+2} = 6$$



$$c_d = 4 \qquad\qquad\qquad d \geq r + 3, \ c_d = 6$$

*Figure 12.*  Clique patterns of the case $a_{r+1} = 3$ (2).

Note that for $x = \{\alpha, \beta, \gamma\} \in \Delta$,

$$\tilde{\partial}(x, \delta) = 2i + 1 \quad \text{if and only if} \quad \delta \in \Gamma \ \text{ and } \ \partial(x, \delta) = i.$$
$$\tilde{\partial}(x, y) = 2i + 2 \quad \text{if and only if} \quad y \in \Delta \ \text{ and } \ \partial(x, y) = i, \ x \neq y.$$

We only give the values of $\tilde{c}_i$'s in each case.

$$\tilde{c}_j = \begin{cases} 1, & j = 1, 2, \ldots, 2r + 2 \\ 2, & j = 2r + 3, \ldots, 2d - 1 \end{cases}$$
$$\tilde{c}_{2d} = 3 \quad \text{if} \quad c_d = 6$$
$$\tilde{c}_{2d} = 2 \quad \text{and} \quad \tilde{c}_{2d+1} = 3 \quad \text{if} \quad c_d = 4.$$

Therefore we have (3) or (5) if $a_{r+1} = 3$.

**Case 3.**  $a_{r+1} = 2$.
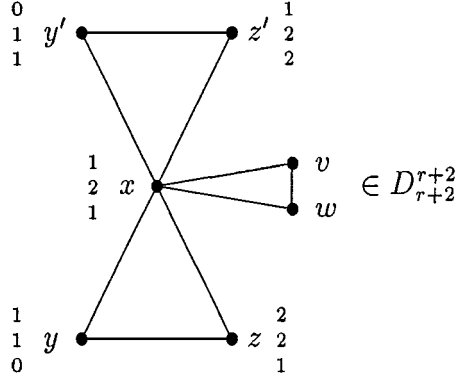
We start from a lemma.

*Figure 13.*   Clique pattern of the case $a_{r+1} = 2$ (1).

**Lemma 4.3**   *Suppose* $(c_{r+1}, a_{r+1}, b_{r+1}) = (1, 2, 3)$.
(1) *There is no triangle in $\Gamma_{r+1}(u)$ for every vertex $u \in \Gamma$, i.e., there is no vertex of type $(1, 1, 1)$. Moreover, there exists a triangle $\{y, z, w\}$ with $y \in \Gamma_{r+1}(x)$, $z, w \in \Gamma_{r+2}(x)$. In particular, $a_{r+2} \neq 0$.*
(2) *If $c_{r+2} = 3$ and $d \geq r + 3$, then $(c_{r+2}, a_{r+2}, b_{r+2}) = (3, 2, 1)$. Moreover there is no triangle in $\Gamma_{r+2}(x)$ for every vertex $x \in \Gamma$, i.e., there is no vertex of type $(2, 2, 2)$.*
(3) *Let $x$ be a vertex of type $(1, 2, 1)$. If $c_{r+2} \leq 3$ and $d \geq r + 3$, then the clique patterrn of the vertex $x$ is as in figure 13.*
(4) *If $c_{r+2} = 2$ and $d \geq r + 3$, then $(c_{r+2}, a_{r+2}, b_{r+2}) = (2, 3, 1)$.*

**Proof:**

(1) This follows easily from the fact that $\Gamma(x) \simeq 3 \cdot K_2$ for every vertex $x \in \Gamma$. See figure 5.
(2) There is a vertex $u$ of type $(2, 1, 1)$ by figure 5. $u$ is adjacent to a vertex in $\Gamma_{r+3}(\alpha)$, which must be of type $(3, 2, 2)$.

   If $b_{r+2} = 2$, then every $b_{r+2}$-graph is a clique. So there is no vertex of type $(3,2,2)$. This is a contradiction. Thus we have $(c_{r+2}, a_{r+2}, b_{r+2}) = (3, 2, 1)$.
(3) Let $\{y\} = D_r^{r+1} \cap \Gamma(x)$ and $\{y'\} = D_r^{r+1} \cap \Gamma(x)$. Since $e(y, D_{r+1}^{r+1}) = 1$, $\{z\} = D(x, y) \subset D_{r+1}^{r+2}$. Similarly, $\{z'\} = D(x, y') \subset D_{r+2}^{r+1}$. Since $(c_{r+1}, a_{r+1}, b_{r+1}) = (1, 2, 3)$, there are $v, w \in D_{r+2}^{r+2} \cap \Gamma(x)$.

   Suppose $\vec{z} = (2, 1, 1)$. Then $\{y, y', z\} \subset \Gamma_{r+1}(\gamma) \cap \Gamma(x)$ and thus $(c_{r+2}, a_{r+2}, b_{r+2}) = (3, 2, 1)$ from (2). Since $|\Gamma_{r+2}(\gamma) \cap \Gamma(x)| = a_{r+2} = 2$, we may assume $\vec{v} = (2, 2, 2)$. This is a contradiction from (2). Hence $\vec{z} = (2, 2, 1)$. By symmetry we have $\vec{z}' = (1, 2, 2)$.
(4) $(c_{r+2}, a_{r+2}, b_{r+2}) = (2, 1, 3)$ contradicts Lemma 2.2.

   Suppose $(c_{r+2}, a_{r+2}, b_{r+2}) = (2, 2, 2)$. Let $x$ be a vertex of type $(1, 2, 1)$. Then the clique patterrn of the vertex $x$ is as in figure 13.

Let $\{z, z_1, z_2\} = D_{r+1}^{r+2} \cap \Gamma(y)$. Then $z_1 \sim z_2$. Since $y \in \Gamma_{r+1}(\gamma)$ and $b_{r+1} = 3$, we may assume that $z_1 \in \Gamma_{r+2}(\gamma)$. Suppose there exists $x_1 \in \Gamma(z_1) \cap D_{r+1}^{r+1}$. By our observation above, $x_1$ has the same clique pattern as $x$. Hence $x_1 \sim y$, which is impossible. Hence we have a vertex $z'' \in D_{r+2}^{r+1} \cap \Gamma(z_1)$. Since $b_{r+2} - b_{r+1} = 1$, we cannot locate the vertex in $D(z_1, z'')$. Thus $(c_{r+2}, a_{r+2}, b_{r+2}) \neq (2, 2, 2)$.

The lemma is proved.                                                      □

By Lemma 4.3 (1), $c_{r+2} \neq 6$.

Suppose $c_{r+2} = 4$ and $d \geq r + 3$. Then every $c_{r+2}$-graph is a union of $K_2$'s. On the other hand, every $b_{r+1}$-graph is a union of a $K_1$ and a $K_2$. This is impossible.

Suppose $c_{r+2} = 3$ and $d \geq r + 3$. Then $(c_{r+2}, a_{r+2}, b_{r+2}) = (3, 2, 1)$ by Lemma 4.3 (2). Let $x$ be a vertex of type $(1, 2, 1)$. The clique pattern at the vertex $x$ is as in figure 13. by Lemma 4.3 (3). Since $(c_{r+2}, a_{r+2}, b_{r+2}) = (3, 2, 1)$ we may assume that $v \in \Gamma_{r+1}(\gamma)$, $w \in \Gamma_{r+3}(\gamma)$. This is impossible. Thus we have $d = r + 2$ if $c_{r+2} = 3$.

Finally assume $c_{r+2} = 2$ and $d \geq r + 3$.

Then we have $(c_{r+2}, a_{r+2}, b_{r+2}) = (2, 3, 1)$ from Lemma 4.3 (4).

We now determine clique patterns in this case. Note that there is no vertex of type $(1, 1, 1)$ by Lemma 4.3 (1).

Let $\delta$ be a vertex in $D_r^{r+1}$. Then $\delta$ is of type $(1, 1, 0)$. And we can determine the types of vertices in $\Gamma(\delta)$. We call a type-$(2, 2, 1)$-vertex $x$ of type A when $D(\delta, x) \subset D_{r+1}^{r+1}$, and of type B when $D(\delta, x) \subset D_{r+1}^{r+2}$. See figure 12.

We have the patterns of cliques at a vertex of type $(1, 2, 1)$ from Lemma 4.3 (3), where $\vec{v} = (2, 3, 2)$ and $\vec{w} = (2, 2, 2)$.

Converting the base points, we also have clique patterns of vertices of types $(1, 0, 1)$, $(0, 1, 1)$, $(2, 1, 1)$, $(1, 1, 2)$.

Let $\eta$ be of type $(2, 2, 1)$-A. By the clique patterns of a vertex of type $(2, 1, 1)$, $\eta$ is not adjacent to a vertex of type $(2, 1, 1)$. Since $b_{r+2} = 1$, there is no vertex of type $(3, 3, 2)$.

Hence if $u_1, u_2, u_3, u_4, u_5, u_6$ be vertices in $\Gamma(\eta)$, we may assume that

$u_1$ is of type $(1, 1, 0)$,    $u_2$ is of type $(1, 2, 1)$,    $u_3$ is of type $(2, 2, 1)$,

$u_4$ is of type $(2, 3, 2)$,    $u_5$ is of type $(3, 2, 2)$,    and $u_6$ is of type $(2, 1, 2)$.

Since $u_6 \in \Gamma_{r+1}(\gamma)$, there exists $v \in \Gamma_r(\gamma) \cap \Gamma(u_6)$ which must be of type $(1, 0, 1)$. If $u_3 \sim u_6$, then $\{v, \eta, u_3\} \subset \Gamma_{r+1}(\beta) \cap \Gamma(u_6)$. This contradicts $c_{r+2} = 2$. Hence

$u_1 \sim u_2$,        $u_3 \sim u_4$,        $u_5 \sim u_6$.

Let $\xi$ be of type $(2, 2, 1)$-B. Then $e(\xi, D_{r+1}^{r+1}) = 0$ as before. So $e(\xi, D_{r+2}^{r+1}) = 1$. Let $v_1, v_2, v_3, v_4, v_5, v_6$ be vertices in $\Gamma(\xi)$, we may assume that

$v_1$ is of type $(1, 1, 0)$,    $v_2$ is of type $(2, 1, 1)$,    $v_3$ is of type $(1, 2, 2)$,

$v_4$ is of type $(2, 3, 2)$,    $v_5$ is of type $(2, 2, 1)$,    and $v_6$ is of type $(3, 2, 2)$.

Since $e(v_3, D_{r+1}^{r+2}) = 1$,

$v_1 \sim v_2$,        $v_3 \sim v_4$,        $v_5 \sim v_6$.

Moreover $u_3$ in $\Gamma(\eta)$ is of type A, and $v_5$ in $\Gamma(\xi)$ is of type B.

Converting the base points, we can obtain the clique patterns of vertices of type $(2, 1, 2)$. See figure 14.

Take a circuit $x_0 \sim x_1 \sim \cdots \sim x_{2r+4} \sim x_0$ of length $2r + 5$ such that

$$x_0 \in D_1^0, \qquad x_i \in D_{i-1}^i, \quad i = 1, \ldots, r + 1,$$

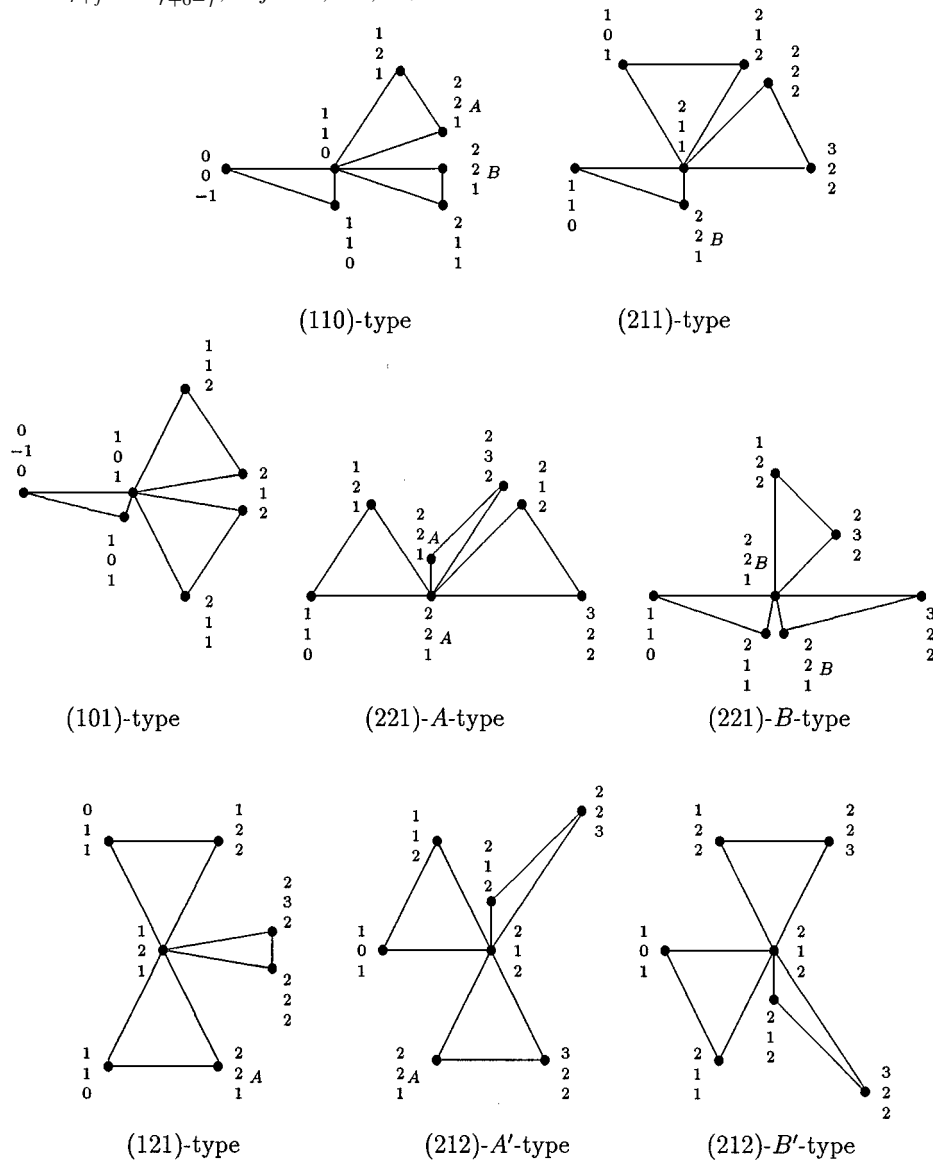$$x_{r+j} \in D_{r+6-j}^{r+5-j}, \quad j = 5, \ldots, r + 4.$$



(110)-type          (211)-type



(101)-type     (221)-$A$-type     (221)-$B$-type



(121)-type     (212)-$A'$-type     (212)-$B'$-type

*Figure 14.* Clique patterns of the case $a_{r+1} = 2$ (2).

Let $\{y_i\} = D(x_i, x_{i+1})$. We define three types as follows.

| | | | |
|---|---|---|---|
| Type I. | $\vec{x}_{r+2} = (2, 2, 1)$-$A$, | $\vec{x}_{r+3} = (2, 3, 2)$, | $\vec{x}_{r+4} = (1, 2, 2)$, |
| | $\vec{y}_{r+2} = (2, 2, 1)$-$A$, | $\vec{y}_{r+3} = (1, 2, 2)$, | $\vec{y}_{r+4} = (1, 2, 1)$ |
| Type II. | $\vec{x}_{r+2} = (2, 2, 1)$-$A$, | $\vec{x}_{r+3} = (2, 2, 1)$-$A$, | $\vec{x}_{r+4} = (1, 2, 1)$, |
| | $\vec{y}_{r+2} = (2, 3, 2)$, | $\vec{y}_{r+3} = (1, 1, 0)$, | $\vec{y}_{r+4} = (1, 2, 2)$ |
| Type III. | $\vec{x}_{r+2} = (1, 2, 1)$, | $\vec{x}_{r+3} = (1, 2, 2)$, | $\vec{x}_{r+4} = (1, 2, 2)$, |
| | $\vec{y}_{r+2} = (0, 1, 1)$, | $\vec{y}_{r+3} = (2, 3, 2)$, | $\vec{y}_{r+4} = (1, 2, 1)$ |

Note that each of these circuits does not contain triangles. In the following, we determine the type of the circuit with respect to $x_1$, $x_2$ for each type.
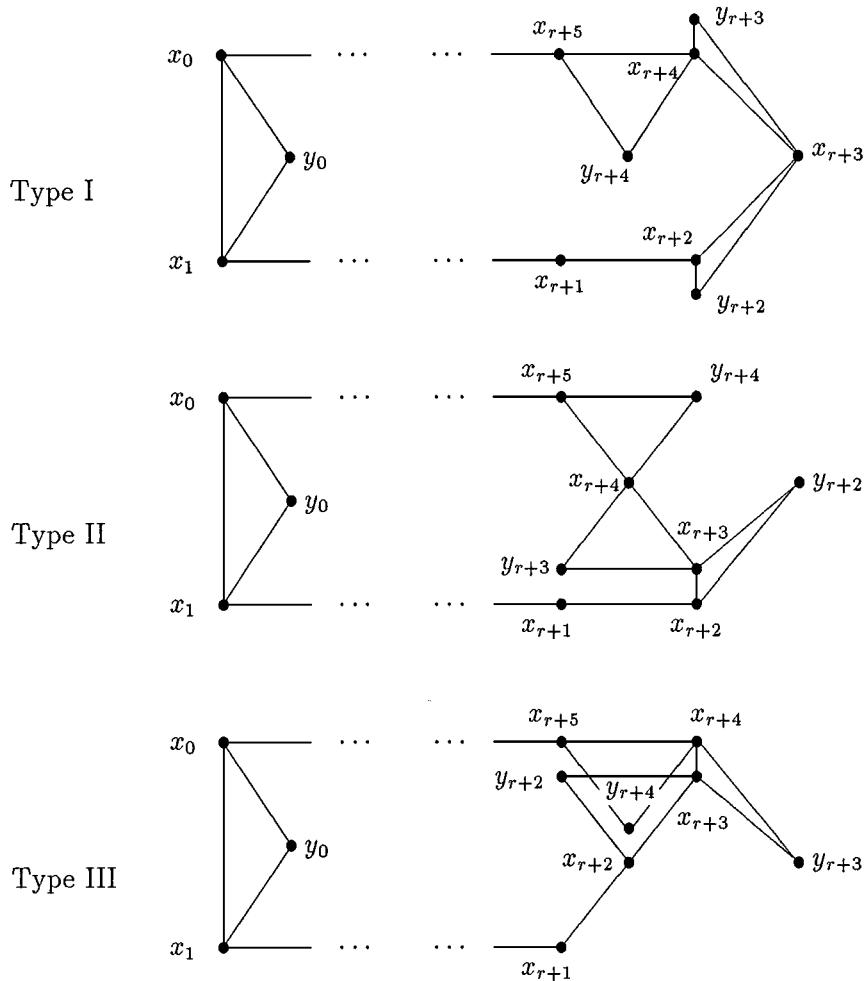


*Figure 15.* Circuit of length $2r + 5$.

Suppose the circuit is of type I. Then $x_{r+3} \in \Gamma_{r+2}(x_1)$, $y_{r+2} \in \Gamma_{r+1}(x_1)$. So $x_{r+3}$ is of $(2, 2, 1)$-A type, and $x_{r+4} \in \Gamma_{r+2}(x_1)$, $y_{r+3} \in \Gamma_{r+2}(x_1)$. Hence there are two possibilities.

$$\vec{x}_{r+4} = (2, 3, 2), \vec{y}_{r+3} = (2, 2, 1)\text{-}A, \quad \text{or} \quad \vec{x}_{r+4} = (2, 2, 1)\text{-}A, \vec{y}_{r+3} = (2, 3, 2)$$

Thus we have either type I or type II. Note that in the first case $e(x_{r+4}, D_{r+1}^{r+1}) = 0$, and $\Gamma(x_{r+4}) \cap D_{r+2}^{r+1}$ is a clique $K_2$.

Suppose the circuit is of type II. Then $x_{r+3} \in \Gamma_{r+1}(x_1)$. So $x_{r+3}$ is of $(1, 2, 1)$ type. $x_{r+4} \in \Gamma_{r+1}(x_1)$, $y_{r+3} \in \Gamma_r(x_1)$ implies that we have type III, as this circuit does not have triangles.

Suppose the circuit is of type III. Then $x_{r+3} \in \Gamma_{r+2}(x_1)$, $y_{r+2} \in \Gamma_{r+1}(x_1)$. So $x_{r+3}$ is of $(2, 2, 1)$-A type, and $x_{r+4} \in \Gamma_{r+2}(x_1)$, $y_{r+3} \in \Gamma_{r+2}(x_1)$. Suppose $\partial(y_{r+3}, y_1) = r + 3$. Then

$$\{y_0, y_1\} \subset \Gamma(x_1) \cap \Gamma_{r+3}(y_{r+3}).$$

This contradicts $b_{r+2} = 1$. Hence $y_{r+3}$ is of $(2, 2, 1)$-A type and we have a circuit of type I.

Therefore by induction, we can conclude that for each $i$, this circuit is either of type I, II, or III with respect to every pair of adjacent vertices $x_i$, $x_{i+1}$ in the circuit. Moreover, it is easy to see that there is a circuit of type III.

Take a circuit of type III with respect to $x_0, x_1$. It is of type I with respect to $x_1, x_2$. Changing the base points to $x_{r+3}, x_{r+4}$, we have that $x_0 \in D_{r+1}^{r+1}$ and $x_1 \in D_{r+2}^{r+2}$. This is absurd because with respect to these base points, this circuit is of type different from I, II, or III.

This completes the proof of Theorem 4.1. □

**Lemma 4.4** *Let $(c_{r+1}, a_{r+1}, b_{r+1}) = (1, 4, 1)$, $c_{r+2} = 6$. Then $r \equiv 0$ (mod 3).*

**Proof:** Let $u \in D_{r+1}^{r+1}$ such that $\partial(u, \gamma) \geq r + 1$. We claim that there are three possible clique patterns.
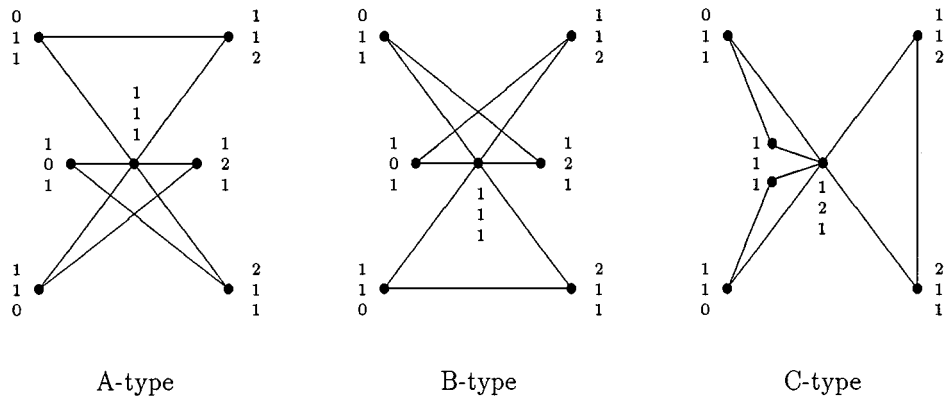


Figure 16.    Clique patterns of the case $a_{r+1} = 4$.

Let $u$ be of type $(1, 1, 1)$ and $\{v_1, v_2, \ldots, v_6\} = \Gamma(u)$. Since $c_{r+1} = b_{r+1} = 1$, we may assume that

$$\vec{v}_1 = (1, 1, 0) \quad \vec{v}_2 = (1, 0, 1) \quad \vec{v}_3 = (0, 1, 1)$$
$$\vec{v}_4 = (1, 1, 2) \quad \vec{v}_5 = (1, 2, 1) \quad \vec{v}_6 = (2, 1, 1).$$

Note that $D(v_2, u) = \{v_4\}$ or $\{v_6\}$. Now we have either A-type or B-type depending on the location of $D(x, u)$.

Now we can determine the clique pattern at a vertex of type $(1, 2, 1)$ as well without difficulty.

Therefore we can use the first circuit used in Case 1 Step 4 in the proof of Theorem 4.1 to conclude $r \equiv 0 \pmod{3}$.                                       □

**Remark**   We can also show that in case (4) in Theorem 4.1, $c_{r+2} \neq 4$.

In the next section, we apply Theorem 4.1 to give a bound of the diameter. We summarize the information we need as a corollary as follows.

**Corollary 4.5**   *Let $\Gamma$ be a distance-regular graph of valency 6, $a_1 = 1$. If $\Gamma$ is not a bipartite half of a bipartite distance-regular graph of valency 3, then $d \leq r + 2$ and the following hold.*
(1) *If $d = r + 1$, then $c_{r+1} \leq 3$.*
(2) *If $(c_{r+1}, a_{r+1}, b_{r+1}) = (2, 2, 2)$, then $c_{r+2} \leq 3$.*
(3) *If $(c_{r+1}, a_{r+1}, b_{r+1}) = (2, 3, 1)$, then $c_{r+2} \geq 4$.*
(4) *If $(c_{r+1}, a_{r+1}, b_{r+1}) = (1, 2, 3)$, then $c_{r+2} \leq 4$.*

## 5.   An upper bound of $r(\Gamma)$

We apply eigenvalue technique to give an upper bound of $r(\Gamma)$ assuming that $d \leq r(\Gamma) + 2$.

**Theorem 5.1**   *Let $\Gamma$ be a distance-regular graph of valency 6 with $a_1 = 1$. Let $r = r(\Gamma)$. If $d(\Gamma) \leq r + 2$, then $r \leq 17$.*

We start by the notational conventions, which mostly follow those used in [1–4].

Let $\Gamma$ be a distance-regular graph of diameter $d$, valency $k$ and parameters $a_i$, $b_i$, $c_i$. Let $A$ be the adjacency matrix of a graph $\Gamma$. Let

$$k = \theta_0 > \theta_1 > \cdots > \theta_{d-1} > \theta_d$$

be the eigenvalues of $A$ and $m(\theta_i)$ the multiplicity of $\theta_i$.

The polynomials $v_i(x)(0 \leq i \leq d + 1)$ are defined by the recurrence relation

$$x v_i(x) = b_{i-1} v_{i-1}(x) + a_i v_i(x) + c_{i+1} v_{i+1}(x)$$

for $0 \leq i \leq d$ with $v_{-1}(x) = 0$, $v_0(x) = 1$, and $c_{d+1} = 1$.

The $F_i(x)$ $(0 \le i \le d)$ are the monic polynomials defined by

$$F_i(x) = c_1 c_2 \cdots c_i (v_0(x) + v_1(x) + \cdots + v_i(x)).$$

They satisfy the recurrence relation

$$F_i(x) = (x - k + b_{i-1} + c_i) F_{i-1}(x) - b_{i-1} c_{i-1} F_{i-2}(x)$$

for $2 \le i \le d$ with $F_0(x) = 1$, $F_1(x) = x + 1$.
  We put

$$S_i(x) = \sum_{j=0}^{i} \frac{v_j(x)^2}{k_j}.$$

It is well known that

$$m(\theta_i) = |\Gamma| / S_d(\theta_i).$$

In the following we assume that $k = 6$, $a_1 = 1$, $r = r(\Gamma) = l(1, 1, 4)$, and $d \le r + 2$.

**Lemma 5.2**  *Let $\theta \ne 6$ be an eigenvalue of A. Then $-3 \le \theta < 5$.*

**Proof:**  Since $a_1 = 1$, the size of maximal cliques is always 3. Hence by Proposition 4.4.6 in [8], $\theta \ge -3$. We now find an upper bound by a Sturm series.
  For $2 \le i \le r$, the recurrence relation of $F_i$'s yields

$$F_i(x) = (x - 1) F_{i-1}(x) - 4 F_{i-2}(x).$$

So for $0 \le i \le r$,

$$F_i(5) = 2^i (2i + 1) \quad \text{and}$$
$$F_{r+1}(5) = (3 + c_{r+1}) F_r(5) - 4 F_{r-1}(5)$$
$$= 2^r (2r + 5 + (2r + 1) c_{r+1}).$$

Moreover, if $d = r + 2$, then

$$F_{r+2}(5) = (-1 + b_{r+1} + c_{r+2}) F_{r+1}(5) - b_{r+1} c_{r+1} F_r(5)$$
$$= 2^r ((-1 + b_{r+1} + c_{r+2})(2r + 5 + (2r + 1) c_{r+1}) - b_{r+1} c_{r+1}(2r + 1))$$
$$= 2^r ((2r + 5)(b_{r+1} + c_{r+2} - 1) + (2r + 1)(c_{r+2} - 1) c_{r+1}).$$

Since $F_0(x), \ldots, F_d(x)$ is a Sturm series, $F_i(5) > 0, i = 0, 1, \ldots, d$, implies that $\theta < 5$ as $\theta$ is a root of $F_d(x) = 0$.
  Thus we have the assertion.                                            $\square$

**Lemma 5.3**  *Let $\theta_1$ be the second largest eigenvalue. Then*

$$\theta_1 > 1 + 4 \cos \frac{2\pi}{2r+1}.$$

**Proof:**  Let $x = 1 + 4 \cos \alpha < 5$. Then by the recurrence relation,

$$F_r(x) = \frac{2^r}{\sin \alpha} (\sin(r+1)\alpha + \sin r\alpha).$$

Let $\alpha = \frac{2\pi}{2r+1}$. Then

$$
\begin{aligned}
\sin(r+1)\,\alpha + \sin r\alpha &= \sin\left( \frac{r+1}{2r+1} \cdot 2\pi \right) + \sin\left( \frac{r}{2r+1} \cdot 2\pi \right) \\
&= \sin\left( \pi + \frac{\pi}{2r+1} \right) + \sin\left( \pi - \frac{\pi}{2r+1} \right) \\
&= 0.
\end{aligned}
$$

Hence a root of $F_d(x)$ is greater than $1 + 4 \cos \frac{2\pi}{2r+1}$.                           $\square$

**Lemma 5.4**  *The following hold.*
(1) *Let $x = 1 + 4 \cos \phi$. Then*

$$
\begin{aligned}
S_r(x) = 1 + \frac{2}{3}r + \frac{1 + \cos \phi}{6 \sin^2 \phi}r &- \frac{\sin r\phi}{12 \sin^3 \phi}(4 \cos(r+3)\phi \\
&+ 4 \cos(r+2)\phi - 3 \cos(r+1)\phi - 2 \cos r\phi + \cos(r-1)\phi).
\end{aligned}
$$

(2) *If $r \geq 18$, then*

$$S_r(\theta_1) > 1 + \frac{2}{3}r + \frac{r^2(r+1)}{3\pi^2}.$$

(3) *If $|x - 1| < \sqrt{14}$, then*

$$S_r(x) < 1 + \frac{2}{3}r + \frac{1}{3}(4 + \sqrt{14})r + \frac{4}{3}(5 + 4\sqrt{2}).$$

**Proof:**  For (1), see [15, Proposition 2.4].
   (2) By Proposition 2.5 in [15], we know that the largest root of

$$v_r(x) = \frac{2^{r-1}}{\sin \phi}(2 \sin(r+1)\phi + \sin r\phi - \sin(r-1)\phi)$$

with $x - 1 = 4 \cos \phi$ is in the interval corresponding to $0 < \phi < \pi/r$.

Firstly we will improve the lower bound above. Let $a = 1.09$, $\phi = \pi/ar$. Then

$$2 \sin(r+1)\phi + \sin r\phi - \sin(r-1)\phi$$

$$= 2 \sin\left(\frac{\pi}{a} + \frac{\pi}{ar}\right) + \sin\frac{\pi}{a} - \sin\left(\frac{\pi}{a} - \frac{\pi}{ar}\right)$$

$$= \sin\frac{\pi}{a}\left(1 + \cos\frac{\pi}{ar}\right) + 3 \cos\frac{\pi}{a} \sin\frac{\pi}{ar}$$

$$\geq \sin\frac{\pi}{a}\left(1 + \cos\frac{\pi}{18}\right) + 3 \cos\frac{\pi}{a} \sin\frac{\pi}{18} > 0.$$

Since $v_i(x)$'s form a Strum sequence, all roots of $v_i(x)$'s are less than $\xi = 1 + 4 \cos\frac{\pi}{ar}$
Hence $S_r(x)$ is increasing in the interval $(\xi, \infty)$.

Thus we may assume that

$$\theta_1 = 1 + 4 \cos\alpha, \quad \frac{\pi}{ar} < \alpha < \frac{2\pi}{2r+1}$$

by Lemma 5.3. Since

$$\frac{\pi}{a} < r\alpha < \frac{2r\pi}{2r+1} < \pi, \qquad -1 < \cos r\alpha < \cos\frac{\pi}{a}.$$

Using the mean value theorem,

$$-1 \leq \cos(r+h)\alpha \leq \cos r\alpha + |h|\alpha$$

for every $h$. Hence for $r \geq 18$,

$$-4 \cos(r+3)\alpha - 4 \cos(r+2)\alpha + 3 \cos(r+1)\alpha + 2 \cos r\alpha - \cos(r-1)\alpha$$

$$\geq 4(-\cos r\alpha - 3\alpha) + 4(-\cos r\alpha - 2\alpha) - 3 + 2 \cos r\alpha - \cos r\alpha - \alpha$$

$$\geq -7 \cos r\alpha - 3 - 21\alpha$$

$$\geq -7 \cos\frac{\pi}{a} - 3 - 21 \cdot \frac{2\pi}{2r+1} > 0.$$

Since $\sin r\alpha > 0$,

$$S_r(\theta_1) > 1 + \frac{2}{3}r + \frac{1 + \cos\alpha}{6 \sin^2\alpha}r$$

$$\geq 1 + \frac{2}{3}r + \frac{r}{6(1 - \cos\alpha)}.$$

As

$$1 - \cos\alpha < \frac{\alpha^2}{2} < \frac{2\pi^2}{(2r+1)^2},$$

$$S_r(\theta_1) > 1 + \frac{2}{3}r + \frac{(2r+1)^2 r}{12\pi^2}$$

$$> 1 + \frac{2}{3}r + \frac{r^2(r+1)}{3\pi^2}.$$

(3) Firstly,

$$4 \cos(r + 3)\phi + 4 \cos(r + 2)\phi - 3 \cos(r + 1)\phi - 2 \cos r\phi + \cos(r - 1)\phi$$
$$= \cos(r + 3)\phi - 6 \sin(r + 2)\phi \sin\phi$$
$$+2 \ \cos(r + 2)\phi - 4 \sin(r + 1)\phi \ \sin\phi + \cos(r - 1)\phi$$
$$= \sin\phi(-6 \sin(r + 2)\phi - 4 \ \sin(r + 1)\phi)$$
$$+ \cos(r + 3)\phi + 2 \ \cos(r + 2)\phi + \cos(r - 1)\phi.$$

Let $x - 1 = 4 \cos\beta$. Since $16 \cos^2\beta < 14$,

$$|\sin\beta| > \frac{1}{2\sqrt{2}}, \quad \text{and} \quad |\cos\beta| < \frac{\sqrt{14}}{4}.$$

We have

$$S_r(x) = 1 + \frac{2}{3}r + \frac{1 + \cos\beta}{6 \sin^2\beta}r + \frac{\sin r\beta}{6 \sin^2\beta}(3 \ \sin(r + 2)\beta + 2 \ \sin(r + 1)\beta)$$
$$- \frac{\sin r\beta}{12 \sin^3\beta}(\cos(r + 3)\beta + 2 \ \cos(r + 2)\beta + \cos(r - 1)\beta)$$
$$< 1 + \frac{2}{3}r + \frac{1}{3}(4 + \sqrt{14})r + \frac{8}{6} \cdot 5 + \frac{16\sqrt{2}}{12} \cdot 4$$
$$= 1 + \frac{2}{3}r + \frac{1}{3}(4 + \sqrt{14})r + \frac{4}{3}(5 + 4\sqrt{2}).$$

This completes the proof of Lemma 5.4.                                          □

**Lemma 5.5** *If $|x - 1| \leq \sqrt{14}$, then the following hold.*
(1) $|v_i(x)| \leq 2^i(1 + 2\sqrt{2}), \quad i = 0, 1, \ldots, r.$
(2) $\frac{v_{r+1}(x)^2}{k_{r+1}} \leq \frac{2}{3c_{r+1}}(9 + 4\sqrt{2}) < 10.$
(3) $\frac{v_{r+2}(x)^2}{k_{r+2}} < 123.$

**Proof:**

(1) Let $x - 1 = 4 \cos\beta$. Then

$$v_i(x) = \frac{2^i}{\sin\beta}\left(\sin(i + 1)\beta + \frac{1}{2} \sin i\beta - \frac{1}{2} \sin(i - 1)\beta\right)$$
$$= 2^i\left(\cos i\beta + \frac{1}{2 \sin\beta}(\sin(i + 1)\beta + \sin i\beta)\right).$$

So $|v_i(x)| \leq 2^i(1 + 2\sqrt{2})$.

(2) $c_{r+1}v_{r+1}(x) = (x-1)v_r(x) - 4v_{r-1}(x)$.

Since the right hand side equals $v_{r+1}(x)$ for a distance-regular graph with $l(1, 1, 4) = r + 1$,

$$|c_{r+1}v_{r+1}(x)| \le 2^{r+1}(1 + 2\sqrt{2})$$

by (1). Hence we have

$$
\begin{aligned}
\frac{v_{r+1}(x)^2}{k_{r+1}} &= \frac{(c_{r+1}v_{r+1}(x))^2}{c_{r+1}b_r k_r} \\
&\le \frac{4^{r+1}(1 + 2\sqrt{2})^2}{c_{r+1} \cdot 6 \cdot 4^r} \\
&\le \frac{2}{3c_{r+1}}(9 + 4\sqrt{2}) < 10.
\end{aligned}
$$

(3) $c_{r+2}v_{r+2}(x) = (x - a_{r+1})v_{r+1}(x) - b_r v_r(x)$.

Assume $c_{r+1} = 1$. Then

$$
\begin{aligned}
c_{r+2}|v_{r+2}(x)| &= |(x-1)v_{r+1}(x) - 4v_r(x) + (1 - a_{r+1})v_{r+1}(x)| \\
&\le 2^{r+2}(1 + 2\sqrt{2}) + 3 \cdot 2^{r+1}(1 + 2\sqrt{2}) \\
&\le 2^{r+1} \cdot 5(1 + 2\sqrt{2}).
\end{aligned}
$$

We have

$$
\begin{aligned}
\frac{v_{r+2}(x)^2}{k_{r+2}} &= \frac{(c_{r+2}v_{r+2}(x))^2}{c_{r+2}b_{r+1}k_{r+1}} \\
&\le \frac{4^{r+1} \cdot 25(9 + 4\sqrt{2})}{2 \cdot 6 \cdot 4^r} \\
&\le \frac{25}{3}(9 + 4\sqrt{2}) < 123.
\end{aligned}
$$

Suppose $c_{r+1} = 2$, $a_{r+1} = 2$. In this case,

$$
\begin{aligned}
c_{r+2}|v_{r+2}(x)| &= |(x - a_{r+1})v_{r+1}(x) - b_r v_r(x)| \\
&\le |x - 2|2^r(1 + 2\sqrt{2}) + 4 \cdot 2^r(1 + 2\sqrt{2}) \\
&\le 2^r(5 + \sqrt{14})(1 + 2\sqrt{2}). \\
\frac{v_{r+2}(x)^2}{k_{r+2}} &= \frac{(c_{r+2}v_{r+2}(x))^2}{c_{r+2}b_{r+1}k_{r+1}} \\
&\le \frac{4^r(5 + \sqrt{14})^2(1 + 2\sqrt{2})^2 \cdot 2}{2 \cdot 2 \cdot 4^r \cdot 6} < 94.
\end{aligned}
$$

If $c_{r+1} = 2$, $a_{r+1} = 3$, then $c_{r+2} \geq 4$. So

$$c_{r+2}|v_{r+2}(x)| \leq 2^r(6 + \sqrt{14})(1 + 2\sqrt{2})$$

$$\frac{v_{r+2}(x)^2}{k_{r+2}} \leq \frac{4^r(6 + \sqrt{14})^2(1 + 2\sqrt{2})^2 \cdot 2}{4 \cdot 4^r \cdot 6} < 116.$$

This competes the proof of Lemma 5.5.                                      □

**Proof of Theorem 5.1:**   Suppose $r \geq 18$. Then by Lemma 5.2 and 5.3,

$$5 > \theta_1 > 1 + 4 \cos \frac{2\pi}{2r + 1} > 1 + \sqrt{15},$$

as $r \geq 18 \geq 12$. Let

$$\eta = \prod((\theta - 1)^2 - 15),$$

where the product is taken for all algebraic conjugates $\theta$ of $\theta_1$. In particular $\eta$ is a non-zero integer.

Since $0 < |(\theta_1 - 1)^2 - 15| < 1$, there is an algebraic conjugate $\theta'$ of $\theta_1$ such that $|(\theta' - 1)^2 - 15| > 1$. So

$$|\theta' - 1| > 4 \quad \text{or} \quad |\theta' - 1| < \sqrt{14}.$$

By Lemma 5.2, the first case is impossible.

Moreover, $m(\theta') = m(\theta_1)$. So

$$S_d(\theta') = S_d(\theta_1) > S_r(\theta_1)$$

$$> 1 + \frac{2}{3}r + \frac{r^2(r + 1)}{3\pi^2},$$

by Lemma 5.4.

On the other hand

$$S_d(\theta') = S_r(\theta') + \frac{v_{r+1}(\theta')^2}{k_{r+1}} + \frac{v_{r+2}(\theta')^2}{k_{r+2}}$$

$$< 1 + \frac{2}{3}r + \frac{1}{3}(4 + \sqrt{14})r + 15 + 10 + 123.$$

Thus

$$\frac{r^2(r + 1)}{3\pi^2} < \frac{1}{3}(4 + \sqrt{14})r + 148.$$

This implies $r \leq 17$.

Therefore we conclude that $r \leq 17$ as desired.                                      □

**Proof of Theorem 1.1:**   If $\Gamma$ is a bipartite half of a bipartite distance-regular graph of valency 3, then we can use the classification of such graphs given by Ito in [11]. We have (1), (2) and (4) in this case.

Now we can assume $d(\Gamma) \leq r(\Gamma) + 2$ and $r(\Gamma) \leq 17$ by Theorem 3.1, 4.1, and 5.1. It is not hard to check the integrality condition of multiplicities of eigenvalues. Actually, just by testing them for the second largest eigenvalues, we see that the only possible arrays are those of (1), (2), (3) or the following.

$$\left\{ \begin{array}{cccc} * & 1 & 1 & 6 \\ 0 & 1 & 4 & 0 \\ 6 & 4 & 1 & * \end{array} \right\}$$

The nonexistence of a distance-regular graph with the last array follows from Lemma 4.4.

Since the characterization of graphs by parameters same as (1), (2), (3) are known ([8]), we have a desired result.                                                                    □

## 6.   Distance-regular graphs of girth 3, $k \leq 7$

In this section, we give a classification of the graphs in the title above. As we noted in Introduction, except the case $k = 6, a_1 = 1$, the result may be known to some specialists. We decided to include this section for the convenience of the reader. See the table.

Here we only determine the arrays. For the description and the uniqueness, refer the readers to [8]. For GD(3, 1) it seems that the uniqueness problem is not settled yet.

**Lemma 6.1**   *Let $\Gamma$ be a distance-regular graph of valency $k$.*
(1) *If $d(\Gamma) = 1$, then $\Gamma \simeq K_{k+1}$.*
(2) *If $b_1 = 1, k > 2$, then $c_2 = k = 2(m-1)$ and $\Gamma \simeq K_{m \times 2} \simeq$ the complement of $m \cdot K_2$.*

**Proof:**   It is easy and well-known. See [8, Proposition 1.1.5].                          □

**Proposition 6.2**   *Let $\Gamma$ be a distance-regular graph of order $(s, 1)$, i.e., of valency $k = 2(a_1 + 1)$ without an induced subgraph isomorphic to $K_{2,1,1}$. Then $\Gamma$ is a line graph and one of the following holds.*
(1) *$\Gamma \simeq C_n$; an n-gon.*
(2) *$\Gamma \simeq H(2, a_1 + 2)$; a Hamming graph.*
(3) *$\Gamma \simeq$ a collinearity graph of a generalized 2d-gon of order $(s, 1), d = 3, 4,$ or 6.*
(4) *$\Gamma \simeq$ the line graph of a Moore graph.*

$$\iota(\Gamma) = \left\{ \begin{array}{cccc} * & 1 & 1 & 4 \\ 0 & \kappa - 2 & \kappa - 1 & 2\kappa - 6 \\ 2\kappa - 2 & \kappa - 1 & \kappa - 2 & * \end{array} \right\}, \quad \kappa = 3, 7, \text{ or } 57.$$

**Proof:**   See [8, Proposition 4.3.4, Theorem 4.2.16].                                       □

**Lemma 6.3** *Let $\Gamma$ be a distance-regular graph of valency $k = 7$ with $a_1 = 2, c_2 = 2$.*
*Then*

$$\iota(\Gamma) = \begin{Bmatrix} * & 1 & 2 & 7 \\ 0 & 2 & 4 & 0 \\ 7 & 4 & 1 & * \end{Bmatrix}.$$

**Proof:** We name a fixed vertex $\infty$. Since $c_2 = 2$, $\Gamma(\infty) \simeq C_7$. We identify $\Gamma(\infty)$ with $\mathbf{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$, and $i \sim i + 1 \pmod 7$. Each vertex in $\Gamma_2(\infty)$ can be represented by a pair of vertices in $\Gamma(\infty)$, which are adjacent to it.

Let $A = \{(i, i + 1)|i = 0, 1, \ldots, 6\}$, $B = \{(i, i + 3)|i = 0, 1, \ldots, 6\}$. Then $\Gamma_2(\infty) = A \cup B$. Since $\Gamma(i) \simeq C_7$, we have

$$(i, j) \sim (i, k), \quad j \neq k \quad \text{for all } (i, j), (i, k) \in B.$$

Moreover, either $(i, i + 1) \sim (i, i + 3)$ or $(i, i + 1) \sim (i, i + 4)$. If $(i, i + 1) \sim (i, i + 4)$ for some $i$,

$$\partial(i + 1, (i, i + 4)) = 2, \quad \text{while}$$

$$\Gamma(i + 1) \cap \Gamma((i, i + 4)) \ni i, \ (i, i + 1), \ (i + 1, i + 4).$$

This is a contradiction. Hence $(i, i + 1) \sim (i, i + 3)$, and $(i, i - 1) \sim (i, i - 3)$. In particular, we have $a_2 \geq 4$.

If $a_2 = 5$, then we have a contradiction by counting the number of triangles. (See [8, Lemma 4.3.1].) So $a_2 = 4$. Since $c_2 \neq 1$, $c_2 < c_3$. (See [8, 5, 4, 1].) We have $c_3 = 7$, as $k_2 = 14$. We have a desired conclusion.                                                            □

With a little more effort, it is not hard at all to show the uniqueness of the graph. (See [8, p. 386].)

**Lemma 6.4** *If $k = 6$, $a_1 = 2$ and $c_2 \geq 2$, then $d = 2$ and $c_2 = 2$ or 3.*

**Proof:** $\Gamma(x) \simeq 2 \cdot K_3$ or $C_6$. So $a_2 \neq 0$. Hence $c_2 = 2$ or 3.

Assume $d \geq 3$ and derive a contradiction. By Proposition 6.2, we may assume that for some vertex $x$, $\Gamma(x) \simeq C_6$. Since $\Gamma$ is connected, $\Gamma(x) \simeq C_6$ for every $x \in \Gamma$.

Suppose $c_2 = 3$. Since $k_1 = k_2$, $\Gamma$ is an antipodal 2-cover. (See [8, Lemma 5.1.2].) So $\Gamma_2(x) \simeq C_6$ and for each $y \in \Gamma(x)$, $\Gamma(y) \simeq C_6$ and $\Gamma(y) \cap \Gamma_2(x)$ is a path of length 2. We easily obtain a contradiction.

Suppose $c_2 = 2$. Let $\infty$ be a fixed vertex. We identify $\Gamma(\infty)$ with $\mathbf{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, with $i \sim i + 1$ as in the previous lemma. Then $\Gamma_2(\infty) = A \cup B$, where $A = \{(i, i + 1)|i = 0, 1, \ldots, 5\}$, $B = \{(0, 3), (1, 4), (2, 5)\}$. Considering the structure of $\Gamma(i)$, we easily have $(i, i + 1) \sim (i, i + 3) \sim (i, i - 1)$. In particular $a_2 \geq 4$.                                    □

Let $\Gamma$ be a distance-regular graph of valency $k$, girth 3 and diameter $d$. Suppose $k \leq 7$. By Lemma 6.1, we may assume that $k \geq 3$ and $a_1 \leq k - 3$.

If $k$ is odd, $a_1$ must be even as $\Gamma(x)$ is $a_1$-regular. In particular, $k \neq 3$.

Let $k = 5$. Then by our assumption, $a_1 = 2$. Hence $\Gamma(x) \simeq C_5$ and $c_2 \geq 2$, $a_2 \neq 0$. So $c_2 = 2$. Since $k_1 = k_2$ and $a_2$ is even, we have an antipodal 2-cover with $d = 3$. (See [8, Proposition 1.1.4].)

Let $k = 7$. Then by our assumption $a_1 = 2$ or 4.

If $a_1 = 4$, then $\Gamma(x)$ is a complement of $C_7$ or $C_3 \cup C_4$. In either case we have $c_2 \geq 4$ and $a_2 \neq 0$. This is impossible.

Suppose $a_1 = 2$. Then $\Gamma(x) \simeq C_7$ or $C_3 \cup C_4$. So $c_2 \geq 2$ and $a_2 \geq 2$. If $c_2 \geq 3$, then $c_2 = 4$ and $\Gamma$ must be an antipodal 2-cover with 16 vertices. We can eliminate this case by counting the number of triangles. On the other hand, if $c_2 = 2$, we can apply Lemma 6.3.

Let $k = 4$. By our assumption, $a_1 = 1$. Hence we can apply Proposition 6.2, in this case.

Let $k = 6$. If $a_1 = 1$, the results follow from Theorem 1.1. So $a_1 = 2$ or 3. If $a_1 = 2$ and $c_2 = 1$, then we can apply Proposition 6.2. On the other hand, if $a_1 = 2$, $c_2 \geq 2$, we can apply Lemma 6.4.

Suppose $a_1 = 3$. Then $\Gamma(x)$ is a complement of $C_6$ or $2 \cdot K_3$. Hence $c_2 \geq 3$. So $c_2 = 3, 4$ or 6 and $d = 2$. The nonexistence of the first case can be shown easily, for example by counting the number of 5-cycles in the complement.

Thus we have the arrays in the following table.

*Table.* Distance-Regular Graphs of Girth, $k \leq 7$.

$$k = 2 \qquad \left\{\begin{matrix} * & 1 \\ 0 & 1 \\ 2 & * \end{matrix}\right\} \qquad K_3$$

$$k = 3 \qquad \left\{\begin{matrix} * & 1 \\ 0 & 2 \\ 3 & * \end{matrix}\right\} \qquad K_4$$

$$k = 4 \ (a) \quad \left\{\begin{matrix} * & 1 \\ 0 & 3 \\ 4 & * \end{matrix}\right\} \qquad K_5$$

$$(b) \quad \left\{\begin{matrix} * & 1 & 4 \\ 0 & 2 & 0 \\ 4 & 1 & * \end{matrix}\right\} \qquad K_{2,2,2}$$

$$(c) \quad \left\{\begin{matrix} * & 1 & 2 \\ 0 & 1 & 2 \\ 4 & 2 & * \end{matrix}\right\} \qquad H(2, 3)$$

$$(d) \quad \left\{\begin{matrix} * & 1 & 1 & 4 \\ 0 & 1 & 2 & 0 \\ 4 & 2 & 1 & * \end{matrix}\right\} \quad L(O_3)$$

*Table.*    *(Continued).*

$(e)$ $\left\{\begin{array}{cccc} * & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 4 & 2 & 2 & * \end{array}\right\}$    $GH(2, 1)$

$(f)$ $\left\{\begin{array}{ccccc} * & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 2 \\ 4 & 2 & 2 & 2 & * \end{array}\right\}$    $GO(2, 1)$

$(g)$ $\left\{\begin{array}{ccccccc} * & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 4 & 2 & 2 & 2 & 2 & 2 & * \end{array}\right\}$ $GD(2, 1)$

$k = 5$ $(a)$ $\left\{\begin{array}{cc} * & 1 \\ 0 & 4 \\ 5 & * \end{array}\right\}$    $K_6$

$(b)$ $\left\{\begin{array}{cccc} * & 1 & 2 & 5 \\ 0 & 2 & 2 & 0 \\ 5 & 2 & 1 & * \end{array}\right\}$    icosahedron

$k = 6$ $(a)$ $\left\{\begin{array}{cc} * & 1 \\ 0 & 5 \\ 6 & * \end{array}\right\}$    $K_7$

$(b)$ $\left\{\begin{array}{ccc} * & 1 & 6 \\ 0 & 4 & 0 \\ 6 & 1 & * \end{array}\right\}$    $K_{2,2,2,2}$

$(c)$ $\left\{\begin{array}{ccc} * & 1 & 6 \\ 0 & 3 & 0 \\ 6 & 2 & * \end{array}\right\}$    $K_{3,3,3}$

$(d)$ $\left\{\begin{array}{ccc} * & 1 & 4 \\ 0 & 3 & 2 \\ 6 & 2 & * \end{array}\right\}$    $J(5, 2)$

$(e)$ $\left\{\begin{array}{ccc} * & 1 & 3 \\ 0 & 2 & 3 \\ 6 & 3 & * \end{array}\right\}$    a conference graph

$(f)$ $\left\{\begin{array}{ccc} * & 1 & 2 \\ 0 & 2 & 4 \\ 6 & 3 & * \end{array}\right\}$    $H(2, 4)$,  Shrikhande graph

$(g)$ $\left\{\begin{array}{cccc} * & 1 & 1 & 2 \\ 0 & 2 & 2 & 4 \\ 6 & 3 & 3 & * \end{array}\right\}$    $GH(3, 1)$

$(h)$ $\left\{\begin{array}{ccccc} * & 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & 2 & 4 \\ 6 & 3 & 3 & 3 & * \end{array}\right\}$    $GO(3, 1)$

*Table.* *(Continued).*

$$(i) \quad \begin{Bmatrix} * & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 2 & 2 & 2 & 2 & 2 & 4 \\ 6 & 3 & 3 & 3 & 3 & 3 & * \end{Bmatrix} \quad GD(3,1)$$

$$(j) \quad \begin{Bmatrix} * & 1 & 3 \\ 0 & 1 & 3 \\ 6 & 4 & * \end{Bmatrix} \quad GQ(2,2)$$

$$(k) \quad \begin{Bmatrix} * & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 6 & 4 & 4 & * \end{Bmatrix} \quad GH(2,2) \text{ (two graphs)}$$

$$(l) \quad \begin{Bmatrix} * & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 6 & 4 & 2 & * \end{Bmatrix} \quad H(3,3)$$

$$(m) \quad \begin{Bmatrix} * & 1 & 1 & 4 & 6 \\ 0 & 1 & 3 & 1 & 0 \\ 6 & 4 & 2 & 1 & * \end{Bmatrix} \quad \text{3-cover of } GQ(2,2)$$

$$k = 7 \quad (a) \quad \begin{Bmatrix} * & 1 \\ 0 & 6 \\ 7 & * \end{Bmatrix} \quad K_8$$

$$(b) \quad \begin{Bmatrix} * & 1 & 2 & 7 \\ 0 & 2 & 4 & 0 \\ 7 & 4 & 1 & * \end{Bmatrix} \quad \text{3-cover of } K_8$$

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