Journal of Algebraic Combinatorics **11** (2000), 5–16


# A Root System Criterion for Fully Commutative and Short Braid-Avoiding Elements in Affine Weyl Groups

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*Received December 30, 1997; Revised July 31, 1998*

**Abstract.** We provide simple characterizations of short-braid avoiding and fully commutative elements in an affine Weyl group $W$, generalizing results of Fan and Stembridge for finite Weyl groups. Our results rely on the combinatorics of the *compatible* subsets of the root system of $W$.

**Keywords:** affine Weyl group, short-braid avoiding element, fully commutative element

## Introduction and basic definitions

In his paper [4] Fan introduces after Zelevinski the following notion for elements of a Coxeter system $(W, S)$.

**Definition 1** An element $w \in W$ is short-braid avoiding if no reduced expression of $w$ contains a substring of the form $sts,\ s, t \in S$.

The notion of short-braid avoiding element is strictly related to the following definitions, due to Fan [3] and Stembridge [8].

Consider $s, t \in S$ and denote by $m(s, t)$ the order of $st \in W$; we call the string $\underbrace{st\dots}_{m(s,t)}$ the *long braid of* $s$ *and* $t$.

**Definition 2** For $w \in W$ we say that $w$ is commutative if no reduced expression of $w$ contains a substring of the form $sts$, $s, t$ being non-commuting generators in $S$ such that the simple root corresponding to $t$ is at least as long as the simple root corresponding to $s$.

**Definition 3** For $w \in W$ we say that $w$ is fully commutative if no reduced expression of $w$ contains the long braid of some pair of non-commuting generators.

**Remark** Denote by $W_s,\ W_c,\ W_{fc}$ the sets of short-braid avoiding, commutative and fully commutative elements in $W$ respectively.

It turns out that $W_s = W_c = W_{fc}$ for simply-laced Coxeter groups. The relation $W_s = W_{fc}$ holds since the only defining relations which are not commutation relations are those of type $sts = tst,\ s, t \in S$. The equality $W_c = W_{fc}$ is obvious. Moreover Fan and Stembridge [3], [5] provide the following remarkable root-theoretic characterization of these elements. Let $\Delta$ be the canonical root system of $(W, S)$ and set

$$N(w) = \{\alpha \in \Delta^+ \mid w^{-1}(\alpha) \in -\Delta^+\}.$$

Then $w$ is commutative if and only if $\alpha, \beta \in N(w) \Rightarrow \alpha + \beta \notin N(w)$. In the general case, the three definitions introduced differ (although the inclusion relation $W_s \subseteq W_c \subseteq W_{fc}$ holds). Let us work out explicitly the example of a Weyl group $W$ of type $G_2$. If $s_2$ denotes the simple reflection corresponding to the short simple root, then we have

$$\begin{aligned} W_s &= \{1, s_1, s_2, s_1s_2, s_2s_1\} \\ W_c &= \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\} \\ W_{fc} &= W \setminus \{s_1s_2s_1s_2s_1s_2 (= s_2s_1s_2s_1s_2s_1)\} \end{aligned}$$

In [3], [4], [8] the types of $W$ for which $W_s, W_c, W_{fc}$ are finite are determined; in each of these cases, their cardinalities are also determined. Moreover, when $W$ is a finite Weyl group, Fan provides the following remarkable criterion for $w \in W$ to be short-braid avoiding: *$w \in W$ is short-braid avoiding if and only if any reduced expression of $w$ remains reduced when a simple reflection is deleted in any possible way.*

This result has interesting applications since it gives a simple smoothness criterion for Schubert varieties attached to braid-avoiding elements. On the other hand, as noticed in [4, 5], the criterion does not hold for affine Weyl groups (a counterexample in type $\tilde{A}_2$ is $s_1s_2s_3s_1s_2$).

In this paper we provide a combinatorial characterization of the elements in $W_s,\ W_{fc}$ for *affine* Weyl groups $W$ in terms of the subsets $N(w)$ which encode the elements of $W$.

Before stating our results we fix the notation and we give some preliminary definitions.

Let $W$ be an irreducible Weyl group (possibly affine) and let $\Delta$ be the associated root system (lying in a real vector space $V$). Fix a positive system $\Delta^+$ in $\Delta$ and let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a corresponding basis of simple roots; then we have $\Delta = W\Pi$ and $\Delta = \Delta^+ \sqcup -\Delta^+$ ($\sqcup$ denotes the disjoint union). We list some standard notation relative to these data.

| | |
|---|---|
| $A = (a_{ij})_{i,j=1}^l$ | (generalized) Cartan matrix corresponding to $\Delta$, |
| $s_i$ | fundamental reflection relative to $\alpha_i \in \Pi$, |
| $S = \{s_1, \dots, s_l\}$ | set of Coxeter generators for $W$, |
| $\ell$ | length function w.r.t. $S$, |
| $s_\alpha$ | reflection relative to $\alpha \in \Delta^+$, |
| $(\ ,\ )$ | standard W-invariant bilinear form on $V$, positive definite if $\Delta$ is finite, positive semidefinite with kernel $\mathbb{R}\delta$ if $\Delta$ is affine, |
| $\langle \alpha, \beta \rangle = \dfrac{2(\alpha, \beta)}{(\beta, \beta)}$ | $(\alpha,\ \beta \in \Delta)$, |

| | |
|---|---|
| $Supp(\alpha)$ | support of $\alpha$: if $\alpha = \sum_{i=1}^{n} a_i\alpha_i$, then $Supp(\alpha) = \{\alpha_i \mid a_i \neq 0\}$. |
| $\mathbb{N}(\alpha, \beta) = (\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta$ | $(\alpha, \beta \in \Delta^+)$. |

For a root $\alpha \in \Delta$ as usual $\alpha > 0$ means $\alpha \in \Delta^+$. We recall that for each $v, w \in W$ we have $\ell(v) = |N(v)|$ and $N(vw) = N(v) + vN(w)$, where $+$ denotes the symmetric difference; moreover $N(vw) = N(v) \sqcup vN(w)$ if and only if $\ell(vw) = \ell(v) + \ell(w)$.

For any $R \subseteq \Delta$ set $W(R) = \langle s_\beta \mid \beta \in R\rangle$. We say that $R$ is a *subsystem* of $\Delta$ if it is $W(R)$-invariant. Note that $R^+ := R \cap \Delta^+$ is a set of positive roots for $R$. We say that $\mathfrak{R} \subseteq \Delta^+$ is a *p-subsystem* if $\mathfrak{R} = R \cap \Delta^+$ for some subsystem $R$. Equivalently, $\mathfrak{R}$ is a p-subsystem if $\mathfrak{R} \subseteq \Delta^+$ and $\mathfrak{R} \cup -\mathfrak{R}$ is a subsystem.

If $R \subseteq \Delta$ is a subsystem, we denote the cardinality of a root basis for $R$ by $rk(R)$, and we call it the *rank* of $R$.

Moreover we say that $R$ is *parabolic* if

$$\Delta \cap Span_{\mathbb{Q}}(R) = R.$$

As usual we say that a subsystem $R$ of $\Delta$ is *standard parabolic* if $\Pi \cap R$ is a basis for $R$. Clearly a standard parabolic subsystem is parabolic. Moreover it is easily seen that a subsystem $R$ is parabolic if and only if $R^+ = vR'^+$ for some standard parabolic subsystem $R'$ and $v \in W$ (see [1, VI, 1.7, Prop. 24]).

If $\Delta$ is an irreducible affine root system and $\Delta^0$ the associated finite root system, then $\Delta^0$ is irreducible (in particular it has a unique highest root); moreover (cf. [6])

$$\Delta^+ = ((\Delta^0)^+ + \mathbb{N}\delta) \cup (-(\Delta^0)^+ + \mathbb{Z}^+\delta)$$

We call the elements of $\Delta^0$ (resp. $(\Delta^0)^+$) *finite* roots (resp. *finite* positive roots).

Let $R_0$ be any subsystem of $\Delta^0$. Then

$$R = \{\beta + k\delta \mid \beta \in R_0, \quad k \in \mathbb{Z}\}$$

is clearly a subsystem of $\Delta$ and it is the affine root system associated to $R_0$. For $\alpha \in \Delta^0$ set:

$$\underline{\alpha} := \begin{cases} \{\alpha + n\delta \mid n \in \mathbb{N}\} & \text{if } \alpha \in (\Delta^0)^+, \\ \{\alpha + m\delta \mid m \in \mathbb{Z}^+\} & \text{if } -\alpha \in (\Delta^0)^+; \end{cases}$$

we call $\underline{\alpha}$ the *$\delta$-string* of $\alpha$. When considering an affine root $\beta_0 + k\delta,\ \beta_0 \in \Delta^0$, we write $k \in \mathbb{N}'$ to mean $k \in \mathbb{N}$ if $\beta_0 > 0$ and $k \in \mathbb{Z}^+$ if $\beta_0 < 0$.

Moreover we say that a root $\beta$ is *parallel* to $\alpha$ ($\beta \parallel \alpha$) if $\beta + \alpha \in \mathbb{Z}\delta$ or $\beta - \alpha \in \mathbb{Z}\delta$.

**Definition 4** We say that $L \subseteq \Delta^+$ is *dependent* if there exist pairwise non-parallel roots $\alpha, \beta, \gamma \in L$ and $k \in \mathbb{Z}^+$ such that $\alpha + \beta = k\gamma$; we say that $L$ is independent if it is not dependent.

Our main theorems are the following.

**Theorem 1** *Assume $\Delta \ncong \tilde{A}_1$. Then $w \in W_s$ if and only if $N(w)$ is independent.*

**Remark** If $\Delta \cong \tilde{A}_1$, then $w \notin W_s$ if and only if $\ell(w) \geq 3$. This is equivalent to the condition of dependence in Definition 4 without any requirement about parallelism.

**Theorem 2** *Let $w \in W$. Then $w \in W_{fc}$ if and only if $N(w)$ does not contain any irreducible parabolic p-subsystem of rank 2.*

## Preliminaries

We introduce now the main tools for the proof of the main theorems.

**Definition 5** Let $L \subseteq \Delta^+$ and $<$ be a total order on $L$. We say that $L$ is associated to $w \in W$ if $L = N(w)$. We say that $(L, <)$ is associated to the reduced expression $s_{i_1} \cdots s_{i_m} = w$ $(w \in W)$ if

$$L = \{\alpha_{i_1} < s_{i_1}(\alpha_{i_2}) < \cdots < s_{i_1} \cdots s_{i_{m-1}}(\alpha_{i_m})\}$$

(in particular $L = N(w)$).

The following Proposition 1 is the easy part of a well known theorem of Dyer [2]; it holds for any Coxeter system. We prove it here for completeness.

**Proposition 1** *Assume that $(L, <)$ is associated to some reduced expression of some element of $W$. Then, for each $\alpha, \beta \in \Delta^+$, $q, r \in \mathbb{R}^+$, the following conditions hold:*
(I) *if $\alpha, \beta \in L$, $\alpha < \beta$, and $q\alpha + r\beta \in \Delta$, then $q\alpha + r\beta \in L$ and $\alpha < q\alpha + r\beta < \beta$.*
(II) *if $q\alpha + r\beta \in L$ and $\beta \notin L$, then $\alpha \in L$ and $\alpha < q\alpha + r\beta$.*

*In particular, if $L$ is associated to some element of $W$, then, for $\alpha, \beta \in \Delta^+$, $q, r \in \mathbb{R}^+$ we have:*
(I′) *if $\alpha, \beta \in L$, $q\alpha + r\beta \in \Delta$, then $q\alpha + r\beta \in L$.*
(II′) *if $q\alpha + r\beta \in L$ and $\beta \notin L$, then $\alpha \in L$.*

**Proof:** Assume that $(L, <) = \{\beta_1 < \cdots < \beta_n\}$ with $\beta_1 = \alpha_{i_1}$, $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$, $1 < j \leq n$, $w = s_{i_1} \cdots s_{i_n}$ reduced expression. First assume $\alpha, \beta \in L$, $q, r \in \mathbb{R}^+$, $q\alpha + r\beta \in \Delta$. Then by definition $w^{-1}(\alpha), w^{-1}(\beta) < 0$; thus $w^{-1}(q\alpha + r\beta) = qw^{-1}(\alpha) + rw^{-1}(\beta) < 0$, i.e., $q\alpha + r\beta \in L$. Assume $\alpha = \beta_j$ and $\beta = \beta_k$, with $j < k$ and put $v = s_{i_1} \cdots s_{i_{j-1}}$. Then $(L', <') := \{v^{-1}(\beta_j) < \cdots < v^{-1}(\beta_k)\}$ is associated to the reduced expression $s_{i_j} \cdots s_{i_k}$ and has $v^{-1}(\alpha)$ and $v^{-1}(\beta)$ as its first and last element, respectively. By the first part of our proof we have $v^{-1}(q\alpha + r\beta) \in L'$, and clearly $v^{-1}(\alpha) <' v^{-1}(q\alpha + r\beta) <' v^{-1}(\beta)$. It follows directly that $\alpha < q\alpha + r\beta < \beta$.

Next assume $q\alpha + r\beta \in L$; $q\alpha + r\beta = \beta_m$, $m \leq n$. Then $\{\beta_1 < \cdots < \beta_m\}$ is associated to the reduced expression $u = s_{i_1} \cdots s_{i_m}$ and has $q\alpha + r\beta$ as its last element. Assume towards a contradiction that neither $\alpha \in N(u)$ nor $\beta \in N(u)$. Then, by definition, $u^{-1}(\alpha), u^{-1}(\beta) > 0$; thus $u^{-1}(q\alpha + r\beta) = qu^{-1}(\alpha) + ru^{-1}(\beta) > 0$, against the assumption, so that we have necessarily $\alpha \in N(u)$ or $\beta \in N(u)$. Moreover, clearly, if $\alpha$ or $\beta$ belongs to $N(u)$, then it precedes $q\alpha + r\beta = \beta_m$. This implies the assertion for $L$. □

**Remark** For $k \in \mathbb{N}$ we have $\alpha + k\delta = \frac{1}{k+1}\alpha + \frac{k}{k+1}(\alpha + (k+1)\delta)$; if $\alpha \in \Delta^+$ and $\alpha + k\delta \in L$ for some $k \in \mathbb{N}$, then since $L$ is finite, by condition (II) of Proposition 1 we obtain that $\alpha \in L$. Moreover, by condition (I), $\alpha + h\delta \in L$ if $0 \leq h \leq k$, $h \in \mathbb{N}$. Similarly, since for each $k$ $\alpha + k\delta = (k+1)\alpha + k(-\alpha + \delta)$, if $\alpha \in L$, then $-\alpha + \delta \notin L$. In fact if $\underline{\alpha} \cap L \neq \emptyset$, then $\underline{-\alpha} \cap L = \emptyset$.

The conditions of Proposition 1 are also sufficient for $(L, <)$ to be associated to some reduced expression of some $w \in W$ [2]. Indeed, for the root system of an (affine) Weyl group they can be weakened [7], as we shall see below. We denote by $\bar{\Delta}$ (resp. $\bar{\Delta}^+$) the generalized root system (resp. positive root system) [8] associated to the Cartan matrix $A$ of $\Delta$,

$$\bar{\Delta} = \Delta \sqcup \pm\mathbb{Z}^+\delta, \quad \bar{\Delta}^+ = \Delta^+ \sqcup \mathbb{Z}^+\delta.$$

**Theorem A** [7] *Let $L \subseteq \Delta^+$ be finite and $<$ be a total order on $L$. $(L, <)$ is associated to some reduced expression of some element of $W$ if and only if, for each $\alpha, \beta \in \bar{\Delta}^+$, the following conditions hold:*

(1) *if $\alpha, \beta \in L$, $\alpha < \beta$, and $\alpha + \beta \in \bar{\Delta}$, then $\alpha + \beta \in L$ and $\alpha < \alpha + \beta < \beta$.*

(2) *If $\alpha + \beta \in L$ and $\beta \notin L$, then $\alpha \in L$ and $\alpha < \alpha + \beta$.*

*$L$ is associated to some element of $W$ if and only if for each $\alpha, \beta \in \bar{\Delta}^+$:*

(1′) *if $\alpha, \beta \in L$, $\alpha + \beta \in \bar{\Delta}$, then $\alpha + \beta \in L$*

(2′) *if $\alpha + \beta \in L$ and $\beta \notin L$, then $\alpha \in L$.*

**Corollary 1** *Let $L \subseteq \Delta^+$ be finite. Then the following are equivalent:*

i) *$L$ is associated to some element of $W$;*

ii) *$L$ satisfies conditions* (I′) *and* (II′) *of Proposition* 1*;*

iii) *$L$ satisfies conditions* (1′) *and* (2′) *of Theorem A.*

**Corollary 2** *Let $L \subseteq \Delta^+$ be finite and $<$ be a total order on $L$. Then the following are equivalent:*

i) *$(L, <)$ is associated to some reduced expression of some element of $W$;*

ii) *$(L, <)$ satisfies conditions* (I) *and* (II) *of Proposition* 1*;*

iii) *$(L, <)$ satisfies conditions* (1) *and* (2) *of Theorem A.*

**Definition 6** Let $L \subseteq \Delta^+$ and $<$ be a total order on $L$. $L$ is called compatible if it satisfies one of the three equivalent conditions of Corollary 1. $(L, <)$ is called compatible, if it satisfies one of the three equivalent conditions of Corollary 2 (in particular $L$ is compatible). In such a case we also say that $<$ is a compatible order.

Note that $N(w)$ determines $w \in W$, thus Theorem A establishes a bijection between $W$ and the compatible finite subsets of $\Delta^+$; moreover it gives a bijection between the compatible orders on $N(w)$ and the reduced expressions of $w$, for any fixed $w \in W$.

**Proofs of the main theorems**

If $<$ is a compatible order on $N(w)$, then, by condition (I) of Proposition 1, or (1) of Theorem A, we get that $\min(N(w), <)$ is a simple root. Indeed any simple root in $N(w)$ can be taken as the least root for some compatible order on $N(w)$; in the following lemma we state this and other basic properties of compatible sets and orders in a convenient form for our next developments.

**Lemma 1** *Let $L$ be a finite compatible set, $\alpha \in L$ be a simple root, and $L' = s_\alpha(L \setminus \{\alpha\})$. Then:*

i) *$L'$ is compatible.*
ii) *If $<'$ is a compatible order on $L'$, then the total order defined on $L$ by:*
*$\alpha = \min(L, <)$ and $\beta < \beta'$ if and only if $s_\alpha(\beta) <' s_\alpha(\gamma)$ for $\beta, \gamma \in L \setminus \{\alpha\}$ $(*)$*
*is compatible. In particular there exists a compatible order $<$ on $L$ such that $\alpha = \min(L, <)$.*
iii) *Conversely, if $<$ is a compatible order on $L$ such that $\alpha = \min(L, <)$, then the total order $<'$ defined on $L'$ by $(*)$ is compatible. In particular if $\beta$ is the successor of $\alpha$ in $(L, <)$, then $s_\alpha(\beta)$ is a simple root.*

**Proof:**

i) By assumption there exists $w \in W$ such that $L = N(w)$. Set $w' = s_\alpha w$; then $\ell(w') = \ell(w) - 1$, hence $N(w) = N(s_\alpha) \sqcup s_\alpha N(w') = \{\alpha\} \sqcup s_\alpha N(w')$ and therefore $N(w') = s_\alpha(N(w) \backslash \alpha) = L'$. It follows that $L'$ is compatible.
ii) If $(L', <')$ is associated to the reduced expression $s_{i_1} \cdots s_{i_k}$, then $(L, <)$ is associated to the reduced expression $s_\alpha s_{i_1} \cdots s_{i_k}$, therefore it is compatible.
iii) $(L, <)$ is associated to the some reduced expression starting with $s_\alpha$, say $s_\alpha s_{i_1} \cdots s_{i_k}$; then $(L', <')$ is associated to the reduced expression $s_{i_1} \cdots s_{i_k}$, therefore it is compatible. □

**Lemma 2** *Suppose that $N(w)$ is endowed with a compatible order $<$. If $R^+$ is a finite p-subsystem, then $M = R^+ \cap N(w)$ is compatible as a subset of $R = R^+ \cup -R^+$, and the restriction of $<$ to $M$ is compatible.*

**Proof:** Suppose $\alpha, \beta \in M$, $\alpha + \beta \in R^+ = \bar{\Delta}^+ \cap R = \Delta^+ \cap R \subseteq \Delta^+$; then the compatibility of $N(w)$ implies $\alpha + \beta \in N(w)$ and in turn $\alpha + \beta \in M$. If now $\alpha + \beta \in M$, then as above the compatibility of $N(w)$ and the relation $R^+ = \Delta^+ \cap R$ imply $\alpha \in M$ or $\beta \in M$ as desired. The claim regarding the order is proved in the same way. □

**Lemma 3** *Let $\xi$ be a positive root in an affine root system $\Delta$; then $\xi$, $\xi + \delta$ can be consecutive in a compatible order on some compatible subset of $\Delta^+$ if and only if $\Delta$ is of type $\tilde{A}_1$.*

**Proof:** Assume that $\xi,\ \xi+\delta$ are consecutive in a compatible order. By Lemma 1 iii) there exists $w \in W$ such that both $\alpha = w(\xi)$ and $s_\alpha w(\xi+\delta) = -\alpha+\delta$ are simple roots; this clearly implies $\Delta \cong \tilde{A}_1$. The converse is also clear. □

**Theorem 1.** *Assume $\Delta \ncong \tilde{A}_1$. Then $w \in W_s$ if and only if $N(w)$ is independent.*

**Proof:** Set $N = N(w)$. Assume $w \notin W_s$; we have to prove that $N$ is dependent. By hypothesis, $w$ can be written in reduced form as $w = us_is_js_iv$ where $s_i, s_j \in S,\ u, v \in W$. Then in the ordering induced by such a reduced expression, $u(\alpha_i), u(s_i(\alpha_j)), u(s_is_j(\alpha_i))$ are consecutive. But $u(\alpha_i) + u(s_is_j(\alpha_i)) = -a_{ij}u(s_i(\alpha_j))$ and moreover, since $\Delta \ncong \tilde{A}_1$, $\alpha_i$ and $\alpha_j$ are not parallel; therefore $N$ is dependent.

Conversely, assume that $N$ is dependent. Endow $N$ with an arbitrary compatible order $<$ and consider the set

$$I_< = \{(\alpha,\gamma,\beta) \in (N)^{\times 3} \mid \alpha < \gamma < \beta,\ \alpha \nparallel \beta,\ \exists k \in \mathbb{Z}^+\ \alpha+\beta = k\gamma\}.$$

For a triple $(\alpha,\gamma,\beta) \in I_<$ set

$$\rho_<(\alpha,\gamma,\beta) = |\{x \in N \mid \alpha < x < \beta\}|$$

Take any triple $(\alpha,\gamma,\beta)$ such that $\rho_<(\alpha,\gamma,\beta)$ is minimal. By repeated applications of Lemma 1 we may assume $\alpha = \min N$, so that $\alpha$ is simple. Since $\alpha, \beta$, hence $\gamma$, are not mutually parallel, they are contained in a uniquely determined finite parabolic subsystem $R$ of rank 2. Let $\beta'$ be the only root in $\Delta^+$ which completes $\alpha$ to a root basis for $R^+$, so that $R^+ = \mathbb{N}(\alpha,\beta')$. By Lemma 2, $(R^+ \cap N, <)$ is compatible in $R$. Since it has $\alpha$ as its first element, it is associated to some expression of type $s_\alpha s_{\beta'} s_\alpha \ldots$, so that $(R^+ \cap N, <) = \{\alpha < s_\alpha(\beta') < s_\alpha s_{\beta'}(\alpha) < \cdots\}$. Therefore, by the minimality of $\rho_<$ we have $\gamma = s_\alpha(\beta')$ and $\beta = s_\alpha s_{\beta'}(\alpha)$. If $\gamma$ is the successor of $\alpha$ in $N$, then by Lemma 1 $s_\alpha(\gamma) = \beta'$ is simple in $\Delta$. Also $s_{\beta'}s_\alpha(\beta) = \alpha$ is simple, hence, again by Lemma 1, there exists a compatible order on $N$ starting with $\alpha < \gamma < \beta$, which corresponds to a reduced expression of $w$ starting with the braid $s_\alpha s_{\beta'} s_\alpha$. Assume that there exists $x \in N$, such that $\alpha < x < \gamma$. We define $\gamma_0 = \gamma$ and, for $i \geq 1$, $\gamma_i = \max_<\{\eta \in N \mid \eta < \gamma_{i-1},\ (\eta,\gamma_{i-1}) \neq 0\}$ if $\{\eta \in N \mid \eta < \gamma_{i-1},\ (\eta,\gamma_{i-1}) \neq 0\}$ is non empty. Let $\gamma_n$ be the last element we can define in such a way. If $\gamma_n \neq \alpha$, then we can replace $<$ with a suitable compatible order in which $\gamma_n$ precedes $\alpha$; therefore without loss of generality we may assume that $\gamma_n = \alpha$. Moreover, if $n = 1$, then for each $x \in N$ such that $\alpha < x < \gamma$ we have $x \perp \gamma$; thus we can bring $\gamma$ adjacent to $\alpha$ and we are done. Assume by contradiction $n \geq 2$. We claim that $\gamma_i \nparallel \gamma_{i+1}\ \forall\, i = 0,\ldots,n-1$. Otherwise, since $N$ is compatible, $\gamma_i - \gamma_{i+1} = \delta$. Moreover, by the definition of $\gamma_{i+1}$, there exists a compatible order in which $\gamma_{i+1},\ \gamma_i$ appear in consecutive positions: this contradicts Lemma 3. Now remark that, since $\gamma_i + \gamma_{i+1}$ is a not a root by the minimality of $\rho_<$, the definition of the $\gamma_i$'s forces $\gamma_i - \gamma_{i+1}$ to be a root. Such a root must be positive, otherwise we get $\gamma_{i+1} - \gamma_i \in N$ and $\gamma_{i+1} - \gamma_i < \gamma_{i+1} < \gamma_i$, against the the minimality of $\rho_<$. Then we define $k_i = \max\{h \in \mathbb{Z}^+ \mid h\gamma_i - \gamma_{i+1} \in \Delta^+\}$. Since we are assuming $n \geq 2$, we have $w^{-1}(k_i\gamma_i - \gamma_{i+1}) > 0$: this follows from the minimality of $\rho_<$ (if $\gamma_{i+1} = \alpha$ then $\gamma_i \neq \gamma$ and by our previous remarks $k_i\gamma_i - \gamma_{i+1} \neq \beta$). Adding up such relations we get that $w^{-1}(k_{n-1}\cdots k_0\gamma-\alpha)$ is a sum of positive roots with non-negative coefficients. This is a contradiction if $k_{n-1}\cdots k_0\geq k$, since $w^{-1}(k\gamma-\alpha)=w^{-1}(\beta)<0$ and $w^{-1}(\gamma)<0$; in particular we get a contradiction if $k=1$. So we assume $k>1$. We remark that if $\Delta\ncong G_2,\ \tilde{G}_2$, any finite rank 2 indecomposable subsystem of $\Delta$ is of type $A_2$ or $B_2$. If $\Delta\cong G_2,\ \tilde{G}_2$, any finite rank 2 subsystem of $\Delta$ is of type $A_2$ or $G_2$. Therefore, if $k>1$ then $k=2$ if $\alpha,\gamma,\beta$ are included in a subsystem of type $B_2$, and $k=3$ if $\alpha,\gamma,\beta$ are included in a subsystem of type $G_2$. It follows that for each $k_i>1$, we have $k_i=k$. Moreover, if $\eta,\eta'$ are non parallel and non orthogonal roots in $\Delta$, then $\langle\eta,\eta'\rangle=\pm1$ if $\eta$ and $\eta'$ have the same length or $\eta$ is short; $\langle\eta,\eta'\rangle=\pm k$ if $\eta$ is long and $\eta'$ is short. Thus if $k>1$, we get in particular that $\alpha$ is long and $\gamma$ is short. Now we remark that $(\gamma_i,\gamma_{i+1})>0$. If $k_i=1$ for each $0\leq i\leq n-1$, then $\gamma_0,\ldots,\gamma_n$ all have the same length: this is impossible since $\gamma_0=\gamma$ is short and $\gamma_n=\alpha$ is long. Thus for some $i$ we have $k_i=k$ and thus $k_{n-1}\cdots k_0\geq k$. □

**Lemma 4** *Assume $\Delta\ncong G_2,\tilde{G}_2$ and $\alpha,\beta\in\Delta^+$. If $(\alpha,\beta)<0$ then either $\alpha\parallel\beta$ or $\mathbb{N}(\alpha,\beta)$ is a p-subsystem. In the latter case $\mathbb{N}(\alpha,\beta)\cup-\mathbb{N}(\alpha,\beta)$ is an irreducible parabolic subsystem having $\{\alpha,\beta\}$ as a basis.*

**Proof:** Assume that $\alpha,\beta$ are not parallel and set $R=(\mathbb{Q}\alpha+\mathbb{Q}\beta)\cap\Delta$. Then $R$ is clearly a finite parabolic rank 2 subsystem of $\Delta$. Since $\Delta\ncong G_2,\tilde{G}_2$ the type of $R$ is one of $A_1\times A_1$, $A_2$, $B_2$. Indeed it cannot be $A_1\times A_1$, since $R$ contains $\alpha$ and $\beta$ which are not orthogonal, hence it is $A_2$ or $B_2$. But if $R'$ is a root system of type $A_2$ or $B_2$, then any two roots with negative scalar product are a basis of $R'$, thus $\{\alpha,\beta\}$ is a basis for $R$ and $R^+=\mathbb{N}(\alpha,\beta)$. □

**Lemma 5** *Assume that the simple roots $\alpha_i$, $\alpha_j$ belong to $N(w)$. Then some reduced expression of $w$ starts with the long braid of $s_i$ and $s_j$.*

**Proof:** Set $X_{ij}=\{w\in W\mid w^{-1}(\alpha_i)>0$ and $w^{-1}(\alpha_j)>0\}$ and $W_{ij}=\langle s_i,s_j\rangle$. By [1, IV, ex. 1.3] there exist unique $u\in W_{ij}$ and $v\in X_{ij}$ such that $w=uv$; moreover $\ell(w)=\ell(u)+\ell(v)$ so that $N(w)=N(u)\sqcup uN(v)$. By definition $\alpha_i,\alpha_j\notin N(v)$ hence $\alpha\notin N(v)$ for each $\alpha\in R(\alpha_i,\alpha_j)$. Now $W_{ij}$ permutes $R(\alpha_i,\alpha_j)$ and therefore it permutes $\Delta\backslash R(\alpha_i,\alpha_j)$. Since $s_k$ permutes $\Delta^+\backslash\{\alpha_k\}$ for $k=i,j$, it follows that $W_{ij}$ permutes the positive roots out of $R(\alpha_i,\alpha_j)$. Therefore we have $\alpha_i,\alpha_j\notin uN(v)$. On the other hand $\alpha_i,\alpha_j\in N(w)$, thus $\alpha_i,\alpha_j\in N(u)$. Therefore $u$ is the longest element in $W_{ij}$ and its reduced expressions are the long braids of $s_i$ and $s_j$. □

Given any finite parabolic subsystem $R$ in $\Delta$, there always exists a compatible pair $(L,<)$ with $L$ finite, in which the roots of the p-subsystem $R^+$ are consecutive. In fact, there exist a standard parabolic p-subsystem $R'$ and $w\in W$ such that $wR'^+=R^+$. Let $u$ be the longest element of $W(R')$ and consider $wu$. Then $N(wu)=N(w)+wN(u)=N(w)+w(R'^+)=N(w)+R^+$; since $w^{-1}(R^+)=R'^+$, we have indeed $N(wu)=N(w)\sqcup R^+$. Therefore the join of a reduced expression of $w$ and a reduced expression of $u$ is a reduced expression of $wu$; in the order induced on $N(wu)$ by any such reduced expression $R^+$ appears as a final section.

On the other hand if we fix a compatible set $L$ including a parabolic p-subsystem $R^+$, then it may happen that there is no compatible order on $L$ of which $R^+$ is a section. For instance in type $D_4$, consider $w = s_2s_1s_3s_2s_4s_2s_3s_1s_2$. Then $\alpha_2, \alpha_1+2\alpha_2+\alpha_3+\alpha_4, \alpha_1+\alpha_2+\alpha_3+\alpha_4 \in N(w)$ and they form a parabolic p-subsystem of type $A_2$, but they can be consecutive in no compatible order of $N(w)$. Nonetheless for the case $A_n$ we have the following "strong" result:

**Proposition 2** *Suppose $W = S_{n+1}$, the symmetric group on $n+1$ letters (so that $\Delta$ is a root system of type $A_n$); consider $w \in W$. Then, for any triple of roots $\{\alpha, \alpha+\beta, \beta\}$ in $N(w)$, there exists a compatible order in $N(w)$ in which these elements are consecutive.*

**Proof:** We proceed by induction on $\ell(w)$. Consider a triple $\{\alpha, \alpha+\beta, \beta\} \subseteq N(w)$. If there exists a simple root $\gamma \in N(w)$ different from $\alpha$ and $\beta$ then we consider the triple $\{s_\gamma(\alpha), s_\gamma(\alpha+\beta), s_\gamma(\beta)\} \subseteq N(s_\gamma w)$. Since $\ell(s_\gamma w) < \ell(w)$, by induction there exists a compatible order on $N(s_\gamma w)$ in which $s_\gamma(\alpha), s_\gamma(\alpha+\beta), s_\gamma(\beta)$ are consecutive; by Lemma 1 this order comes from a compatible order on $N(w)$ in which $\alpha, \alpha+\beta, \beta$ are consecutive. We have two more cases to consider: either both $\alpha$ and $\beta$ are simple roots or one of the two—say $\alpha$—is the only simple root in $N(w)$. In the first case we are done by Lemma 5; in the other case we get a contradiction, since by compatibility $\beta$ should contain $\alpha$ in its support but this would imply $\alpha+\beta \notin \Delta$. □

**Theorem 2.** *Let $w \in W$. Then $w \in W_{fc}$ if and only if $N(w)$ does not contain any irreducible parabolic p-subsystem of rank 2.*

**Proof:** We show that $w \notin W_{fc}$ if and only if $N(w)$ contains an irreducible p-subsystem of rank 2; if $w \notin W_{fc}$, then for some $i, j \in \{1, \ldots, l\}$, $w = u\underbrace{s_is_j\ldots}_{m(i,j)}v$, $u, v \in W$, $\ell(w) = \ell(u) + m(i,j) + \ell(v)$.

Then

$$N(w) = N(u) \sqcup u(N(s_is_j\ldots)) \sqcup us_is_j\ldots(N(v)).$$

But $N(s_is_j\ldots) = \mathbb{N}(\alpha_i, \alpha_j)$ is clearly an irreducible p-subsystem of rank 2, hence $u(\mathbb{N}(\alpha_i, \alpha_j)) = \mathbb{N}(u(\alpha_i), u(\alpha_j))$ is too.

Next assume that the set of irreducible rank 2 p-subsystems contained in $N(w)$ is non-empty; this implies in particular $\Delta \not\cong \tilde{A}_1$. Set $N = N(w)$. Fix any compatible order $<$ on $N$. For any parabolic p-subsystem $R^+ \subseteq N$ set $R = R^+ \cup -R^+$ and

$$d_<(R) = |\{x \in N \mid \min R^+ < x < \max R^+\}|,$$

where the maximum and minimum are taken with respect to the restriction of $<$ to $R^+$. Choose a finite parabolic irreducible p-subsystem of rank 2, $R^+ \subseteq N$ such that $d_<(R)$ is minimal. Then set $\alpha = \min R^+$, $\beta = \max R^+$, and $\gamma = \min(R^+ \setminus \{\alpha\})$, the successor of $\alpha$ in $R^+$. Consider the set $\{x \in N \mid \alpha < x < \gamma\}$: if it is empty, then by Lemma 1 *iii*), $\beta = s_\alpha(\gamma)$ is simple and we conclude using Lemma 5. We shall prove that if $x \in N$ and $\alpha < x < \gamma$, then $x$ is orthogonal to $\alpha$. From this, by Lemma 1 *iii*), it follows that we can bring $\alpha$ adjacent to $\gamma$, still obtaining a compatible order on $N$, and we can conclude by Lemma 5. It is enough to prove it for $x$ the successor of $\alpha$ in $N$ since if $x \perp \alpha$, then we can exchange $\alpha$ and $x$ in $(N, <)$, still obtaining a compatible order $<'$ on $N$ in which $d_{<'}(R)$ is minimal. As in the proof of theorem 1 we may assume that $\alpha = \min N$ and that $\alpha$ is simple. Moreover we may assume $\beta = \max N$. So let $x = \min(N \backslash \{\alpha\})$, $x \neq \gamma$.

Remark that a subsystem of type $A_2$ is parabolic unless it is contained in a (sub)system of type $G_2$; moreover, if $\Delta$ contains a subsystem of type $G_2$, then $\Delta$ is of type $G_2$ or $\tilde{G}_2$. A subsystem of type $B_2$ or $G_2$ is always parabolic. These remarks lead us to consider separately the $G_2$, $\tilde{G}_2$ cases.

*First case:* $\Delta \ncong G_2, \tilde{G}_2$

We first prove that if $\Delta$ is an affine system, then for each $\xi \in R^+$, $\xi$ is the least root (w.r.t. $<$) in its $\delta$-string. Since $\alpha$ is simple, we have $\alpha \in \Delta^0$ or $\alpha = -\theta + \delta$, $\theta$ being the highest root in $\Delta^0$. Assume $\beta = \beta_0 + k\delta$ with $\beta_0 \in \Delta^0$ and $k \in \mathbb{N}'$. Since $N$ is compatible, if $\beta_0 > 0$ then $\beta_0 \in N$ and if $\beta_0 < 0$ then $\beta_0 + \delta \in N$. We have $\langle \alpha, \beta \rangle = \langle \alpha, \beta_0 + k\delta \rangle = \langle \alpha, \beta_0 \rangle$ and $\langle \beta, \alpha \rangle = \langle \beta_0 + k\delta, \alpha \rangle = \langle \beta_0, \alpha \rangle$ for each $k \in \mathbb{Z}$; therefore, since $\Delta \ncong \tilde{G}_2$, $\{\alpha, \beta_0 + k\delta\}$ is a basis for a parabolic irreducible subsystem of rank 2 in $\Delta$ if and only if $\{\alpha, \beta_0\}$ is. Thus, by the minimality of $d_<(R)$, if $\beta_0 > 0$ then $\beta = \beta_0$, and if $\beta_0 < 0$ then $\beta = \beta_0 + \delta$. Since by assumption $\alpha + \beta$ is a root, if $\beta = \beta_0 + \delta$ with $\beta_0 < 0$, then $\alpha$ is a finite simple root (recall that $\theta + \eta \notin \Delta \ \ \forall \eta \in (\Delta^0)^+$); similarly, if $\alpha = -\theta + \delta$ then $\beta$ is finite positive. In both cases if $\xi \in R^+$, $\xi \neq \alpha, \beta$ then $\xi = \xi_0 + \delta$ with $-\xi_0 \in (\Delta^0)^+$. If $\alpha$ and $\beta$ are positive finite, then the same holds for any $\xi \in R^+$. In any case, each $\xi \in R^+$ has the required minimality condition.

From the above result we get that $x$ does not belong to the same $\delta$-string of any root in $R^+$ other than $\alpha$. Indeed, by compatibility, $x$ is not parallel to any root in $R^+$ other than $\alpha$; moreover, by Lemma 3, $x$ is not parallel to $\alpha$.

Henceforth $\Delta$ may be finite or not. We distinguish several cases.

I. $R \cong A_2$. Then $R^+ = \{\alpha, \gamma, \beta\}$, $\gamma = \alpha + \beta$.
*First suppose that $x$ has the same length as $\alpha, \beta, \gamma$.* Remark that $(x, \alpha) \geq 0$: otherwise $x + \alpha \in \Delta$ and $\alpha < \alpha + x < x$ against the choice of $x$. Thus $\langle x, \alpha \rangle = \langle \alpha, x \rangle = 1$. If $\langle x, \gamma \rangle = 0$, then $\langle x, \beta \rangle = -1$ and we get a contradiction by Lemma 4. Similarly $\langle x, \gamma \rangle \neq -1$, therefore $\langle x, \gamma \rangle = \langle \gamma, x \rangle = 1$. It follows that $x - \alpha$ and $\gamma - x$ are roots. The compatibility of the order forces both $x - \alpha$ and $\gamma - x$ to be positive. Now $\{\alpha, x, x - \alpha\}$ and $\{x, \gamma, \gamma - x\}$ are parabolic $p$-subsystems of $\Delta$ of type $A_2$, thus, by the minimality of $d_<(R^+)$, $x - \alpha, \gamma - x \notin N$. But $\beta = (x - \alpha) + (\gamma - x) \in N$, against the compatibility of $N$.
*Next assume that $\alpha, \beta, \gamma$ are long and $x$ is short.* Then $(\langle \alpha, x \rangle, \langle x, \alpha \rangle) = (2, 1)$. As above $x \not\perp \gamma$, thus $(\langle \gamma, x \rangle, \langle x, \gamma \rangle) = (2, 1)$. Then $\alpha, x, 2x - \alpha, x - \alpha$ are roots and the compatibility of $<$ forces them to be positive; thus they form a parabolic p-subsystem of $\Delta$ of type $B_2$. Similarly, $x, \gamma, \gamma - x, \gamma - 2x$ form a parabolic p-subsystem of $\Delta$. By our choice of minimality we have $x - \alpha, \gamma - 2x \notin N$.
But $\beta = (\gamma - x) + (x - \alpha) = (2x - \alpha) + (\gamma - 2x)$, therefore by compatibility, $\gamma - x, 2x - \alpha \in N$. Now $\langle \gamma - 2x, \alpha \rangle = -1$, thus $\gamma - 2x + \alpha$ is a (positive) root. We have $\gamma = (\gamma - 2x + \alpha) + (2x - \alpha)$; it is easily seen that $\gamma, \gamma - 2x + \alpha$, and

$2x - \alpha$ are all long, thus they form a parabolic p-subsystem of $\Delta$ of type $A_2$. Since $\gamma, 2x - \alpha \in N$, by minimality we get $\gamma - 2x + \alpha \notin N$. But then the decomposition $\gamma - x = (\gamma - 2x + \alpha) + (x - \alpha)$ contradicts the compatibility of $N$.

*Finally assume that* $\alpha, \beta, \gamma$ *are short and* $x$ *is long.* Then we have $(\langle\alpha, x\rangle, \langle x, \alpha\rangle) = (\langle\gamma, x\rangle, \langle x, \gamma\rangle) = (1, 2)$. As above we get that $\{\alpha, x, x - \alpha, x - 2\alpha\}$ and $\{x, \gamma, 2\gamma - x, \gamma - x\}$ are parabolic p-subsystems of $\Delta$ of type $B_2$. By minimality, $x - 2\alpha, \gamma - x \notin N$ and by compatibility, $x - \alpha, 2\gamma - x \in N$. Now $\langle\gamma - x, \alpha\rangle = -1$, thus $\gamma - x + \alpha$ is a (positive) root. Now $\gamma - x + \alpha, x - \alpha$, and $\gamma$ are all short and $(\gamma - x + \alpha) + (x - \alpha) = \gamma$, thus $\gamma - x + \alpha, x - \alpha$, and $\gamma$ form a parabolic p-subsystem of $\Delta$ of type $A_2$; since $\gamma, x - \alpha \in N$, by minimality $\gamma - x + \alpha \notin N$. But then we get a contradiction: $\beta = (x - 2\alpha) + (\gamma - x + \alpha) \in N$ and $x - 2\alpha, \gamma - x + \alpha \notin N$.

II. a) $R \cong B_2$ and $\alpha$ is long. Then $R^+ = \{\alpha, \alpha + \beta, \alpha + 2\beta, \beta\}$. Set $\gamma = \alpha + \beta$ and $\gamma' = \alpha + 2\beta$.

*Assume that* $x$ *is long.* Then $\langle x, \alpha\rangle = \langle\alpha, x\rangle = 1$; as above $(x, \gamma) \neq 0$, otherwise $(x, \beta) < 0$. Since also $(x, \gamma) \not< 0$, we get thus $(\langle x, \gamma\rangle, \langle\gamma, x\rangle) = (2, 1)$. It follows that $\{\alpha, x, x - \alpha\}$ is a parabolic p-subsystem of $\Delta$ of type $A_2$ and $\{x, \gamma, 2\gamma - x, \gamma - x\}$ is a parabolic p-subsystem of type $B_2$. By minimality, $x - \alpha, \gamma - x \notin N$, whereas $(x - \alpha) + (\gamma - x) = \beta \in N$: a contradiction.

*Next assume that* $x$ *is short.* Then $(\langle x, \alpha\rangle, \langle\alpha, x\rangle) = (1, 2)$ and $\langle x, \gamma\rangle = \langle\gamma, x\rangle = 1$; $\{\alpha, x, 2x - \alpha, x - \alpha\}$ is a parabolic $p$-subsystem of type $B_2$ and $\{x, \gamma, \gamma - x\}$ is a parabolic p-subsystem of type $A_2$. As above we get a contradiction since by minimality $x - \alpha, \gamma - x \notin N$.

II. b) $R \cong B_2$ and $\alpha$ is short. Then $R^+ = \{\alpha, 2\alpha + \beta, \alpha + \beta, \beta\}$. Set $\gamma = 2\alpha + \beta$ and $\gamma' = \alpha + \beta$.

*Assume that* $x$ *is short.* Arguing as above we get $\langle x, \alpha\rangle = \langle\alpha, x\rangle = 1$ and $\langle x, \gamma'\rangle = \langle\gamma', x\rangle = 1$. Thus $\{\alpha, x, x - \alpha\}$ and $\{x, \gamma', \gamma' - x\}$ are parabolic p-subsystems of $\Delta$ of type $A_2$. By minimality $\gamma' - x, x - \alpha \notin N$ and as above we get a contradiction.

*Finally assume that* $x$ *is long.* Then $(\langle x, \alpha\rangle, \langle\alpha, x\rangle) = (\langle x, \gamma'\rangle, \langle\gamma', x\rangle) = (2, 1)$; it follows $(x, \beta) = 0$ and thus $\langle x, \gamma\rangle = \langle\gamma, x\rangle = 1$. Then $\{\alpha, x, x - \alpha, x - 2\alpha\}$ is a parabolic p-subsystem of type $B_2$ and $\{x, \gamma, \gamma - x\}$ is a parabolic p-subsystem of type $A_2$. Since $\beta = (x - 2\alpha) + (\gamma - x)$, we get a contradiction arguing as in the previous cases. This concludes the proof for all types of $\Delta$ other than $G_2, \tilde{G}_2$.

*Second case:* $\Delta \cong G_2, \tilde{G}_2$

The case $\Delta = G_2$ is trivial, since there are no irreducible proper parabolic p-subsystems of rank 2. So we assume $\Delta \cong \tilde{G}_2$.

*First assume that* $R \cong G_2$. We can argue as in the general case and get that each element in $\mathbb{N}^+(\alpha, \beta)$ is minimal in its $\delta$-string, with respect to $<$. Clearly $x$ must be parallel to some root in $\mathbb{N}^+(\alpha, \beta)$; but it is not parallel to $\alpha$, being consecutive to $\alpha$, and it cannot be parallel to any other root in $R^+$, since each element in such a set is minimal in its $\delta$-string. Therefore we must have $d_<(R) = 0$.

*Next assume* $R \cong A_2$. First we prove the following criterion.

*Suppose* $a = a_0 + h\delta, b = b_0 + k\delta$*, with* $a_0, b_0 \in \Delta^0, h, k \in \mathbb{Z}$*.* $\{a, b\}$ *is a basis for a parabolic subsystem of type* $A_2$ *if and only if* $a_0$ *and* $b_0$ *are long,* $(a_0, b_0) < 0$*, and* $3 \nmid (2h + k), (h + 2k)$*.*

Assume that $a, b \in \Delta$ are a basis for a parabolic subsystem $R \cong A_2$. Then $R = (\mathbb{Z}a + \mathbb{Z}b) \cap \Delta = (\mathbb{Q}a + \mathbb{Q}b) \cap \Delta$. Clearly we have $\langle a_0, b_0 \rangle = \langle b_0, a_0 \rangle = -1$; moreover, from the Dynkin diagram, we see that $a, b$, hence $a_0, b_0$ must be long. Then we have $\frac{1}{3}(2\alpha_0 + b_0) \in \Delta^0$, therefore, if $3 \mid (2h + k)$, also $\frac{1}{3}(2a + b) \in \Delta$: this would imply $(\mathbb{Z}a + \mathbb{Z}b) \cap \Delta \neq (\mathbb{Q}a + \mathbb{Q}b) \cap \Delta$ against the assumption. Therefore $3 \nmid (2h + k)$ and similarly $3 \nmid (h + 2k)$. Conversely, assume that $a, b$ are long roots such that $\langle a, b \rangle = \langle b, a \rangle = -1$. If $\mathbb{N}(a, b)$ is not parabolic, $a, b$ are included in a parabolic subsystem of type $G_2$. Then $\frac{1}{3}(2a + b), \frac{1}{3}(a + 2b) \in \Delta$; therefore $3 \mid (2h + k), (h + 2k)$.

Now we go on by a direct inspection.

I. $\alpha = \alpha_1$. By the above criterion, together with our choice of minimality, we get either $\beta = (\alpha_1 + 3\alpha_2) + \delta$, or $\beta = -\theta + \delta$; we distinguish the two cases.

a) $\beta = (\alpha_1 + 3\alpha_2) + \delta$. By compatibility, $\alpha_1 + 3\alpha_2 \in N$; moreover, since $\beta = (\alpha_1 + \delta) + 3\alpha_2$, at least one of $\alpha_2$, $\alpha_1 + \delta$ belongs to $N$. In the first case we get a contradiction since $\alpha_1, \alpha_2$ clearly generate a parabolic p-subsystem; in the latter case we get a contradiction since $\alpha_1 + 3\alpha_2, \alpha_1 + \delta$ generate a parabolic p-subsystem of type $A_2$.

b) $\beta = -\theta + \delta$. Then $\gamma = -\alpha_1 - 3\alpha_2 + \delta$. In this case $\beta, \gamma$ are minimal in their $\delta$-string, thus $x$ cannot be parallel to any of $\alpha, \beta, \gamma$ and therefore it cannot be long. Moreover $(\alpha_1, x) > 0$ and $x$ must be minimal in its $\delta$ string, therefore we have either $x = \alpha_1 + \alpha_2$ or $x = -\alpha_2 + \delta$. In the first case we get a contradiction since $\{x, \beta\}$ is a basis for a parabolic subsystem of type $G_2$; the second case is not possible, since $x = \beta + 2(\alpha_1 + \alpha_2)$ and neither $\beta$, nor $\alpha_1 + \alpha_2$ precede $x$ in $N$.

II. $\alpha = -\theta + \delta$. Then $\beta = \alpha_1$ or $\beta = \alpha_1 + 3\alpha_2$. In both subcases, $\beta, \gamma$ are minimal in their $\delta$-string and therefore $x$ is short; since $(x, \alpha) > 0$, we get $x = -\alpha_1 - \alpha_2 + \delta$ or $x = -\alpha_1 - 2\alpha_2 + \delta$.

a) $\beta = \alpha_1$. Then $\gamma = -\alpha_1 - 3\alpha_2 + \delta$. If $x = -\alpha_1 - \alpha_2 + \delta$ we get a contradiction since $\{x, \beta\}$ is a basis for a $G_2$ subsystem. The case $x = -\alpha_1 - 2\alpha_2 + \delta$ is not possible since $x = \gamma + \alpha_2$ and neither $\gamma$, nor $\alpha_2$ precede $x$ in $N$.

b) $\beta = \alpha_1 + 3\alpha_2$. Then $\gamma = -\alpha_1 + \delta$. We get $x \neq -\alpha_1 - 2\alpha_2 + \delta$, otherwise $x, \beta$ would be a basis for a $G_2$ subsystem. Finally the case $x = -\alpha_1 - \alpha_2 + \delta$ is not possible since $2x = (-\theta + 2\delta) + \alpha_2$ and neither $-\theta + 2\delta$, nor $\alpha_2$ precede $x$ in $N$. □

## References

1. N. Bourbaki, *Groupes et algebres de Lie*, Hermann, Paris, 1968
2. M. Dyer, "Hecke algebras and shellings of Bruhat intervals," *Comp. Math.* **89** (1993), 91–115
3. C.K. Fan, "A Hecke Algebra Quotient and Properties of Commutative Elements of a Weyl Group," MIT Thesis, 1995.
4. C.K. Fan, "Schubert varieties and short braidedness," *Transformation Groups* **3** (1998), 51–57.
5. C.K. Fan and J.R. Stembridge, "Nilpotent orbits and commutative elements," *J. Alg.* **196** (1997), 490–498.
6. V.G. Kac, *Infinite Dimensional Lie Algebras*, Birkhäuser, Boston, 1983.
7. P. Papi, "A characterization of a special ordering in a root system," *Proc. Amer. Math. Soc.* **120** (1994), 661–665.
8. J. Stembridge, "On the fully commutative elements of a Coxeter group," *J. Alg. Combin.* **5** (1996), 353–385.