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The Martin Boundary of the Young-Fibonacci Lattice

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Abstract. In this paper we find the Martin boundary for the Young-Fibonacci lattice $\mathbb{YF}$. Along with the lattice of Young diagrams, this is the most interesting example of a differential partially ordered set. The Martin boundary construction provides an explicit Poisson-type integral representation of non-negative harmonic functions on $\mathbb{YF}$. The latter are in a canonical correspondence with a set of traces on the locally semisimple Okada algebra. The set is known to contain all the indecomposable traces. Presumably, all of the traces in the set are indecomposable, though we have no proof of this conjecture. Using an explicit product formula for Okada characters, we derive precise regularity conditions under which a sequence of characters of finite-dimensional Okada algebras converges.

Keywords: differential poset, harmonic function, Martin boundary, Okada algebra, non-commutative symmetric function

1. Introduction

The Young-Fibonacci lattice $\mathbb{YF}$ is a fundamental example of a *differential partially ordered set* which was introduced by Stanley [11] and Fomin [3]. In many ways, it is similar to another major example of a differential poset, the Young lattice $\mathbb{Y}$. Addressing a question posed by Stanley, Okada has introduced [9] two algebras associated to $\mathbb{YF}$. The first algebra $\mathcal{F}$ is a locally semisimple algebra defined by generators and relations, which bears the same relation to the lattice $\mathbb{YF}$ as does the group algebra $\mathbb{C}\mathfrak{S}_\infty$ of the infinite symmetric group to Young's lattice. The second algebra R is an algebra of non-commutative polynomials, which bears the same relation to the lattice $\mathbb{YF}$ as does the ring of symmetric functions to Young's lattice.

The purpose of the present paper is to study some combinatorics, both finite and asymptotic, of the lattice $\mathbb{YF}$. Our object of study is the compact convex set of *harmonic functions* on $\mathbb{YF}$ (or equivalently the set of positive normalized traces on $\mathcal{F}$ or certain positive linear functionals on R.) We address the study of harmonic functions by determining the *Martin boundary* of the lattice $\mathbb{YF}$. The Martin boundary is the (compact) set consisting of those

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harmonic functions which can be obtained by finite rank approximation. There are two basic facts related to the Martin boundary construction: (1) every harmonic function is represented by the integral of a probability measure on the Martin boundary, and (2) the set of extreme harmonic functions is a subset of the Martin boundary (see, e.g., [1]).

This paper gives a parametrization of the Martin boundary for $\mathbb{YF}$ and a description of its topology.

The Young-Fibonacci lattice is described in Section 2, and preliminaries on harmonic functions are explained in Section 3. A first rough description of our main results is given at the end of Section 3. (A precise description of the parametrization of harmonic functions is found in Section 7, and the proof, finally, is contained in Section 8.) Section 4 contains some general results on harmonic functions on differential posets.

The main tool in our study is the Okada ring R and two bases of this ring, introduced by Okada, which are in some respect analogous to the Schur function basis and the power sum function basis in the ring of symmetric functions (Section 5). We describe the Okada analogs of the Schur function basis by non-commutative determinants of tridiagonal matrices with monomial entries. We obtain a simple and explicit formula for the transition matrix (character matrix) connecting the s-basis and the p-basis, and also for the value of (the linear extension of) harmonic functions evaluated on the p-basis. This is done in Sections 6 and 7.

The explicit formula allows us to study the regularity question for the lattice $\mathbb{YF}$, that is the question of convergence of extreme traces of finite dimensional Okada algebras $\mathcal{F}_n$ to traces of the inductive limit algebra $\mathcal{F} = \varinjlim \mathcal{F}_n$. The regularity question is studied in Section 8.

The analogous questions for Young's lattice $\mathbb{Y}$ (which is also a differential poset) were answered some time ago. The parametrization of the Martin boundary of $\mathbb{Y}$ has been studied in [14], [15]. A different approach was recently given in [10].

A remaining open problem for the Young-Fibonacci lattice is to characterize the set of extreme harmonic functions within the Martin boundary. For Young's lattice, the set of extreme harmonic functions coincides with the entire Martin boundary.

2. The Young-Fibonacci lattice

In this Section we recall the definition of Young-Fibonacci modular lattice (see figure 1) and some basic facts related to its combinatorics. See Section A.1 in the Appendix for the background definitions and notations related to graded graphs and differential posets. We refer to [3–4], [11–13] for a more detailed exposition.

A simple recurrent construction

The simplest way to define the graded graph $\mathbb{YF} = \bigcup_{n=0}^{\infty} \mathbb{YF}_n$ is provided by the following recurrent procedure.

Let the first two levels $\mathbb{YF}_0$ and $\mathbb{YF}_1$ have just one vertex each, joined by an edge. Assuming that the part of the graph $\mathbb{YF}$, up to the nth level $\mathbb{YF}_n$, is already constructed, we define the set of vertices of the next level $\mathbb{YF}_{n+1}$, along with the set of adjacent edges, by

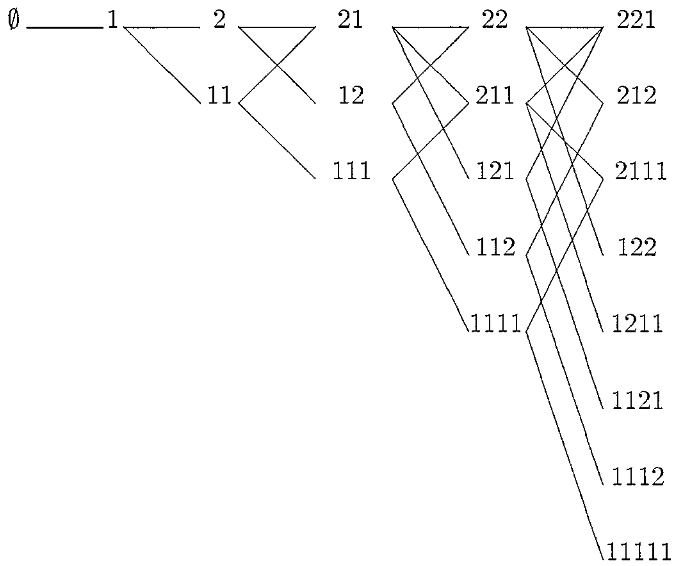

Figure 1. The Young-Fibonacci lattice.

first *reflecting* the edges in between the two previous levels, and then by *attaching* just one new edge leading from each of the vertices on the level $\mathbb{YF}_n$ to a corresponding new vertex at level $n+1$.

In particular, we get two vertices in the set $\mathbb{YF}_2$, and two new edges: one is obtained by reflecting the only existing edge, and the other by attaching a new one. More generally, there is a natural notation for new vertices which helps to keep track of the inductive procedure. Let us denote the vertices of $\mathbb{YF}_0$ and $\mathbb{YF}_1$ by an empty word $\varnothing$ and 1 correspondingly. Then the endpoint of the reflected edge will be denoted by 2, and the end vertex of the new edge by 11. In a similar way, all the vertices can be labeled by words in the letters 1 and 2. If the left (closer to the root $\varnothing$) end of an edge is labeled by a word v, then the endvertex of the reflected edge is labeled by the word $2v$. Each vertex w of the nth level is joined to a vertex $1w$ at the next level by a new edge (which is not a reflection of any previous edge).

Clearly, the number of vertices at the nth level $\mathbb{YF}_n$ is the nth Fibonacci number f_n.

Basic definitions

We now give somewhat more formal description of the Young-Fibonacci lattice and its Hasse diagram.

Definition A finite word in the two-letter alphabet $\{1, 2\}$ will be referred to as a *Fibonacci word*. We denote the sum of digits of a Fibonacci word w by $|w|$, and we call it the *rank* of w. The set of words of a given rank n will be denoted by $\mathbb{YF}_n$, and the set of all Fibonacci words by $\mathbb{YF}$. The *head* of a Fibonacci word is defined as the longest contiguous subword of 2's at its left end. The *position* of a 2 in a Fibonacci word is one more than the rank of the subword to the right of the 2; that is if $w = u2v$, then the position of the indicated 2 is $|v|+1$.

Next we define a partial order on the set $\mathbb{YF}$ which is known to make $\mathbb{YF}$ a modular lattice. The order will be described by giving the covering relations on $\mathbb{YF}$ in two equivalent forms.

Given a Fibonacci word v, we first define the set $\bar{v} \subset \mathbb{YF}$ of its successors. By definition, this is exactly the set of words $w \in \mathbb{YF}$ which can be obtained from v by one of the following three operations:

(i) put an extra 1 at the left end of the word v;
(ii) replace the first 1 in the word v (reading left to right) by 2;
(iii) insert 1 anywhere in between 2's in the head of the word v, or immediately after the last 2 in the head.

Example Take 222121112 for the word v of rank 14. Then the group of 3 leftmost 2's forms its head, and v has 5 successors, namely

$$\bar{v} = \{\mathbf{1}222121112,\ 2\mathbf{1}22121112,\ 22\mathbf{1}2121112,\ 222\mathbf{1}121112,\ 222\mathbf{2}21112\}.$$

The changing letter is shown in boldface. Note that the ranks of all successors of a Fibonacci word v are one bigger than that of v.

The set $\underline{v}$ of predecessors of a non-empty Fibonacci word v can be described in a similar way. The operations to be applied to v in order to obtain one of its predecessors are as follows:

(i) the leftmost letter 1 in the word v can be removed;
(ii) any one of 2's in the head of v can be replaced by 1.

Example The word $v = 222121112$ has 4 predecessors, namely

$$\underline{v} = \{122121112,\ 212121112,\ 221121112,\ 22221112\}.$$

We write $u \nearrow v$ to show that v is a successor of u (and u is a predecessor of v). This is a covering relation which determines a partial order on the set $\mathbb{YF}$ of Fibonacci words. As a matter of fact, it is a modular lattice, see [11]. The initial part of the Hasse diagram of the poset $\mathbb{YF}$ is represented in figure 1.

The Young-Fibonacci lattice as a differential poset

Assuming that the head length of v is k, the word v has $k+2$ successors and $k+1$ predecessors, if v contains at least one letter 1. If $v = 22\cdots 2$ is made of 2's only, it has $k+1$ successors and k predecessors. Note that the number of successors is always one bigger than that of predecessors. Another important property of the lattice $\mathbb{YF}$ is that, for any two different Fibonacci words v_1, v_2 of the same rank, the number of their common successors equals that of common predecessors (both numbers can only be 0 or 1). These are exactly the two characteristic properties (D1), (D2) of differential posets, see Section A.1. In what follows we shall frequently use the basic facts on differential posets, surveyed for the reader's convenience in the Appendix. Much more information on differential posets and their generalizations can be found in [3], [11].

The Okada algebra

Okada [9] introduced a (complex locally semisimple) algebra $\mathcal{F}$, defined by generators and relations, which admits the Young-Fibonacci lattice $\mathbb{YF}$ as its branching diagram. The Okada algebra has generators $(e_i)_{i\geq 1}$ satisfying the relations:

$$e_i^2 = e_i \quad \text{for all } i \geq 1; \tag{O1}$$

$$e_i e_{i-1} e_i = \frac{1}{i}\, e_i \quad \text{for all } i \geq 2; \tag{O2}$$

$$e_i e_j = e_j e_i \quad \text{for } |i - j| \geq 2. \tag{O3}$$

The algebra $\mathcal{F}_n$ generated by the first $n-1$ generators $e_1, \dots, e_{n-1}$ and these identities is semisimple of dimension $n!$, and has simple modules M_v labelled by elements $v \in \mathbb{YF}_n$. For $u \in \mathbb{YF}_{n-1}$ and $v \in \mathbb{YF}_n$, one has $u \nearrow v$ if, and only if, the simple $\mathcal{F}_n$-module M_v, restricted to the algebra $\mathcal{F}_{n-1}$ contains the simple $\mathcal{F}_{n-1}$-module M_u. As a matter of fact, the restrictions of simple $\mathcal{F}_n$-modules to $\mathcal{F}_{n-1}$ are multiplicity free.

3. Harmonic functions on graphs and traces of *AF*-algebras

In this Section, we recall the notion of harmonic functions on a *graded graph* and the classical Martin boundary construction for graded graphs and *branching diagrams*. We discuss the connection between harmonic functions on branching diagrams and traces on the corresponding AF-algebra. Finally, we give a preliminary statement on our main results on the Martin boundary of the Young-Fibonacci lattice.

We refer the reader to Appendix A.1 for basic definitions on graded graphs and branching diagrams and to [2], [7] for more details on the combinatorial theory of *AF*-algebras.

The Martin boundary of a graded graph

A function $\varphi : \Gamma \to \mathbb{R}$ defined on the set of vertices of a graded graph Γ is called *harmonic*, if the following variant of the "mean value theorem" holds for all vertices $u \in \Gamma$:

$$\varphi(u) = \sum_{w : u \nearrow w} \varphi(w). \tag{3.1}$$

We are interested in the problem of determining the space $\mathcal{H}$ of all non-negative harmonic functions normalized at the vertex $\varnothing$ by the condition $\varphi(\varnothing) = 1$. Since $\mathcal{H}$ is a compact convex set with the topology of pointwise convergence, it is interesting to ask about its set of extreme points. (Recall that an *extreme point* φ in a convex set K is a point which cannot be written as a non-trivial convex combination of points of K; that is, whenever $\varphi = s\varphi_1 + (1-s)\varphi_2$, with $0 < s < 1$ and $\varphi_1, \varphi_2 \in K$, it follows that $\varphi_1 = \varphi_2 = \varphi$.)

A general approach to the problem of determining the set of extreme points is based on the Martin boundary construction (see, for instance, [1]). One starts with the *dimension function* $d(v, w)$ defined as the number of all oriented paths from v to w. We put $d(w) = d(\emptyset, w)$.

From the point of view of potential theory, $d(v, w)$ is the Green function with respect to "Laplace operator"

$$(\Delta\varphi)(u) = -\varphi(u) + \sum_{w:u\nearrow w} \varphi(w). \tag{3.2}$$

This means that if $\psi_w(v) = d(v, w)$ for a fixed vertex w, then $-(\Delta\psi_w)(v) = \delta_{vw}$ for all $v \in \Gamma$. The ratio

$$K(v, w) = \frac{d(v, w)}{d(w)} \tag{3.3}$$

is usually called the *Martin kernel*.

Consider the space $\mathrm{Fun}(\Gamma)$ of all functions $f : \Gamma \to \mathbb{R}$ with the topology of pointwise convergence, and let $\tilde{E}$ be the closure of the subset $\tilde{\Gamma} \subset \mathrm{Fun}(\Gamma)$ of functions $v \mapsto K(v, w)$, $w \in \Gamma$. Since those functions are uniformly bounded, $0 \le K(v, w) \le 1$, the space $\tilde{E}$ (called the *Martin compactification*) is indeed compact. One can easily check that $\tilde{\Gamma} \subset \tilde{E}$ is a dense open subset of $\tilde{E}$. Its boundary $E = \tilde{E} \setminus \tilde{\Gamma}$ is called the *Martin boundary* of the graph Γ.

By definition, the Martin kernel (3.3) may be extended by continuity to a function $K : \Gamma \times \tilde{E} \to \mathbb{R}$. For each boundary point $\omega \in E$ the function $\varphi_\omega(v) = K(v, \omega)$ is non-negative, harmonic, and normalized. Moreover, harmonic functions have an integral representation similar to the classical Poisson integral representation for non-negative harmonic functions in the disk:

Theorem (cf. [1]). *Every normalized non-negative harmonic function $\varphi \in \mathcal{H}$ admits an integral representation*

$$\varphi(u) = \int_E K(u, \omega)\; M(d\omega), \tag{3.4}$$

where M is a probability measure. Conversely, for every probability measure M on E, the integral (3.4) *provides a non-negative harmonic function $\varphi \in \mathcal{H}$.*

All *indecomposable* (i.e., *extreme*) elements of $\mathcal{H}$ can be represented in the form $\varphi_\omega(v) = K(v, \omega)$, for appropriate boundary point $\omega \in E$, and we denote by $E_{\min}$ the set of all such points. It is known that $E_{\min}$ is a non-empty G_δ subset of E. One can always choose the measure M in the integral representation (3.4) to be supported by $E_{\min}$. Under this assumption, the measure M representing a function $\varphi \in \mathcal{H}$ via (3.4) is unique.

Given a concrete example of a graded graph, one looks for an appropriate "geometric" description of the abstract Martin boundary. The purpose of the present paper is to give an explicit description for the Martin boundary of the Young–Fibonacci graph $\mathbb{YF}$.

The traces on locally semisimple algebras

We next discuss the relation between harmonic functions on a graded graph and traces on locally semisimple algebras. A *locally semisimple complex algebra A* (or *AF*-algebra) is the union of an increasing sequence of finite dimensional semisimple complex algebras, $A = \varinjlim A_n$. The *branching diagram* or *Bratteli diagram* $\Gamma(A)$ of a locally semisimple algebra A (more precisely, of the approximating sequence $\{A_n\}$) is a graded graph whose vertices of rank n correspond to the simple A_n-modules. Let M_v denote the simple A_n-module corresponding to a vertex $v \in \Gamma_n$. Then a vertex v of rank n and a vertex w of rank $n+1$ are joined by $\varkappa(v, w)$ edges if the simple A_{n+1} module M_w, regarded as an A_n module, contains M_v with multiplicity $\varkappa(v, w)$. We will assume here that all multiplicities $\varkappa(v, w)$ are 0 or 1, as this is the case in the example of the Young-Fibonacci lattice with which we are chiefly concerned. Conversely, given a branching diagram Γ—that is, a graded graph with unique minimal vertex at rank 0 and no maximal vertices—there is a locally semisimple algebra A such that $\Gamma(A) = \Gamma$.

A *trace* on a locally semisimple algebra A is a complex linear functional ψ satisfying

$$\begin{aligned} \psi(e) &\geq 0 \quad \text{for all idempotents } e \in A; \\ \psi(1) &= 1; \\ \psi(ab) &= \psi(ba) \quad \text{for all } a, b \in A. \end{aligned} \tag{3.5}$$

To each trace ψ on A, there corresponds a positive normalized harmonic function $\tilde{\psi}$ on $\Gamma = \Gamma(A)$ given by

$$\tilde{\psi}(v) = \psi(e) \tag{3.6}$$

whenever v has rank n and e is a minimal idempotent in A_n such that $eM_v \neq (0)$ and $eM_w = 0$ for all $w \in \Gamma_n \setminus \{v\}$. The trace property of ψ implies that $\tilde{\psi}$ is a well defined non-negative function on Γ, and harmonicity of $\tilde{\psi}$ follows from the definition of the branching diagram $\Gamma(A)$. Conversely, a positive normalized harmonic function $\tilde{\psi}$ on $\Gamma = \Gamma(A)$ defines a trace on A; in fact, a trace on each A_n is determined by its value on minimal idempotents, so the assignment

$$\psi^{(n)}(e) = \tilde{\psi}(v), \tag{3.7}$$

whenever e is a minimal idempotent in A_n such that $eM_v \neq (0)$, defines a trace on A_n. The harmonicity of $\tilde{\psi}$ implies that the $\psi^{(n)}$ are *coherent*, i.e., the restriction of $\psi^{(n+1)}$ from A_{n+1} to the subalgebra A_n coincides with $\psi^{(n)}$. As a result, the traces $\psi^{(n)}$ determine a trace of the limiting algebra $A = \varinjlim A_n$.

The set of traces on A is a compact convex set, with the topology of pointwise convergence; an *extreme* or *indecomposable* trace is an extreme point in this compact convex set.

The map $\tilde{\psi} \mapsto \psi$ is an affine homeomorphism between the space of positive normalized harmonic functions on $\Gamma = \Gamma(A)$ and the space of traces on A. From the point of view of traces, the Martin boundary of Γ consists of traces ψ which can be obtained as limits of a sequence ψ_n, where ψ_n is an extreme trace on A_n. All extreme traces on A are in the Martin boundary, so determination of the Martin boundary is a step towards determining the set of extreme traces on A.

The locally semisimple algebra corresponding to the Young-Fibonacci lattice $\mathbb{YF}$ is the Okada algebra $\mathcal{F}$ introduced in Section 2.

The main result

We can now give a description of the Martin boundary of the Young-Fibonacci lattice (and consequently of a Poisson-type integral representation for non-negative harmonic functions on $\mathbb{YF}$).

Definition Let w be an infinite word in the alphabet $\{1, 2\}$ (infinite Fibonacci word), and let $d_1, d_2, \ldots$ denote the positions of 2's in w. The word w is said to be *summable* if, and only if, the series $\sum_{j=1}^{\infty} 1/d_j$ converges, or, equivalently, the product

$$\pi(w) = \prod_{j:d_j \geq 2} \left(1 - \frac{1}{d_j}\right) > 0 \tag{3.8}$$

converges.

As for any differential poset, the lattice $\mathbb{YF}$ has a distinguished harmonic function φ_P, called the *Plancherel harmonic function;* φ_P is an element of the Martin boundary. The complement of $\{\varphi_P\}$ in the Martin boundary of $\mathbb{YF}$ can be parametrized with two parameters (β, w); here β is a real number, $0 < \beta \leq 1$, and w is a summable infinite word in the alphabet $\{1, 2\}$.

We denote by Ω the parameter space for the Martin boundary:

Definition Let the space Ω be the union of a point P and the set

$$\{(\beta, w) : 0 < \beta \leq 1,\ w \text{ a summable infinite word in the alphabet } \{1, 2\}\},$$

with the following topology: A sequence $(\beta^{(n)}, w^{(n)})$ converges to P iff

$$\beta^{(n)} \to 0 \quad \text{or} \quad \pi\big(w^{(n)}\big) \to 0.$$

A sequence $(\beta^{(n)}, w^{(n)})$ converges to (β, w) if, and only if,

$$w^{(n)} \to w \text{ (digitwise)} \quad \text{and} \quad \beta^{(n)}\pi\big(w^{(n)}\big) \to \beta\pi(w).$$

We will describe in Section 7 the mapping $\omega \mapsto \varphi_\omega$ from Ω to the set of normalized positive harmonic functions on $\mathbb{YF}$.

We are in a position now to state the main result of the paper.

Theorem 3.2 *The map $\omega \mapsto \varphi_\omega$ is a homeomorphism of the space Ω onto the Martin boundary of the Young-Fibonacci lattice. Consequently, for each probability measure M on Ω, the integral*

$$\varphi(v) = \int_\Omega \varphi_\omega(v) M(d\omega), \quad v \in \mathbb{YF} \tag{3.9}$$

provides a normalized, non-negative harmonic function on the Young-Fibonacci lattice $\mathbb{YF}$. Conversely, every such function admits an integral representation with respect to a measure M on Ω (which may not be unique).

In general, for all differential posets, we show that there is a flow

$$(t, \varphi) \mapsto C_t(\varphi)$$

on $[0, 1] \times \mathcal{H}$ with the properties

$$C_t(C_s(\varphi)) = C_{ts}(\varphi) \quad \text{and} \quad C_0(\varphi) = \varphi_P. \tag{3.10}$$

For the Young-Fibonacci lattice, one has $C_t(\varphi_{\beta,w}) = C_{t\beta,w}$ and $C_t(\varphi_P) = \varphi_P$. In particular, the flow on $\mathcal{H}$ preserves the Martin boundary. *It is not clear whether this is a general phenomenon for differential posets.*

We have not yet been able to characterize the extreme points within the Martin boundary of $\mathbb{YF}$. In a number of similar examples, for instance the Young lattice, all elements of the Martin boundary are extreme points.

4. Harmonic functions on differential posets

The Young-Fibonacci lattice is an example of a differential poset. In this section, we introduce some general constructions for harmonic functions on a differential poset. Later on in Section 7 we use the construction to obtain the Martin kernel of the graph $\mathbb{YF}$.

Type I harmonic functions

In this subsection we don't need any special assumptions on the branching diagram Γ. Consider an infinite path

$$t = (v_0, v_1, \ldots, v_n, \ldots)$$

in Γ. For each vertex $u \in \Gamma$ the sequence $\{d(u, v_n)\}_{n=1}^{\infty}$ is weakly increasing, and we shall use the notation

$$d(u, t) = \lim_{n\to\infty} d(u, v_n). \tag{4.1}$$

Note that $d(u, t) = d(u, s)$ if the sequences t, s coincide eventually.

Lemma 4.1 *The following conditions are equivalent for a path t in Γ:*

(i) *All but finitely many vertices v_n in the path t have a single immediate predecessor v_{n-1}.*

(ii) $d(\varnothing, t) < \infty$.

(iii) $d(u, t) < \infty$ *for all $u \in \Gamma$.*

(iv) *There are only finitely many paths which eventually coincide with t.*

Proof: It is clear that $d(\varnothing, v_{n-1}) = d(\varnothing, v_n)$ iff v_{n-1} is the only predecessor of v_n. Since $d(u, t) \leq d(\varnothing, t)$ for all $u \in \Gamma$, we have $(i) \to (ii) \to (iii) \to (i)$. The number of paths $s \in T$, equivalent to t is exactly $d(\varnothing, t)$. □

In case $\Gamma = \mathbb{Y}$ is the Young lattice, there are only two paths (i.e. Young tableaux) satisfying these conditions: $t = ((1), \ldots, (n), \ldots)$ and $t = ((1), \ldots, (1^n), \ldots)$. In case of Young-Fibonacci lattice there are countably many paths satisfying the conditions of Lemma 4.1. The vertices of such a path eventually take the form $v_n = 1^{n-m}v$, $n \geq m$, for some Fibonacci word v of rank m. Hence, the equivalence class of eventually coinciding paths in $\mathbb{YF}$ with the properties of Lemma 4.1 can be labelled by infinite words in the alphabet $\{2, 1\}$ with only finite number of 2's. We denote the set of such words as $1^\infty\mathbb{YF}$.

Proposition 4.2 *Assume that a path t in Γ satisfies the conditions of Lemma* 4.1. *Then*

$$\varphi_t(v) = \frac{d(v, t)}{d(\varnothing, t)}, \quad v \in \Gamma \tag{4.2}$$

is a positive normalized harmonic function on Γ.

Proof: Since $d(v, t) = \sum_{w: v\nearrow w} d(w, t)$, the function φ_t is harmonic. Also, $\varphi_t(v) \geq 0$ for all $v \in \Gamma$, and $\varphi_t(\varnothing) = 1$. □

We say that these harmonic functions are *of type* I, since the corresponding *AF*-algebra traces are traces of finite-dimensional irreducible representations (type I factor-representations). It is clear that all the harmonic functions of type I are indecomposable.

Plancherel harmonic function

Let us assume now that the poset Γ is *differential* in the sense of [11] or, equivalently, is a Y-graph in the terminology of [3] or a *self-dual graph* in that of [4]. The properties of differential posets which we need are surveyed in the Appendix.

Proposition 4.3 *The function*

$$\varphi_P(v) = \frac{d(\varnothing, v)}{n!}; \quad v \in \Gamma, \quad n = |v| \tag{4.3}$$

is a positive normalized harmonic function on the differential poset Γ.

Proof: This follows directly from (A.2.1) in the Appendix. □

Note that if $\Gamma = \mathbb{Y}$ is the Young lattice, the function φ_P corresponds to the Plancherel measure of the infinite symmetric group (cf. [7]).

Contraction of harmonic functions on a differential poset

Assume that Γ is a differential poset. We shall show that for any harmonic function φ there is a family of affine transformations, with one real parameter τ, connecting the Plancherel function φ_P to φ.

Proposition 4.4 *For* $0 \le \tau \le 1$ *and a harmonic function* φ, *define a function* $C_\tau(\varphi)$ *on the set of vertices of the differential poset* Γ *by the formula*

$$C_\tau(\varphi)(v) = \sum_{k=0}^{n} \frac{\tau^k (1-\tau)^{n-k}}{(n-k)!} \sum_{|u|=k} \varphi(u)\, d(u, v), \quad n = |v|. \tag{4.4}$$

Then $C_\tau(\varphi)$ *is a positive normalized harmonic function, and the map* $\varphi \mapsto C_\tau(\varphi)$ *is affine.*

Proof: We introduce the notation

$$S_k(v, \varphi) = \sum_{|u|=k} \varphi(u)\ d(u, v). \tag{4.5}$$

First we observe the identity

$$\sum_{w: v \nearrow w} S_k(w, \varphi) = S_{k-1}(v, \varphi) + (n-k+1)\, S_k(v, \varphi),$$

which is obtained from a straightforward computation using (A.2.3) from the Appendix, and the harmonic property (3.1) of the function φ. From this we derive that

$$\begin{aligned} \sum_{w: v \nearrow w} C_\tau(\varphi)(w) &= \sum_{w: v \nearrow w} \sum_{k=0}^{n+1} \frac{\tau^k (1-\tau)^{n-k+1}}{(n-k+1)!} S_k(w, \varphi) \\ &= \sum_{k=1}^{n+1} \frac{\tau^k (1-\tau)^{n-k+1}}{(n-k+1)!} S_{k-1}(v, \varphi) + \sum_{k=0}^{n} \frac{\tau^k (1-\tau)^{n-k+1}}{(n-k)!} S_k(v, \varphi) \\ &= \tau\ C_\tau(\varphi)(v) + (1-\tau)\ C_\tau(\varphi)(v) \\ &= C_\tau(\varphi)(v). \end{aligned} \tag{4.6}$$

This shows that $C_\tau(\varphi)$ is harmonic. It is easy to see that $C_\tau(\varphi)$ is normalized and positive, and that the map $\varphi \mapsto C_t(\varphi)$ is affine. □

Remarks (a) The semigroup property holds: $C_t(C_s(\varphi)) = C_{st}(\varphi)$; (b) $C_0(\varphi) = \varphi_P$, for all φ, and $C_t(\varphi_P) = \varphi_P$ for all t, $0 \le t \le 1$; (c) $C_1(\varphi) = \varphi$. These statements can be verified by straightforward computations.

Example Let φ denote the indecomposable harmonic function on the Young lattice with the Thoma parameters $(\alpha; \beta; \gamma)$, see [7] for definitions. Then the function $C_\tau(\varphi)$ is also indecomposable, with the Thoma parameters $(\tau\alpha; \tau\beta; 1 - \tau(1-\gamma))$.

Central measures and contractions

Recall (see [7]) that for any harmonic function φ on Γ there is a *central measure* M^φ on the space T of paths of Γ, determined by its level distributions

$$M_n^\varphi(v) = d(\varnothing, v)\,\varphi(v), \quad v \in \Gamma_n. \tag{4.7}$$

In particular, $\sum_{v\in\Gamma_n} M_n^\varphi(v) = 1$ for all n.

There is a simple probabilistic description of the central measure corresponding to a harmonic function on a differential poset obtained by the contraction of Proposition 4.4. Define a random vertex $v \in \Gamma_n$ by the following procedure:

(a) Choose a random k, $0 \le k \le n$ with the binomial distribution

$$b(k) = \binom{n}{k}\tau^k(1-\tau)^{n-k} \tag{4.8}$$

(b) Choose a random vertex $u \in \Gamma_k$ with probability

$$M_k^\varphi(u) = d(\varnothing, u)\varphi(u) \tag{4.9}$$

(c) Start a random walk at the vertex u, with the Plancherel transition probabilities

$$p_{x,y} = \frac{d(\varnothing, y)}{(r+1)\,d(\varnothing, x)}; \quad |x| = r, \ \ x \nearrow y. \tag{4.10}$$

Let v denote the vertex at which the random walk first hits the n'th level set Γ_n. We denote by $M_n^{(\tau,\varphi)}$ the distribution of the random vertex v.

Proposition 4.5 *The distribution $M_n^{(\tau,\varphi)}$ is the n'th level distribution of the central measure corresponding to the harmonic function $C_\tau(\varphi)$:*

$$M_n^{(\tau,\varphi)}(v) = d(\varnothing, v)\,C_\tau(\varphi)(v). \tag{4.11}$$

Proof: It follows from (A.2.1) that (4.10) is a probability distribution. By Lemma A.3.2, the probability to hit Γ_n at the vertex v, starting the Plancherel walk at $u \in \Gamma_k$, is

$$p(u,v) = \frac{k!}{n!} \frac{d(u,v)\, d(\varnothing, v)}{d(\varnothing, u)}. \tag{4.12}$$

The Proposition now follows from the definition of the contraction $C_\tau(\varphi)$ written in the form

$$C_\tau(\varphi)(v)\, d(\varnothing, v) = \sum_{k=0}^{n} b(k) \sum_{u \in \Gamma_k} M_k^\varphi(u)\; p(u,v). \tag{4.13}$$

□

Example Let $\Gamma = \mathbb{Y}$ be the Young lattice and let $t = ((1), (2), \ldots, (n), \ldots)$ be the one-row Young tableau. Then the distribution (4.9) is trivial, and the procedure reduces to choosing a random row diagram (k) with the distribution (4.8) and applying the Plancherel growth process until the diagram gains n boxes.

5. Okada clone of the symmetric function ring

In this Section we introduce the Okada variant of the symmetric function algebra, and its two bases analogous to the Schur function basis and the power sum basis. The Young-Fibonacci lattice arises in a Pieri-type formula for the first basis.

The rings R and R_∞

Let $R = \mathbb{R} < X, Y >$ denote the ring of all polynomials in two non-commuting variables X, Y. We endow R with a structure of graded ring, $R = \bigoplus_{n=0}^{\infty} R_n$, by declaring the degrees of variables to be $\deg X = 1$, $\deg Y = 2$. For each word

$$v = 1^{k_t}\, 21^{k_{t-1}} \cdots 1^{k_1} 21^{k_0} \in \mathbb{YF}_n, \tag{5.1}$$

let h_v denote the monomial

$$h_v = X^{k_0} Y X^{k_1} \cdots X^{k_{t-1}} Y X^{k_t} \tag{5.2}$$

Then R_n is a $\mathbb{R}$-vector space with the f_n (Fibonacci number) monomials h_v as a basis.

We let $R_\infty = \varinjlim R_n$ denote the inductive limit of linear spaces R_n, with respect to imbeddings $Q \mapsto QX$. Equivalently, $R_\infty = R/(X-1)$ is the quotient of R by the principal left ideal generated by $X - 1$. Linear functionals on R_∞ are identified with linear functionals φ on R which satisfy $\varphi(f) = \varphi(fX)$. The ring R_∞ has a similar rôle for the Young-Fibonacci lattice and the Okada algebra $\mathcal{F}$ as the ring of symmetric functions has for the Young lattice and the group algebra of the infinite symmetric group $\mathfrak{S}_\infty$ (see [8]).

Non-commutative Jacobi determinants

The following definition is based on a remark which appeared in the preprint version of [9]. We consider two non-commutative n-th order determinants

$$P_n = \begin{vmatrix} X & Y & 0 & 0 & \cdots & 0 & 0 \\ 1 & X & Y & 0 & \cdots & 0 & 0 \\ 0 & 1 & X & Y & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & X \end{vmatrix} \tag{5.3}$$

and

$$Q_{n-1} = \begin{vmatrix} Y & Y & 0 & 0 & \cdots & 0 & 0 \\ X & X & Y & 0 & \cdots & 0 & 0 \\ 0 & 1 & X & Y & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & X \end{vmatrix}. \tag{5.4}$$

By definition, the non-commutative determinant is the expression

$$\det(a_{ij}) = \sum_{w \in \mathfrak{S}_n} \operatorname{sign}(w)\, a_{w(1)1}\, a_{w(2)2} \cdots a_{w(n)n}. \tag{5.5}$$

In other words, the k-th factor of every summand is taken from the k-th column. Note that polynomials (5.3), (5.4) are homogeneous elements of R, $\deg P_n = n$ and $\deg Q_{n-1} = n+1$.

Following Okada, we define elements of R (which we call *Okada-Schur polynomials* or s-functions) by the products

$$s_v = P_{k_0}\, Q_{k_1} \cdots Q_{k_t}, \quad v = \underbrace{1^{k_t}2} \cdots \underbrace{1^{k_1}2}\, 1^{k_0} \in \mathbb{YF}_n \tag{5.6}$$

(cf. [9], Proposition 3.5). The polynomials s_v for $|v| = n$ are homogeneous of degree n, and form a basis of the linear space R_n. We define a scalar product $\langle\, .\, ,\, .\, \rangle$ on the space R by declaring the s-basis to be orthonormal.

The branching of Okada-Schur functions

We will use the formulae

$$P_{n+1} = P_n X - P_{n-1} Y, \quad n \geq 1, \tag{5.7}$$

$$Q_{n+1} = Q_n X - Q_{n-1} Y, \quad n \geq 1, \tag{5.8}$$

obtained by decomposing the determinants (5.3), (5.4) along the last column. The first identity is also true for $n = 0$, assuming $P_{-1} = 0$. The $n = 0$ case of the second identity (5.8) can be written in the form

$$Q_0 X = X Q_0 + Q_1. \tag{5.9}$$

One can think of (5.9) as of a commutation rule for passing X over a factor of type Q_0. It is clear from (5.9) that

$$Q_0^m X = X Q_0^m + \sum_{i=1}^{m} Q_0^{m-i} Q_1 Q_0^{i-1}. \tag{5.10}$$

It will be convenient to rewrite (5.7), (5.8) in a form similar to (5.9):

$$P_n X = P_{n+1} + P_{n-1} Q_0. \tag{5.11}$$

$$Q_n X = Q_{n+1} + Q_{n-1} Q_0, \tag{5.12}$$

The following formulae are direct consequences of (5.10)–(5.12):

$$P_n Q_0^m X = \sum_{i=0}^{m} P_n Q_0^{m-i} Q_1 Q_0^i + P_{n+1} Q_0^m + P_{n-1} Q_0^{m+1}, \tag{5.13}$$

$$Q_n Q_0^m X = \sum_{i=0}^{m} Q_n Q_0^{m-i} Q_1 Q_0^i + Q_{n+1} Q_0^m + Q_{n-1} Q_0^{m+1}. \tag{5.14}$$

It is understood in (5.13), (5.14) that $n \geq 1$.

The formulae (5.10), (5.13) and (5.14) imply

Theorem 5.1 *(Okada)* *For every $w \in \mathbb{YF}_n$ the product of the Okada-Schur determinant s_w by X from the right hand side can be written as*

$$s_w X = \sum_{v: w \nearrow v} s_v. \tag{5.15}$$

This theorem says that the branching of Okada s-functions reproduces the branching law for the Young-Fibonacci lattice. In the following statement, U is the "creation operator" on $\mathrm{Fun}(\mathbb{YF})$, which is defined in the Appendix, (A.1.1).

Corollary 5.2 *The assignment $\Theta : v \mapsto s_v$ extends to a linear isomorphism*

$$\Theta : \oplus_n \mathrm{Fun}(\mathbb{YF}_n) \to R$$

taking $\mathrm{Fun}(\mathbb{YF}_n)$ *to R_n and satisfying $\Theta \circ U(f) = \Theta(f) X$.*

Because of this, we will sometimes write $U(f)$ instead of fX for $f \in R$, and $D(f)$ for $\Theta \circ D \circ \Theta^{-1}(f)$, see (A.1.2) for the definition of D.

Corollary 5.3 *There exist one-to-one correspondences between:*
(a) *Non-negative, normalized harmonic functions on* $\mathbb{YF}$*;*
(b) *Linear functionals* φ *on* $\mathrm{Fun}(\mathbb{YF})$ *satisfying*

$$\varphi \circ U = \varphi, \qquad \varphi(1) = 1, \qquad \textit{and} \qquad \varphi(\delta_v) \geq 0 \quad \textit{for } v \in \mathbb{YF};$$

(c) *Linear functionals* φ *on* R *satisfying*

$$\varphi(f) = \varphi(fX) \quad \textit{for all } f \in R, \qquad \varphi(1) = 1, \qquad \textit{and} \qquad \varphi(s_v) \geq 0, \quad \textit{for } v \in \mathbb{YF};$$

(d) *Linear functionals on* $R_\infty = \varinjlim R_n$ *satisfying*

$$\varphi(1) = 1 \qquad \textit{and} \qquad \varphi(\bar{s}_v) \geq 0, \quad \textit{for } v \in \mathbb{YF},$$

where $\bar{s}_v$ *denotes the image of* s_v *in* R_∞*;*
(e) *Traces of the Okada algebra* $\mathcal{F}_\infty$.

We refer to linear functionals φ on R satisfying $\varphi(s_v) \geq 0$ as *positive* linear functionals.

The Okada p-functions

Following Okada [9], we introduce another family of homogeneous polynomials, labelled by Fibonacci words $v \in \mathbb{YF}$,

$$p_v = \left(X^{k_0+2} - (k_0+2)X^{k_0}Y\right) \cdots \left(X^{k_{t-1}+2} - (k_{t-1}+2)X^{k_{t-1}}Y\right)X^{k_t}, \tag{5.16}$$

where

$$v = 1^{k_t}\underbrace{21^{k_{t-1}}} \cdots \underbrace{21^{k_0}}.$$

One can check that $\{p_v\}_{|v|=n}$ is a $\mathbb{Q}$-basis of R_n. Two important properties of the p-basis which were found by Okada are:

$$U(p_v) = p_v X = p_{1v} \quad \text{and} \quad D(p_{2v}) = 0. \tag{5.17}$$

Since the images of p_u and of p_{1u} in R_∞ are the same, we can conveniently denote the image by $p_{1^\infty u}$. The family of p_v, where v ranges over $1^\infty\mathbb{YF}$, is a basis of R_∞.

Transition matrix from s-basis to p-basis

We denote the transition matrix relating the two bases $\{p_u\}$ and $\{s_v\}$ by X_u^v; thus

$$p_u = \sum_{|v|=n} X_u^v s_v, \quad u, v \in \mathbb{YF}_n. \tag{5.18}$$

The coefficients X_u^v are analogous to the character matrix of the symmetric group $\mathfrak{S}_n$. They were described recurrently in [9], Section 5, as follows:

$$X_{1u}^{1v} = X_u^v; \quad X_{2u}^{1v} = X_{1u}^v; \quad X_{2u}^{2v} = -X_u^v, \tag{5.19}$$

$$X_{11u}^{2v} = (m(u)+1)X_u^v; \quad X_{12u}^{2v} = 0, \tag{5.20}$$

where $m(u)$ is defined in (6.2) below. An explicit product expression for the X_u^v will be given in the next section.

6. A product formula for Okada characters

In this Section we improve Okada's description of the character matrix X_u^v to obtain the product formula (6.11) and its consequences.

Some notation

We recall some notation from [9] which will be used below. Let v be a Fibonacci word:

$$v = 1^{k_t}21^{k_{t-1}} \cdots 1^{k_1}21^{k_0} \in \mathbb{YF}_n.$$

Then:

$\epsilon(v) = +1$ if the rightmost digit of$v$ is 1, and $\epsilon(v) = -1$ otherwise. (6.1)

$m(v) = k_t$ is the number of 1's at the left end of v. (6.2)

The rank of v, denoted $|v|$, is the sum of the digits of v. (6.3)

If $v = v_1 2 v_2$, then the *position* of the indicated 2 is $|v_2| + 1$. (6.4)

$d(v) = \prod_{i=0}^{t-1}(k_0 + \cdots + k_i + 2i + 1)$. In other words, $d(v)$ is the product of the positions of 2's in v. It is easy to check by induction that $d(v) = d(\emptyset, v)$. (6.5)

$z(v) = k_t!\,(k_{t-1}+2)k_{t-1}! \cdots (k_0+2)k_0!$. (6.6)

The *block ranks* of v are the numbers $k_0+2, k_1+2, \ldots, k_{t-1}+2, k_t$. (6.7)

The *inverse block ranks* of v are $k_t+2, k_{t-1}+2, \ldots, k_1+2, k_0$. (6.8)

Consider a sequence $\bar{n} = (n_t, \ldots, n_1, n_0)$ of positive integers with $\sum n_i = n$. We call a word $v \in \mathbb{YF}_n$ *$\bar{n}$-splittable,* if it can be written as a concatenation

$$v = v_t \cdots v_1 v_0, \quad \text{where } |v_i| = n_i \quad \text{for } i = 0, 1, \ldots, t. \tag{6.8}$$

Lemma 6.1 *Let $\bar{n} = (n_t, \ldots, n_1, n_0)$ be the sequence of block ranks in a Fibonacci word u. Then*

(i) $X_u^v \neq 0$ *if, and only if, the word* v *is* $\bar{n}$*-splittable.*
(ii) *If* $v = v_t \cdots v_1 v_t$ *is the* $\bar{n}$*-splitting, then*

$$X_u^v = d(v_t) g(v_{t-1}) \cdots g(v_0), \tag{6.9}$$

where

$$g(w) = \begin{cases} +d(w'), & \text{if } w = 1w' \\ -d(w'), & \text{if } w = 2w'. \end{cases} \tag{6.10}$$

Proof: This is a direct consequence of Okada's recurrence relations cited in the previous section. □

Proposition 6.2 *Let* $u, v \in \mathbb{YF}_n$*. Let* $\delta_1, \delta_2, \ldots, \delta_m$ *be the positions of* 2*'s in the word* u*, and put* $\delta_{m+1} = \infty$*. Let* $d_1, d_2, \ldots, d_r$ *the positions of* 2*'s in the word* v*. Then*

$$X_u^v = \prod_{j=1}^{m} \prod_{\delta_j \le d_s < \delta_{j+1}} (d_s - (\delta_j + 1)). \tag{6.11}$$

Proof: This can also be derived directly from Okada's recurrence relations, or from the previous lemma. Note in particular that $X_u^v = 0$ if, and only if, $d_s = \delta_j + 1$ for some s and j; this is the case if, and only if, v does not split according to the block ranks of u. □

We define $\tilde{X}_u^v = d(v)^{-1} X_u^v$; from Proposition 6.2 and the dimension formula (6.5), we have the expression

$$\tilde{X}_u^v = \prod_{j=1}^{m} \prod_{\delta_j \le d_s < \delta_{j+1}} \left(1 - \frac{\delta_j + 1}{d_s}\right) \tag{6.12}$$

The inverse transition matrix

According to [9], Proposition 5.3, the inverse formula to Eq. (5.18) can be written in the form

$$s_v = \sum_{|u|=n} X_u^v \frac{p_u}{z(u)}, \quad v \in \mathbb{YF}_n. \tag{6.13}$$

We will give a description of a column X_u^v for a fixed v.

Lemma 6.3 *Let* $\bar{n} = (n_t, \ldots, n_1, n_0)$ *be the sequence of inverse block ranks* $n_t = k_t + 2, \ldots, n_1 = k_1 + 2, n_0 = k_0$ *in a word* $v = (1^{k_t} 2 \cdots 1^{k_1} 2 1^{k_0}) \in \mathbb{YF}_n$*. Then*

(i) $X_u^v \neq 0$ *if, and only if, the word* u *is* $\bar{n}$*-splittable.*
(ii) *If* $u = u_t \cdots u_1 u_0$ *is the* $\bar{n}$*-splitting for* u*, then*

$$X_u^v = f_1\, f_2\, \cdots\, f_t, \tag{6.14}$$

where

$$f_j = \begin{cases} -1, & \text{if } \epsilon(u_j) = -1 \\ 1 + m(u_{j-1} \cdots u_1 u_0), & \text{if } \epsilon(u_j) = +1. \end{cases} \tag{6.15}$$

Here $m(u) = m$ *denotes the number of* 1*'s at the left end of* $u = 1^m 2u'$.

Proof: This is another corollary of Okada's recurrence relations cited in Section 5. □

7. The Martin boundary of the Young-Fibonacci lattice

In this section, we examine certain elements of the Martin boundary of the Young-Fibonacci lattice $\mathbb{YF}$. Ultimately we will show that the harmonic functions listed here comprise the entire Martin boundary.

It will be useful for us to evaluate normalized positive linear functionals on the ring R_∞ (corresponding to normalized positive harmonic functions on $\mathbb{YF}$) on the basis $\{p_u\}$. The first result in this direction is the evaluation of the Plancherel functional on these basis elements.

Proposition 7.1 $\varphi_P(p_u) = 0$ *for all Fibonacci words* u *containing at least one* 2.

Proof: It follows from the definition of the Plancherel harmonic function φ_P that for $w \in \mathbb{YF}_n$,

$$\sum_{v : v \nearrow w} \varphi_P(v) = n\varphi_P(w).$$

Therefore, for all $f \in R_n$,

$$\varphi_P(Df) = n\varphi_P(f).$$

If $u = 1^\infty 2v$, and $|2v| = n$, then

$$\varphi_P(p_u) = \varphi_P(p_{2v}) = \frac{1}{n}\, \varphi_P(Dp_{2v}) = 0,$$

since $Dp_{2v} = 0$, by (5.17). □

For each word $w \in \mathbb{YF}_n$, the path $(w, 1w, 1^2w, \ldots)$ clearly satisfies the conditions of Proposition 4.1, and therefore there is a type I harmonic function on $\mathbb{YF}$ defined by

$$\psi_w(v) = \lim_{k\to\infty} \frac{d(v, 1^k w)}{d(\emptyset, 1^k w)}.$$

Proposition 7.2 *Let $w \in \mathbb{YF}_n$, and let $d_1, d_2, \ldots$ be the positions of 2's in w. Let u be a word in $1^\infty\mathbb{YF}$ containing at least one 2, and let $\delta_1, \delta_2, \ldots, \delta_m$ be the positions of 2's in u. Then:*

$$\psi_w(p_u) = \prod_{i=1}^{m} \prod_{\delta_i \le d_j < \delta_{i+1}} \left(1 - \frac{\delta_i + 1}{d_j}\right). \tag{7.1}$$

Proof: Let $u = 1^\infty u_0$, where $u_0 \in \mathbb{YF}_m$. Choose $r, s \ge 1$ such that $|1^s w| = |1^r u_0|$. Then

$$\begin{aligned}\psi_w(p_u) &= (d(1^s w))^{-1}\langle p_{1^r u_0}, s_{1^s w}\rangle \\ &= (d(1^s w))^{-1}\left\langle \sum_v X^v_{1^r u_0} s_v, s_{1^s w}\right\rangle \\ &= (d(1^s w))^{-1} X^{1^s w}_{1^r u_0}.\end{aligned}$$

Thus the result follows from Eq. (6.12). □

Next we describe some harmonic functions which arise from summable infinite words. Given a summable infinite word w, define a linear functional on the ring R_∞ by the requirements $\varphi_w(1) = 1$ and

$$\varphi_w(p_u) = \prod_{i=1}^{m} \prod_{\delta_i \le d_j < \delta_{i+1}} \left(1 - \frac{\delta_i + 1}{d_j}\right), \tag{7.3}$$

where $u \in 1^\infty\mathbb{YF}$. As usual, $\delta_1, \ldots, \delta_m$ are the positions of 2's in u, and the d_j's are the positions of 2's in w. It is evident that $\varphi_w(p_u X) = \varphi_w(p_{1u}) = \varphi_w(p_u)$, so that φ_w is in fact a functional on R_∞.

Proposition 7.3 *If w is a summable infinite Fibonacci word, then φ_w is a normalized positive linear functional on R_∞, so corresponds to a normalized positive harmonic function on $\mathbb{YF}$.*

Proof: Only the positivity needs to be verified. Let w_n be the finite word consisting of the rightmost n digits of w. It follows from the product formula for the normalized characters ψ_{w_n} that $\varphi_w(p_u) = \lim_{n\to\infty} \psi_{w_n}(p_u)$. Therefore also $\varphi_w(s_v) = \lim_{n\to\infty} \psi_{w_n}(s_v) \ge 0$. □

Given a summable infinite Fibonacci word w and $0 \le \beta \le 1$, we can define the harmonic function $\varphi_{\beta,w}$ by contraction of φ_w, namely, $\varphi_{\beta,w} = C_\beta(\varphi_w)$.

For $u \in 1^\infty\mathbb{YF}$, we let $\|u\|$ denote the *essential rank* of u, namely $\|u\| = 1 + \delta$, where δ is the position of the leftmost 2 in u, and $\|u\| = 0$ for $u = 1^\infty$.

Proposition 7.4 *Let w be a summable infinite word and $0 \le \beta \le 1$. Let $u \in 1^\infty\mathbb{YF}$. Then*

$$\varphi_{\beta,w}(p_u) = \beta^{\|u\|}\varphi_w(p_u). \tag{7.4}$$

Proof: The case $u = 1^\infty$ is trivial. Let $u = 1^\infty u_0$, where $\|u\| = |u_0| = n > 0$. Then for any linear functional φ on R_∞, one has $\varphi(p_u) = \varphi(p_{u_0})$. For $0 \le k \le n$, define

$$A^w_{k,n} = \sum_{|v|=k} \sum_{|x|=n} \varphi_w(v)\, d(v, x) s_x.$$

In particular,

$$A^w_{n,n} = \sum_{|v|=n} \varphi_w(v) s_v.$$

Note that $UA^w_{k,n-1} = A^w_{k,n}$, when $k \le n-1$. It follows from the definitions of $\varphi_{\beta,w}$ (cf. (4.4)) and of $A^w_{k,n}$ that

$$\varphi_{\beta,w}(f) = \left\langle \sum_{k=0}^{n} \frac{\beta^k (1-\beta)^{n-k}}{(n-k)!} A^w_{k,n},\, f \right\rangle,$$

for f in R_n, where $\langle \cdot, \cdot \rangle$ denotes the inner product on R with respect to which the Okada s-functions form an orthonormal basis. Recall that the operators U and D are conjugate with respect to this inner product. Consequently,

$$\begin{aligned}
\varphi_{\beta,w}(p_{u_0}) &= \left\langle \sum_{k=0}^{n} \frac{\beta^k (1-\beta)^{n-k}}{(n-k)!} A^w_{k,n},\, p_{u_0} \right\rangle \\
&= \left\langle \sum_{k=0}^{n-1} \frac{\beta^k (1-\beta)^{n-k}}{(n-k)!} UA^w_{k,n-1},\, p_{u_0} \right\rangle + \left\langle \beta^n A^w_{n,n},\, p_{u_0} \right\rangle \\
&= \left\langle \beta^n A^w_{n,n},\, p_{u_0} \right\rangle = \beta^n \varphi_w(p_{u_0}),
\end{aligned}$$

since $\langle UA^w_{k,n-1}, p_{u_0} \rangle = \langle A^w_{k,n-1}, Dp_{u_0} \rangle = 0$ and $\langle A^w_{n,n}, f \rangle = \varphi_w(f)$ for f in R_n. □

Corollary 7.5 *The functionals $\varphi_{\beta,w}$ for $\beta > 0$ and w summable are pairwise distinct, and different from the Plancherel functional φ_P.*

Proof: Suppose that w is a summable word and that A is the set of positions of 2's in w. We set

$$\pi_k(w) = \prod_{j : d_j \ge k-1} \left(1 - \frac{k}{d_j}\right).$$

Then for each $k \ge 2$ and $\beta > 0$,

$$\varphi_{\beta,w}(p_{21^{k-2}}) = \beta^k \pi_k(w)$$

is zero if and only if $k \in A$. In particular $\varphi_{\beta,w} \ne \varphi_P$, by Lemma 7.1, and moreover, $A \setminus \{1\}$ is determined by the sequence of values $\varphi_{\beta,w}(p_{21^{k-2}})$, $k \ge 2$. It is also clear that $\varphi_{\beta,w}(p_2) = \beta^2\pi_2(w)$ is negative iff $1 \in A$, hence the set A and therefore also w are determined by the values $\varphi_{\beta,w}(p_{21^k})$, $k \geq 0$. Finally, β is determined by

$$\beta = \left(\frac{\varphi_{\beta,w}(p_{21^{k-1}})}{\pi_k(w)}\right)^{1/k},$$

for any $k \notin A$. $\square$

Proposition 7.6 *For each summable infinite Fibonacci word w and each β, $0 \leq \beta \leq 1$, there exists a sequence $v^{(n)}$ of finite Fibonacci words such that $\varphi_{\beta,w} = \lim_{n\to\infty} \psi_{v^{(n)}}$.*

Proof: If $\beta = 1$, put $r_n = 0$; if $\beta = 0$, put $r_n = n^2$; and if $0 < \beta < 1$, choose the sequence r_n so that

$$\lim_{n\to\infty} \frac{r_n}{n} = \frac{1-\beta^2}{\beta^2}.$$

Then, in every case,

$$\lim_{n\to\infty} \frac{n}{n+r_n} = \beta^2.$$

Let w_n be the finite word consisting of the rightmost n digits of w, put $s_n = 2n+1-|w_n|$, and

$$v^{(n)} = 2^{r_n}1^{s_n}w_n.$$

Fix $u = 1^\infty u_0 \in 1^\infty\mathbb{YF}$ and let $n \geq |u_0|$. Suppose that u_0 has 2's at positions $\delta_1, \delta_2, \ldots, \delta_m$, and put $k = \|u\| = \delta_m + 1$. Using the product formula for $\psi_{v^{(n)}}$, one obtains

$$\psi_{v^{(n)}}(p_u) = \psi_{w_n}(p_u)\left[\left(1 - \frac{k}{2n+2}\right)\left(1 - \frac{k}{2n+4}\right)\cdots\left(1 - \frac{k}{2n+2r_n}\right)\right].$$

The first factor converges to $\varphi_w(p_u)$, so it suffices, by Proposition 7.4, to show that the second factor converges to β^k. The second factor reduces to

$$\frac{\Gamma(n+r_n-k/2+1)\Gamma(n+1)}{\Gamma(n+1-k/2)\Gamma(n+r_n+1)}.$$

Using the well-known fact that

$$\lim_{n\to\infty} n^{b-a}\frac{\Gamma(n+a)}{\Gamma(n+b)} = 1,$$

one obtains that the ratio of gamma functions is asymptotic to

$$\left(\frac{n}{n+r_n}\right)^{k/2},$$

which, by our choice of r_n, converges to β^k, as desired. □

This proposition shows that φ_P as well as all of the harmonic functions $\varphi_{\beta,w}$ are contained in the Martin boundary of the Young-Fibonacci lattice. In the following sections, we will show that these harmonic functions make up the entire Martin boundary.

8. Regularity conditions

In this Section we obtain a simple criterion for a sequence of characters of finite dimensional Okada algebras to converge to a character of the limiting infinite dimensional algebra $\mathcal{F} = \varinjlim \mathcal{F}_n$. Using this criterion, the *regularity conditions*, we show that the harmonic functions provided by the formulae (7.3) and (7.4) make up the entire Martin boundary of the Young-Fibonacci graph $\mathbb{YF}$. Technically, it is more convenient to work with linear functionals on the spaces R_n and their limits in $R_\infty = \varinjlim R_n$, rather than with traces on $\mathcal{F}$. As it was already explained in Section 5, there is a natural one-to-one correspondence between traces of Okada algebra $\mathcal{F}_n$, and positive linear functionals on the space R_n.

In this Section we shall use the following elementary inequalities:

$$\left(1-\frac{k}{d}\right) \le \left(1-\frac{1}{d}\right)^k, \tag{8.1}$$

for every pair of positive integer numbers $d \ge 2$ and k;

$$\left(1-\frac{1}{d}\right)^{k^2} \le \left(1-\frac{k}{d}\right), \tag{8.2}$$

and

$$1 \le \frac{\left(1-\frac{1}{d}\right)^k}{\left(1-\frac{k}{d}\right)} \le 1+\frac{\binom{k}{2}}{(d-k)^2}, \tag{8.3}$$

for every pair of integers $d > k$. We omit the straightforward proofs of these inequalities.

Convergence to the Plancherel measure

We first examine the important particular case of convergence to the Plancherel character φ_P.

Definition We define the function π of a finite or summable word v by

$$\pi(v) = \prod_{j:\,d_j \geq 2} \left(1 - \frac{1}{d_j}\right),$$

where the d_j are the positions of the 2's in v. We also recall that for each $k \geq 2$ the function π_k was defined as

$$\pi_k(v) = \prod_{j\,:\,d_j \geq k-1} \left(1 - \frac{k}{d_j}\right).$$

Note that if $u = 1^\infty 21^{k-2}$, and v is a summable word, then $\varphi_v(p_u) = \pi_k(v)$, according to Eq. (7.3).

Proposition 8.1 *The following properties of a sequence $w_n \in \mathbb{YF}$, $n = 1, 2, \ldots$, are equivalent:*

(i) *The normalized characters ψ_{w_n} converge to the Plancherel character, i.e.,*

$$\lim_{n\to\infty} \psi_{w_n}(p_u) = \varphi_P(p_u), \text{ for each } u \in 1^\infty\mathbb{YF};$$

(ii) $\lim_{n\to\infty} \pi_k(w_n) = 0,$ *for every $k = 2, 3, \ldots;$*

(iii) $\lim_{n\to\infty} \pi(w_n) = 0.$

The proof is based on the following lemmas.

Lemma 8.2 *For every finite word $v \in \mathbb{YF}$, and for every $u \in 1^\infty\mathbb{YF}$ of essential rank $k = \|u\|$,*

$$|\psi_v(p_u)| \leq |\pi_k(v)|. \tag{8.4}$$

Proof: Let $\delta_1, \ldots, \delta_m$ indicate the positions of 2's in the word u, and let $d_1, \ldots, d_n$ be the positions of 2's in v. The essential rank of u can be written as $k = \|u\| = \delta_m + 1$, so that

$$\pi_k(v) = \prod_{j:d_j \geq \delta_m} \left(1 - \frac{k}{d_j}\right).$$

By the product formula,

$$|\psi_v(p_u)| = |\pi_k(v)| \prod_{i=1}^{m-1} \prod_{\delta_i \leq d_j < \delta_{i+1}} \left|1 - \frac{\delta_i + 1}{d_j}\right| \leq |\pi_k(v)|,$$

since none of the factors in the product exceed 1. In fact, $|1 - (\delta + 1)/d| = 1/\delta \leq 1$ if $d = \delta$, $1 - (\delta + 1)/d = 0$ if $d = \delta + 1$, and $0 \leq 1 - (\delta + 1)/d < 1$ if $d > \delta + 1$. $\square$

Lemma 8.3 *For each $k = 2, 3, \ldots$, and for every word $v \in \mathbb{YF}$,*

$$|\pi_k(v)| \leq (k\pi(v))^k.$$

Proof: It follows from (8.1) that

$$|\pi_k(v)| = \prod_{d_j \geq k-1} \left|1 - \frac{k}{d_j}\right| \leq \prod_{d_j \geq k+1} \left(1 - \frac{k}{d_j}\right) \leq \prod_{d_j \geq k+1} \left(1 - \frac{1}{d_j}\right)^k$$
$$= \pi(v)^k \prod_{2 \leq d_j \leq k} \left(1 - \frac{1}{d_j}\right)^{-k}.$$

Since $(1 - 1/d)^{-1} > 1$ for $d \geq 2$, the last product can be estimated as

$$\prod_{2 \leq d_j \leq k} \left(1 - \frac{1}{d_j}\right)^{-k} \leq \left(\frac{1}{2} \quad \frac{2}{3} \quad \cdots \quad \frac{k-1}{k}\right)^{-k} = k^k,$$

and the lemma follows. □

Lemma 8.4 *If $d_1(v) \neq 2$, then $|\pi_2(v)| \geq \pi(v)^4$, and if $d_1(v) = 2$, then $|\pi_3(v)| \geq \pi(v)^9$.*

Proof: We apply the inequality (8.2). If $d_1(v) \geq 3$, then

$$|\pi_2(v)| = \prod_{j:d_j \geq 3} \left(1 - \frac{2}{d_j}\right) \geq \prod_{j:d_j \geq 3} \left(1 - \frac{1}{d_j}\right)^4 = \pi(v)^4$$

by the inequality (8.2). If $d_1(v) = 1$, then $(1 - 2/d_1) = -1$, and since $d_2 \geq 3$, the inequality holds in this case as well.

In case of $d_1(v) = 2$ we have $d_2(v) \geq 4$, hence

$$|\pi_3(v)| = \frac{1}{2} \prod_{j:d_j \geq 4} \left(1 - \frac{3}{d_j}\right); \qquad |\pi(v)| = \frac{1}{2} \prod_{j:d_j \geq 4} \left(1 - \frac{1}{d_j}\right),$$

so that the second inequality of Lemma also follows from (8.2). □

Proof Proposition 8.1: The implication (i) $\Rightarrow$ (ii) is trivial, since $\pi_k(v) = \psi_v(p_u)$ is a particular character value for $u = 1^\infty 21^{k-2}$. The statement (iii) follows from (ii) by Lemma 8.4. In fact, we can split the initial sequence $\{w_n\}$ into two subsequences, $\{w_n'\}$ and $\{w_n''\}$, in such a way that $d_1(w_n') = 2$ and $d_1(w_n'') \neq 2$. Then we derive from Lemma 8.4 that for both subsequences $\pi(w_n) \to 0$, and (iii) follows.

Now, (ii) follows from (iii) by Lemma 8.3, and (ii) implies (i) by Lemma 8.2. □

General regularity conditions

We now find the conditions for a sequence of linear functionals on the spaces R_n to converge to a functional on the limiting space $R_\infty = \varinjlim R_n$.

Definition (Regularity of character sequences) Let ψ_n be a linear functional on the graded component R_n of the ring $R = \mathbb{Q}\langle X, Y\rangle$, for each $n = 1, 2, \ldots,$ and assume that the sequence converges pointwise to a functional φ on the ring R, in the sense that

$$\lim_{n\to\infty} \psi_n(PX^{n-m}) = \varphi(P) \tag{8.5}$$

for every $m \in \mathbb{N}$ and every polynomial $P \in R_m$. We call such a sequence *regular*.

Our goal in this Section is to characterize the set of regular sequences.

Definition (Convergence of words) Let $\{w_n\}$ be a sequence of finite Fibonacci words, and assume that the ranks $|w_n|$ tend to infinity as $n \to \infty$. We say that $\{w_n\}$ *converges* to an infinite word w, iff the mth letter $w_n(m)$ of w_n coincides with the mth letter $w(m)$ of the limiting word w for almost all n (i.e., for all but finitely many n's), and for all m.

Let us recall that an infinite word w with 2's at positions $d_1, d_2, \ldots$ is summable, if, and only if, the series $\sum_{j=1}^{\infty} 1/d_j$ converges, or, equivalently, if the product

$$\pi(w) = \prod_{j:d_j\geq 2} \left(1 - \frac{1}{d_j}\right) > 0$$

converges.

Consider a sequence $w_1, w_2, \ldots$ of Fibonacci words converging to a summable infinite word w. We denote by w'_n the longest initial (rightmost) subword of w_n identical with the corresponding segment of w, and we call it *stable* part of w_n. The remaining part of w_n will be denoted by w''_n, and referred to as *transient* part of w_n.

Definition (Regularity conditions) We say that a sequence of Fibonacci words $w_n \in \mathbb{YF}_n$ *satisfies regularity conditions*, if either one of the following two conditions holds:

(i) $\lim_{n\to\infty} \pi(w_n) = 0;$

or

(ii) the sequence w_n converges to a summable infinite word w, and a strictly positive limit

$$\beta = \pi(w)^{-1} \lim_{n\to\infty} \pi(w_n) > 0 \tag{8.6}$$

exists.

Theorem 8.5 *Assume that the regularity conditions hold for a sequence $w_n \in \mathbb{YF}_n$. Then the character sequence ψ_{w_n} is regular. If the regularity condition (i) holds, then*

$$\lim_{n\to\infty} \psi_{w_n}(QX^{n-m}) = \varphi_P(Q),$$

and if regularity condition (ii) holds, then

$$\lim_{n\to\infty} \psi_{w_n}(QX^{n-m}) = \varphi_{\beta,w}(Q) \tag{8.7}$$

for every polynomial $Q \in R_m$, $m = 1, 2, \ldots$. *Conversely, if the character sequence* ψ_{w_n} *is regular, then the regularity conditions hold for the sequence* $w_n \in \mathbb{YF}_n$.

This theorem will follow from Proposition 8.1 and the following proposition:

Proposition 8.6 *Assume that a sequence* $w_1, w_2, \ldots$ *of Fibonacci words converges to a summable infinite word* w, *and that there exists a limit*

$$\beta = \pi(w)^{-1} \lim_{n\to\infty} \pi(w_n). \tag{8.8}$$

Then

$$\pi_k(w)^{-1} \lim_{n\to\infty} \pi_k(w_n) = \beta^k \tag{8.9}$$

for every $k = 2, 3, \ldots$. *More generally,*

$$\lim_{n\to\infty} \psi_{w_n}(p_u) = \beta^k \varphi_w(p_u) \tag{8.10}$$

for every element $u \in 1^\infty\mathbb{YF}$ *of essential rank* $\|u\| = k$.

Proof: Let m_n be the length of the stable part of the word w_n, and note that $m_n \to \infty$. In the following ratio, the factors corresponding to 2's in the stable part of w_n cancel out,

$$\frac{\pi(w_n)}{\pi(w)} = \prod_{j:d_j(w_n)>m_n} \left(1 - \frac{1}{d_j(w_n)}\right) \prod_{j:d_j(w)>m_n} \left(1 - \frac{1}{d_j(w)}\right)^{-1},$$

and a similar formula holds for the functional π_k, $k = 2, 3, \ldots$:

$$\frac{\pi_k(w_n)}{\pi_k(w)} = \prod_{j:d_j(w_n)>m_n} \left(1 - \frac{k}{d_j(w_n)}\right) \prod_{j:d_j(w)>m_n} \left(1 - \frac{k}{d_j(w)}\right)^{-1}. \tag{8.11}$$

Consider the ratio

$$\frac{(\pi(w_n)/\pi(w))^k}{(\pi_k(w_n)/\pi_k(w))} = \prod_{j:d_j(w_n)>m_n} \frac{(1 - 1/d_j(w_n))^k}{(1 - k/d_j(w_n))} \prod_{j:d_j(w)>m_n} \frac{(1 - k/d_j(w))}{(1 - 1/d_j(w))^k}.$$

The second product in the right hand side is a tail of the converging infinite product (since the word w is summable), hence converges to 1, as $n \to \infty$. By (8.3), the first product can also be estimated by a tail of a converging infinite product,

$$1 \le \prod_{j:d_j(w_n)>m_n} \frac{(1 - 1/d_j(w_n))^k}{(1 - k/d_j(w_n))} \le \prod_{j:d_j>m_n} \left(1 + \frac{\binom{k}{2}}{(d_j - k)^2}\right),$$

hence converges to 1, as well.

The proof of the formula (8.10) is only different in the way that the ratio in the left hand side of (8.11) should be replaced by $\psi_{w_n}(p_u)/\varphi_w(p_u)$. $\square$

Proof of the Theorem 8.5: It follows directly from Propositions 8.1 and 8.6 that the regularity conditions for a sequence $w_1, w_2, \ldots$ imply convergence of functionals ψ_{w_n} to the Plancherel character φ_P in case (i), and to the character $\varphi_{\beta,w}$ in case (ii). By the Corollary 7.5 we know that the functions $\varphi_{\beta,w}$ are pairwise distinct, and also different from the Plancherel functional φ_P.

Let us now assume that the sequence ψ_{w_n} converges to a limiting functional φ. We can choose a subsequence $\psi_{w_{n_m}}$ in such a way that the corresponding sequence w_{n_m} converges digitwise to an infinite word w. If w is not summable, then $\varphi = \varphi_P$ coincides with the Plancherel functional, and the part (i) of the regularity conditions holds. Otherwise, we can also assume that the limit (8.6) exists, and hence $\varphi = \varphi_{\beta,w}$ by Proposition 8.6. Since the parameter β and the word w can be restored, by Corollary 7.5, from the limiting functional φ, the sequence w_n cannot have subsequences converging to different limits, nor can the sequence $\pi(w_n)$ have subsequences converging to different limits. It follows, that the regularity conditions are necessary. The Theorem is proved. $\square$

Finally, we prove Theorem 3.2, which states that the map

$$(\beta, w) \mapsto \varphi_{\beta,w}, \quad P \mapsto \varphi_P$$

is a homeomorphism of Ω onto the Martin boundary of $\mathbb{YF}$, where Ω is the space defined near the end of Section 3.

Proof of Theorem 3.2: It follows from Corollary 7.5 that the map is an injection of Ω into the Martin boundary, and from Theorem 8.5 that the map is surjective. Furthermore, the proof that the map is a homeomorphism is a straightforward variation of the proof of the regularity statement, Theorem 8.5. $\square$

9. Concluding remarks

The Young-Fibonacci lattice, along with the Young lattice, are the most interesting examples of differential posets. There is a considerable similarity between the two graphs, as well as a few severe distinctions.

Both lattices arise as Bratteli diagrams of increasing families of finite dimensional semisimple matrix algebras, i.e., group algebras of symmetric groups in case of Young lattice, and Okada algebras in case of Young-Fibonacci graph. For every Bratteli diagram, there is a problem of describing the traces of the corresponding inductive limit algebra, which is well-known to be intimately related to the Martin boundary construction for the graph. The relevant fact is that indecomposable positive harmonic functions, which are in one-to-one correspondence with the indecomposable traces, form a part of the Martin boundary.

For the Young lattice the Martin boundary has been known for several decades, and all of the harmonic functions in the boundary are known to be indecomposable (extreme points). In this paper we have found the Martin boundary for the Young-Fibonacci lattice. Unfortunately, we still do not know which harmonic functions in the boundary are decomposable (if any). The method employed to prove indecomposability of the elements of the Martin boundary of the Young lattice can not be applied to Young-Fibonacci lattice, since the K_0-functor ring R of the limiting Okada algebra $\mathcal{F}$ is not commutative, as it is in case of the group algebra of the infinite symmetric group (in this case it can be identified with the symmetric function ring).

Another natural problem related to Okada algebras is to find all non-negative Markov traces. We plan to address this problem in another paper.

Appendix

In this appendix, we survey a few properties of differential posets introduced by R. Stanley in [11], and by S. Fomin (under another name) [3]. Further generalizations were introduced in [12–13], and [4–5].

A.1. Definitions

A graded poset $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_n$ is called *branching diagram* (cf. [7]), if

(B1) The set Γ_n of elements of rank n is finite for all $n = 0, 1, \ldots$

(B2) There is a unique minimal element $\varnothing \in \Gamma_0$

(B3) There are no maximal elements in Γ.

One can consider a branching diagram as an extended phase space of a non-stationary Markov chain, Γ_n being the set of admissible states at the moment n and covering relations indicating the possible transitions.

We denote the rank of a vertex $v \in \Gamma_n$ by $|v| = n$, and the number of saturated chains in an interval $[u, v] \subset \Gamma$ by $d(u, v)$.

Following [11], we define an *r-differential poset* as a branching diagram Γ satisfying two conditions:

(D1) If $u \neq v$ in Γ then the number of elements covered by u and v is the same as the number of elements covering both u and v.

(D2) If $v \in \Gamma$ covers exactly k elements, then v is covered by exactly $k + r$ elements of Γ.

Note that the number of elements in a differential poset covering two distinct elements can be at most 1. In this paper we focus on 1-differential posets.

For any branching diagram Γ one can define two linear operators in the vector space $\mathrm{Fun}(\Gamma)$ of functions on Γ with coefficients in $\mathbb{R}$: the *creation operator*

$$U(f)(v) = \sum_{w: v \nearrow w} f(w), \tag{A.1.1}$$

and the *annihilation operator*

$$D(f)(v) = \sum_{u:u\nearrow v} f(u). \tag{A.1.2}$$

Identifying finitely supported functions on Γ with formal linear combinations of points of Γ and vertices of Γ with the delta functions at the vertices, one can write instead:

$$Uv = \sum_{w:v\nearrow w} w, \tag{A.1.1$'$}$$

$$Dv = \sum_{u:u\nearrow v} u. \tag{A.1.2$'$}$$

One can characterize r-differential posets as branching diagrams for which the operators U, D satisfy the Weyl identity $DU - UD = r\,I$.

A.2. Some properties of differential posets

We review below only a few identities we need in the main part of the paper. For a general algebraic theory of differential posets see [11–13], [3–5]. Assume here that Γ is a 1-differential poset.

The first formula is well known:

$$\sum_{w:v\nearrow w} d(\varnothing, w) = (n+1)\,d(\varnothing, v), \qquad v \in \Gamma. \tag{A.2.1}$$

Proof: Let $d_n = \sum_{|v|=n} d(\varnothing, v)\, v \in \mathrm{Fun}(\Gamma)$. Then $U\,d_n = d_{n+1}$ and (A.2.1) can be written as $D\,d_{n+1} = (n+1)\,d_n$. This is trivial for $n = 0$, and assuming $D\,d_n = n\,d_{n-1}$ we obtain

$$D\,d_{n+1} = DU\,d_n = UD\,d_n + d_n = nU\,d_{n-1} + d_n = (n+1)\,d_n. \qquad \square$$

Our next result is a generalization of (A.2.1).

Lemma A.2.2 *Let Γ be a 1-differential poset, and let $u \le v$ be any vertices of ranks $|u| = k$, $|v| = n$. Then*

$$\sum_{w:v\nearrow w} d(u, w) - \sum_{x:x\nearrow u} d(x, v) = (n-k+1)\; d(u, v). \tag{A.2.3}$$

Proof: Using the notation $d_n(u) = \sum_{|v|=n} d(u, v)v$, one can easily see that $U\,d_n(u) = d_{n+1}(u)$ and that (A.2.3) can be rewritten in the form

$$D\,d_{n+1}(u) = \sum_{x:x\nearrow u} d_n(x) + (n-k+1)\,d_n(u).$$

For $n = k - 1$ the formula is true by the definition of D. By induction,

$$\begin{aligned} D\, d_{n+1}(u) &= DU\, d_n(u) = UD\, d_n(u) + d_n(u) \\ &= U\left(\sum_{x:x\nearrow u} d_{n-1}(x) + (n-k)\, d_{n-1}(u)\right) + d_n(u) \\ &= \sum_{x:x\nearrow u} d_n(x) + (n-k+1)\, d_n(u). \end{aligned}$$

Note that (A.2.3) specializes to (A.2.1) in case $k = 0$, $u = \varnothing$. □

A.3. Plancherel transition probabilities on a differential poset

It follows from (A.2.1) that the numbers

$$p_{v,w} = \frac{d(\varnothing, w)}{(n+1)\, d(\varnothing, v)}; \quad v \nearrow w, \quad n = |v|, \tag{A.3.1}$$

as transition probabilities of a Markov chain on Γ. Generalizing the terminology used in the particular example of Young lattice (see [7]), we call (A.3.1) *Plancherel transition probabilities*.

Lemma A.3.2 *Let $u \leq v$ be vertices of ranks $|u| = k$, $|v| = n$ in a 1-differential poset Γ. Then the Plancherel probability $p(u, v)$ to reach (by any path) the vertex v starting with u is*

$$p(u, v) = \frac{k!}{n!}\, \frac{d(u, v)\, d(\varnothing, v)}{d(\varnothing, u)}. \tag{A.3.3}$$

Proof: We have to check that $\sum_v p(u, v)\, p_{v,w} = p(u, w)$. Since $\sum_v d(u, v)\, d(v, w) = d(u, w)$, we obtain

$$\frac{k!}{n!} \sum_{|v|=n} \frac{d(\varnothing, v)}{d(\varnothing, u)} d(u, v)\, d(v, w) \frac{d(\varnothing, w)}{(n+1)\, d(\varnothing, v)} = \frac{k!}{(n+1)!}\, \frac{d(u, w)\, d(\varnothing, w)}{d(\varnothing, u)},$$

and the Lemma follows. □

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