



Orlik-Solomon Algebras and Tutte Polynomials

CARRIE J. ESCHENBRENNER

eschenbrenner@op.com

Independent Stationers, 9900 Westpoint Drive, Suite 1b, Indianapolis, IN 46256

MICHAEL J. FALK

michael.falk@nau.edu

Department of Mathematics, Northern Arizona University, Flagstaff, AZ 86011-5717

Received July 10, 1997; Revised April 13, 1998

Abstract. The *OS* algebra A of a matroid M is a graded algebra related to the Whitney homology of the lattice of flats of M . In case M is the underlying matroid of a hyperplane arrangement \mathcal{A} in \mathbb{C}^r , A is isomorphic to the cohomology algebra of the complement $\mathbb{C}^r \setminus \bigcup \mathcal{A}$. Few examples are known of pairs of arrangements with non-isomorphic matroids but isomorphic *OS* algebras. In all known examples, the Tutte polynomials are identical, and the complements are homotopy equivalent but not homeomorphic.

We construct, for any given simple matroid M_0 , a pair of infinite families of matroids M_n and M'_n , $n \geq 1$, each containing M_0 as a submatroid, in which corresponding pairs have isomorphic *OS* algebras. If the seed matroid M_0 is connected, then M_n and M'_n have different Tutte polynomials. As a consequence of the construction, we obtain, for any m , m different matroids with isomorphic *OS* algebras. Suppose one is given a pair of central complex hyperplane arrangements \mathcal{A}_0 and \mathcal{A}_1 . Let \mathcal{S} denote the arrangement consisting of the hyperplane $\{0\}$ in \mathbb{C}^1 . We define the parallel connection $P(\mathcal{A}_0, \mathcal{A}_1)$, an arrangement realizing the parallel connection of the underlying matroids, and show that the direct sums $\mathcal{A}_0 \oplus \mathcal{A}_1$ and $\mathcal{S} \oplus P(\mathcal{A}_0, \mathcal{A}_1)$ have diffeomorphic complements.

Keywords: matroid, arrangement, Orlik-Solomon algebra, Tutte polynomial

1. Introduction

Let M be a simple matroid with ground set E . Associated with M is a graded-commutative algebra $A(M)$ called the *Orlik-Solomon (OS)* algebra of M . Briefly, $A(M)$ is the quotient of the free exterior algebra $\Lambda(E)$ on E by the ideal generated by “boundaries” of circuits in M . If \mathcal{A} is an arrangement in \mathbb{C}^r realizing the matroid M , then $A(M)$ is isomorphic to the cohomology algebra of the complement $C(\mathcal{A}) = \mathbb{C}^r \setminus \bigcup \mathcal{A}$. So in the attempt to classify homotopy types of complex hyperplane complements one is led to study graded algebra isomorphisms of *OS* algebras.

The structure of $A(M)$ as a graded vector space is determined uniquely by the characteristic polynomial $\chi_M(t)$ of M . In most cases, even for matroids having the same characteristic polynomial, the *OS* algebras can be distinguished using more delicate invariants of the multiplicative structure [4, 6]. In [5], however, two infinite families of rank three matroids are constructed in which corresponding pairs have isomorphic *OS* algebras, generalizing a result of L. Rose and H. Terao [9, Example 3.77].

Research conducted under an NSF *Research Experiences for Undergraduates* grant.

The Tutte polynomial $T_M(x, y)$ is an invariant of M that specializes to $\chi(t)$ under the substitution $x = 1 - t$, $y = 0$. In the examples referred to above, the associated matroids have identical Tutte polynomials. Furthermore, in [6] it is shown that, under a fairly weak hypothesis which is satisfied in all known cases, the Tutte polynomial of a rank-three matroid M can be reconstructed from $A(M)$. It is natural to conjecture that $A(M)$ determines $T_M(x, y)$ in general. The purpose of this paper is to show that, without additional hypotheses, counterexamples to this conjecture abound. Here is our main result.

Theorem 1.1 *Let M_0 be an arbitrary connected matroid without loops or multiple points. Then for each positive integer $n \geq 3$, there exist matroids M_n and M'_n of rank $\text{rk}(M_0) + n - 1$ satisfying*

- (i) M_0 is a submatroid of M_n and M'_n .
- (ii) $A(M_n)$ is isomorphic to $A(M'_n)$ as a graded algebra.
- (iii) $T_{M_n}(x, y) \neq T_{M'_n}(x, y)$.

In the other direction, we find several examples in [6] of matroids with the same Tutte polynomials and non-isomorphic OS algebras.

The matroid M_n of the theorem is simply the direct sum of M_0 with the polygon matroid C_n of the n -cycle. The matroid M'_n can be taken to be the direct sum of an isthmus with any parallel connection of M_0 and C_n . Thus, by careful choice of M_0 , we obtain the following corollary.

Corollary 1.2 *Given any positive integer $m \geq 2$, there exist m nonisomorphic simple matroids with isomorphic OS algebras.*

Note that the matroids M_n and M'_n have rank greater than three, and neither is connected. So it remains possible that $A(M)$ determines $T_M(x, y)$ for matroids of rank three, or for connected matroids.

The arrangements constructed in [5] were shown to have homotopy equivalent complements, and the isomorphism of OS algebras is a corollary. In the last section we prove a far more general result in the high rank setting of the present work. We define the parallel connection $P(\mathcal{A}_0, \mathcal{A}_1)$ of two arrangements in Section 4, as a natural realization of the parallel connection of the underlying matroids. The direct sum $\mathcal{A}_0 \oplus \mathcal{A}_1$, denoted by $\mathcal{A}_0 \coprod \mathcal{A}_1$ in [9], realizes the direct sum of the underlying matroids.

Theorem 1.3 *Let \mathcal{A}_0 and \mathcal{A}_1 denote arbitrary arrangements. Let S denote the unique nonempty central arrangement of rank 1. Then $\mathcal{A}_0 \oplus \mathcal{A}_1$ and $S \oplus P(\mathcal{A}_0, \mathcal{A}_1)$ have diffeomorphic complements.*

The examples of [5] are generic sections of the arrangements described in Theorem 1.3, with \mathcal{A}_0 and \mathcal{A}_1 of rank two. The fact that their fundamental groups are isomorphic then follows immediately from Theorem 1.3 by the Lefschetz hyperplane theorem. For these particular arrangements, the complements are homotopy equivalent. It is possible that for more general \mathcal{A}_0 and \mathcal{A}_1 , this construction could yield rank-three arrangements whose complements have isomorphic fundamental groups but are not homotopy equivalent. This

phenomenon has not been seen before, and would be of considerable interest. Also worthy of note is the result of [7] that, for arrangements of rank three, the diffeomorphism type of the complement determines the underlying matroid. Theorem 1.3 demonstrates that this result is false in ranks greater than three.

The formulation of Theorem 1.3 affords a really easy proof, providing an alternative for the proof of Theorem 1.1(ii), in case M_0 is a realizable matroid. The proof is based on a simple and well-known relation [2, 9] between the topology of the complement of a central arrangement in \mathbb{C}^r and that of its projective image, which coincides with the complement of an affine arrangement in \mathbb{C}^{r-1} , called the “decone” of \mathcal{A} . The proof of Theorem 1.3 demonstrates that all the known cases where topological invariants coincide even while underlying matroids differ are consequences of this fundamental principle.

Here is an outline of the proof of Theorem 1.1. Once the matroids M_n and M'_n are constructed in the next section, we define a map at the exterior algebra level which is easily seen to be an isomorphism. We show that this map carries relations to relations, hence induces a well-defined map ϕ of OS algebras. This map is automatically surjective. Then in Section 4 we compute the Tutte polynomials of M_n and M'_n . These are shown to be unequal provided M_0 is connected, but they coincide upon specialization to $y = 0$. Thus M_n and M'_n have identical characteristic polynomials. It follows that the OS algebras have the same dimension in each degree, so that ϕ must be injective. In the final section we prove Corollary 1.2 and Theorem 1.3, and close with a few comments and a conjecture.

2. The construction

We refer the reader to [10, 11] for background material on matroid theory and Tutte polynomials, and to [9] for more information on arrangements and OS algebras.

Let C_n be the polygon matroid of the n -cycle. Thus C_n is a matroid of rank $n - 1$ on n points, with one circuit, of size n . This matroid is realized by any arrangement \mathcal{A}_n of n hyperplanes in general position in \mathbb{C}^{n-1} . The ground set of C_n will be taken to be $[n] := \{1, \dots, n\}$ throughout the paper.

Fix a simple matroid M_0 with ground set E_0 disjoint from $[n]$. Thus M_0 has no loops or multiple points. Let $M_n = C_n \oplus M_0$. So the circuits of M_n are those of M_0 together with the unique circuit of C_n . If \mathcal{A}_0 is an arrangement realizing M_0 in \mathbb{C}^r , then M_n is realized by the direct sum of \mathcal{A}_n and \mathcal{A}_0 in \mathbb{C}^{r+n-1} , denoted $\mathcal{A}_n \amalg \mathcal{A}_0$ in [9].

Now fix $\epsilon_0 \in E_0$. Let $P_{\epsilon_0}^n$ denote the parallel connection $P(C_n, M_0)$ of C_n with M_0 along ϵ_0 . Loosely speaking, $P_{\epsilon_0}^n$ is the freest matroid obtained from C_n and M_0 by identifying ϵ_0 with the point 1 of C_n . Here is a precise definition. Define an equivalence relation on $E := [n] \cup E_0$ so that $\{1, \epsilon_0\}$ is the only nontrivial equivalence class. Denote the class of any $p \in E$ by \bar{p} . For $X \subset E$ let \bar{X} be the set of classes of elements of X . Then $P_{\epsilon_0}^n$ is the matroid on the set \bar{E} whose set of circuits is

$$C = \{ \bar{C} \mid C \text{ is a circuit of } C_n \text{ or } M_0 \} \\ \cup \{ \overline{C - 1} \cup \overline{C' - \epsilon_0} \mid 1 \in C \text{ a circuit of } C_n \\ \text{ and } \epsilon_0 \in C' \text{ a circuit of } M_0 \}.$$

Let S denote an isthmus, that is, a matroid of rank one on a single point, which point will be denoted p . Finally, let M'_n be the direct sum $S \oplus P_{\epsilon_0}^n$.

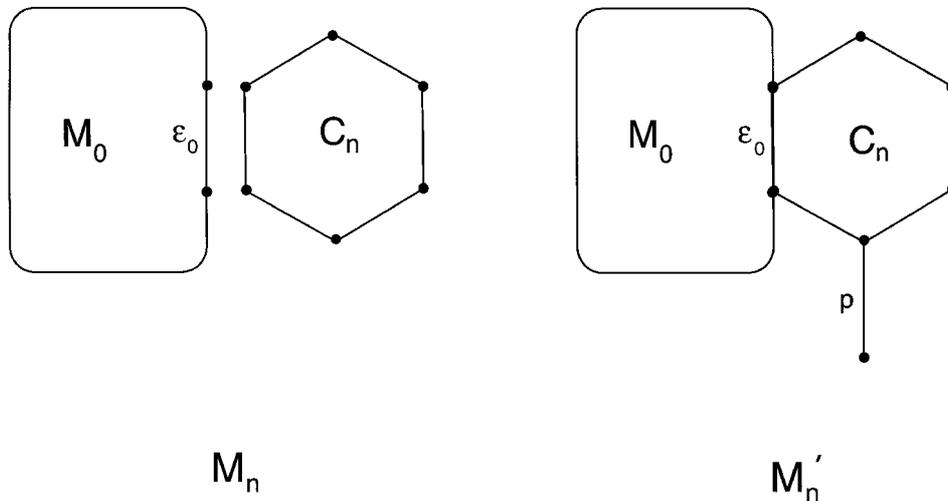


Figure 1. The construction.

These two matroids M_n and M'_n are most easily understood in terms of graphs. If M_0 is a graphic matroid, then M_n is the polygon matroid of the union (with or without a vertex in common) of the corresponding graph G with the n -cycle. The parallel connection $P_{\epsilon_0}^n$ is the matroid of the graph obtained by attaching a path of length $n - 1$ to the vertices of an edge ϵ_0 of G , and M'_n is then obtained by throwing in a pendant edge p . These graphs are illustrated in figure 1, with $n = 6$.

3. An algebra homomorphism

We proceed to define the OS algebra of a matroid. Let M be a simple matroid with ground set E . Let $\Lambda = \Lambda(E)$ be the free exterior algebra generated by degree one elements e_ϵ for $\epsilon \in E$. The results of this paper will hold for coefficients in any commutative ring. Define $\partial : \Lambda(E) \rightarrow \Lambda(E)$ by

$$\partial(e_1 \cdots e_k) = \sum_{i=1}^k (-1)^{i-1} e_1 \cdots \hat{e}_i \cdots e_k,$$

and extending to a linear map. Let $I = I(M)$ be the ideal of $\Lambda(E)$ generated by

$$\{\partial(e_{\epsilon_1} \cdots e_{\epsilon_k}) \mid \{\epsilon_1, \dots, \epsilon_k\} \text{ is a circuit of } M\}.$$

Definition 3.1 The OS algebra $A(M)$ of M is the quotient $\Lambda(E)/I(M)$.

Since Λ is graded and I is generated by homogeneous elements, $A(M)$ is a graded algebra.

The definition of $A(M)$ is motivated by differential topology. Suppose $\mathcal{A} = \{H_1, \dots, H_n\}$ is an arrangement of hyperplanes in \mathbb{C}^r realizing the matroid M . Let $C(\mathcal{A}) = \mathbb{C}^r - \bigcup_{i=1}^n H_i$. Extending work of V.I. Arnol'd and E. Brieskorn, P. Orlik and L. Solomon proved the following theorem [8].

Theorem 3.2 *The cohomology algebra $H^*(C(\mathcal{A}), \mathbb{C})$ of the complement $C(\mathcal{A})$ is isomorphic to $A(M)$.*

We now specialize to the examples constructed in the last section. For simplicity we suppress much of the notation. Consider the integer $n \geq 3$, the matroid M_0 , and the point ϵ_0 to be fixed once and for all. Unprimed symbols M, Λ, I, A will refer to the matroid M_n , and primed symbols M', Λ', I', A' refer to M'_n .

Recall the ground sets of M and M' are $E = [n] \cup E_0$ and $\bar{E} \cup \{p\}$ respectively. The generator of Λ' corresponding to $\bar{\epsilon} \in \bar{E}$ will be denoted by $\bar{e}_{\bar{\epsilon}}$.

We define a homomorphism $\hat{\phi} : \Lambda \rightarrow \Lambda'$ by specifying the images of generators. Specifically,

$$\begin{aligned} \hat{\phi}(e_i) &= \bar{e}_i - \bar{e}_n + e_p & \text{for } i \in [n-1], \\ \hat{\phi}(e_n) &= e_p, & \text{and} \\ \hat{\phi}(e_{\epsilon}) &= \bar{e}_{\epsilon} & \text{for } \epsilon \in E_0. \end{aligned}$$

Lemma 3.3 *The map $\hat{\phi} : \Lambda \rightarrow \Lambda'$ is an isomorphism.*

Proof: Keeping in mind that $\bar{e}_1 = \bar{e}_{\epsilon_0}$ in Λ' , we see that $\hat{\phi}$ has a well-defined inverse in degree one given by

$$\begin{aligned} \bar{e}_i &\mapsto e_i - e_1 + e_{\epsilon_0} & \text{for } 1 \leq i \leq n, \\ \bar{e}_{\epsilon} &\mapsto e_{\epsilon} & \text{for } \epsilon \in E_0, \text{ and} \\ e_p &\mapsto e_n. \end{aligned}$$

It follows that $\hat{\phi}$ is an isomorphism. \square

Lemma 3.4 $\hat{\phi}(I) \subseteq I'$.

Proof: If $\{\epsilon_1, \dots, \epsilon_q\}$ is a circuit of M_0 , then $\hat{\phi}(\partial e_{\epsilon_1} \cdots e_{\epsilon_q}) = \partial \bar{e}_{\epsilon_1} \cdots \bar{e}_{\epsilon_q}$. With the observation that $\hat{\phi}(e_i - e_{i+1}) = \bar{e}_i - \bar{e}_{i+1}$ for $1 \leq i \leq n-2$, and also for $i = n-1$, we calculate

$$\begin{aligned} \hat{\phi}(\partial e_1 \cdots e_n) &= \hat{\phi}((e_1 - e_2)(e_2 - e_3) \cdots (e_{n-1} - e_n)) \\ &= (\bar{e}_1 - \bar{e}_2)(\bar{e}_2 - \bar{e}_3) \cdots (\bar{e}_{n-1} - \bar{e}_n) \\ &= \partial \bar{e}_1 \cdots \bar{e}_n. \end{aligned}$$

Referring to the definitions of M and M' , we see that these computations suffice to prove the lemma. \square

Corollary 3.5 $\hat{\phi} : \Lambda \rightarrow \Lambda'$ induces a surjection $\phi : A \rightarrow A'$.

4. The Tutte polynomials

By the end of this section we will have proved Theorem 1.1. The final ingredient is the computation of Tutte polynomials. The Tutte polynomial $T_M(x, y)$ is defined recursively as follows. $M \setminus e$ and M/e refer to the deletion and contraction of M relative to e .

- (i) $T_M(x, y) = x$ if M is an isthmus; $T_M(x, y) = y$ if M is a loop.
- (ii) $T_M(x, y) = T_e(x, y)T_{M \setminus e}(x, y)$ if e is a loop or isthmus in M .
- (iii) $T_M(x, y) = T_{M \setminus e}(x, y) + T_{M/e}(x, y)$ otherwise.

These properties uniquely determine a polynomial $T_M(x, y)$ which is a matroid-isomorphism invariant of M .

We will use the following standard property of Tutte polynomials.

Lemma 4.1 $T_{M \oplus M'}(x, y) = T_M(x, y)T_{M'}(x, y)$.

The characteristic polynomial $\chi_M(t)$ of M may be defined by

$$\chi_M(t) = T_M(1 - t, 0).$$

The following result of [8] was the initial cause for interest in $A(M)$ among combinatorialists. We will use it to show that ϕ is injective.

Theorem 4.2 *The Hilbert series*

$$\sum_{p=0}^{\infty} \dim(A^p) t^p$$

of $A = A(M)$ is equal to $t^r \chi_M(-t^{-1})$, where $r = rk(M)$.

In fact the OS algebra is isomorphic to the Whitney homology of the lattice of flats of L , equipped with a natural product [1].

The next lemma is easy to prove by induction on n .

Lemma 4.3 For any $n \geq 2$, $T_{C_n}(x, y) = \sum_{i=1}^{n-1} x^i + y$.

Lemma 4.3 and Theorem 4.4 may be deduced from more general results proved in Section 6 of [3]. We include the proof of Theorem 4.4 here for the reader's convenience. Let M and M' be the matroids of the preceding section.

Theorem 4.4 *Let $n \geq 2$, Then*

$$T_M(x, y) = \left(\sum_{i=1}^{n-1} x^i + y \right) T_{M_0}(x, y), \text{ and}$$

$$T_{M'}(x, y) = \left(\sum_{i=1}^{n-1} x^i \right) T_{M_0}(x, y) + xy T_{M_0/\epsilon_0}(x, y).$$

Proof: The first formula is a consequence of Lemmas 4.1 and 4.3. To prove the second assertion, we establish a recursive formula for the Tutte polynomial of $P_{\epsilon_0}^n$. Assume $n \geq 3$, and apply property (iii) above to a point of C_n other than 1. The deletion is the direct sum of M_0 with $n - 2$ isthmuses, and the contraction is $P_{\epsilon_0}^{n-1}$. Thus we have

$$T_{P_{\epsilon_0}^n}(x, y) = x^{n-2} T_{M_0}(x, y) + T_{P_{\epsilon_0}^{n-1}}(x, y).$$

Now consider the case $n = 2$. Deleting the point $\bar{2}$ yields M_0 , while contracting $\bar{2}$ yields the direct sum of M_0/ϵ_0 with a loop. Thus

$$T_{P_{\epsilon_0}^2}(x, y) = T_{M_0}(x, y) + y T_{M_0/\epsilon_0}(x, y).$$

Then one can prove inductively that

$$T_{P_{\epsilon_0}^n}(x, y) = \left(\sum_{i=0}^{n-2} x^i \right) T_{M_0}(x, y) + y T_{M_0/\epsilon_0}.$$

Since M' is the direct sum of $P_{\epsilon_0}^n$ with an isthmus, right-hand side of this formula is multiplied by x to obtain $T_{M'}(x, y)$. \square

Corollary 4.5 $\chi_M(t) = \chi_{M'}(t)$.

Proof: The two formulas in Theorem 4.4 yield the same expression upon setting $y = 0$. The assertion then follows from the definition of $\chi_M(t)$ above. \square

Corollary 4.6 *The map $\phi : A \rightarrow A'$ is an isomorphism.*

Proof: According to Theorem 4.2, the last corollary implies $\dim A^p = \dim(A')^p$. Since ϕ is surjective by Corollary 3.5, and all spaces are finite-dimensional, ϕ must be an isomorphism. \square

In case $n = 3$ and $M_0 = C_3$, the map ϕ is a modified version of the isomorphism discovered by L. Rose and H. Terao [9, Example 3.77] for the rank three truncations of M_3 and M'_3 .

With the next result, we complete the proof of Theorem 1.1.

Corollary 4.7 *If M_0 is connected, then $T_M(x, y) \neq T_{M'}(x, y)$.*

Proof: Assume $T_M(x, y) = T_{M'}(x, y)$. By Theorem 4.4 this implies

$$T_{M_0}(x, y) = xT_{M_0/\epsilon_0}(x, y).$$

By hypothesis ϵ_0 is not an isthmus. Deleting and contracting along ϵ_0 , and evaluating at $(x, y) = (1, 1)$, we obtain

$$T_{M_0 \setminus \epsilon_0}(1, 1) + T_{M_0/\epsilon_0}(1, 1) = T_{M_0/\epsilon_0}(1, 1),$$

which implies $T_{M_0 \setminus \epsilon_0}(1, 1) = 0$. Coefficients of Tutte polynomials are non-negative, so this implies $T_{M_0 \setminus \epsilon_0}(x, y) = 0$, which is not possible. \square

The proof of the last corollary uses only the fact that ϵ_0 is not an isthmus. Thus Theorem 1.1 remains true for any simple matroid M_0 which is not the uniform matroid of rank m on m points (realized by the boolean arrangement of coordinate hyperplanes), in which every point is an isthmus.

Remark The proof of Corollary 4.7 specializes, upon setting $(x, y) = (1 - t, 0)$, to a proof of the result of H. Crapo that a connected matroid has nonzero beta invariant [11].

5. Concluding remarks

We start this section with a proof of Corollary 1.2. Let G_m be the graph with vertex set \mathbb{Z}_{2m} and edges $\{i, i + 1\}$ for $1 \leq i < 2m - 1$ and $\{0, i\}$ for $1 \leq i < 2m$. Then G_m has $2m$ vertices and $4m - 3$ edges. The graph G_4 is illustrated in figure 2.

Theorem 5.1 *Let $n > 2m + 1$. Then the parallel connections of G_m with C_n along the edges $\{0, i\}$ of G_m result in mutually non-isomorphic graphs, for $m \leq i \leq 2m - 1$.*

Proof: Fix i in the specified range. Then the parallel connection $P(C_n, G_m)$ along $\{1, i\}$ has longest circuit of length $(n - 1) + i$. The assertion follows. \square

Proof of Corollary 1.2: Let M_0 be the polygon matroid of G_m . The proof of Theorem 5.1 actually shows that the parallel connections P_i^n of M_0 with C_n along $\{1, i\}$ yield m non-isomorphic matroids as i ranges from m to $2m$. The same holds true when they are extended by an isthmus, resulting in m non-isomorphic matroids $M'_{n,i}$. But the proof of Corollary 4.6 did not depend on the choice of ϵ_0 . So the OS algebra of $M'_{n,i}$ is isomorphic to the OS algebra of $M_n = C_n \oplus M_0$ independent of i . This completes the proof of Corollary 1.2. \square

We close with some topological considerations. We will see that part of Theorem 1.1, in the case that M_0 is realizable over \mathbb{C} , is a consequence of a general topological equivalence. This equivalence follows from a well-known relationship between the complements of a central arrangement and its projective image. The proof is quite trivial, but requires us to introduce explicit realizations, with apologies for the cumbersome notation. We will need a few easy facts about hyperplane complements, which may be found in [9].

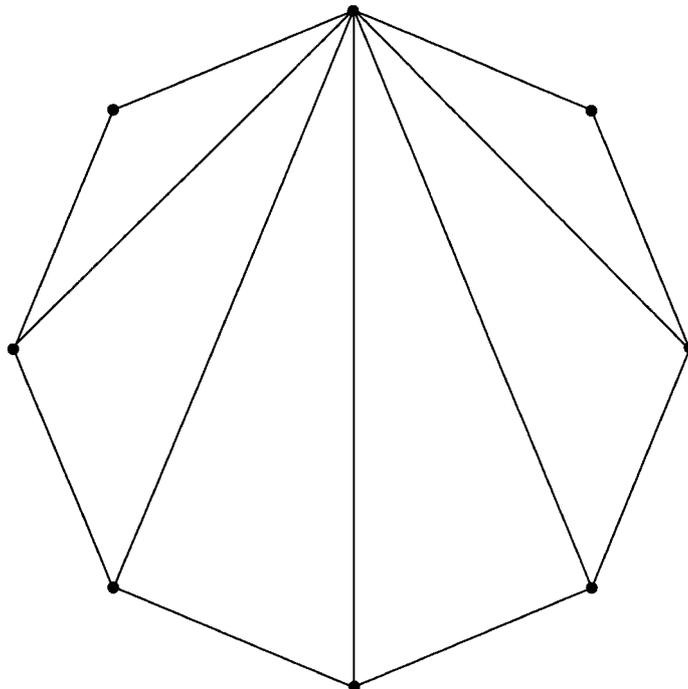


Figure 2. The graph G_4 .

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of affine hyperplanes in \mathbb{C}^r . Let $\phi_i : \mathbb{C}^r \rightarrow \mathbb{C}$ be a linear polynomial function with $H_i = \{x \in \mathbb{C}^r \mid \phi_i(x) = 0\}$. The *defining polynomial* of \mathcal{A} is the product $Q(\mathcal{A}) = \prod_{i=1}^n \phi_i$. If all of the ϕ_i are homogeneous linear forms, \mathcal{A} is said to be a *central arrangement*.

Recall $C(\mathcal{A})$ denotes the complement of $\bigcup \mathcal{A}$ in \mathbb{C}^r . The connection between central arrangements in \mathbb{C}^r and affine arrangements in \mathbb{C}^{r-1} goes as follows. Assume \mathcal{A} is central. Change variables so that $\phi_1(x) = x_1$, and write $Q(\mathcal{A})(x) = x_1 \hat{Q}(x_1, \dots, x_r)$. Consider (x_2, \dots, x_r) to be coordinates on \mathbb{C}^{r-1} . Then let $d\mathcal{A}$ denote the affine arrangement in \mathbb{C}^{r-1} with defining polynomial $\hat{Q}(1, x_2, \dots, x_r)$.

Lemma 5.2 $C(\mathcal{A})$ is diffeomorphic to $\mathbb{C}^* \times C(d\mathcal{A})$.

Suppose \mathcal{A}_0 and \mathcal{A}_1 are affine arrangements with defining polynomials $Q_0(x)$ and $Q_1(y)$ in disjoint sets of variables $x = (x_1, \dots, x_{r_0})$ and $y = (y_1, \dots, y_{r_1})$. Let $\mathcal{A}_0 \oplus \mathcal{A}_1$ be the arrangement in $\mathbb{C}^{r_0+r_1}$ with defining polynomial $Q_0(x)Q_1(y)$.

Lemma 5.3 $C(\mathcal{A}_0 \oplus \mathcal{A}_1)$ is diffeomorphic to $C(\mathcal{A}_0) \times C(\mathcal{A}_1)$.

If \mathcal{A}_0 and \mathcal{A}_1 are central arrangements, with underlying matroids M_0 and M_1 , then $\mathcal{A}_0 \oplus \mathcal{A}_1$ is a realization of the direct sum $M_0 \oplus M_1$.

Now let \mathcal{A}_0 and \mathcal{A}_1 be arbitrary central arrangements, with underlying matroids M_0 and M_1 . To realize the parallel connection $P(M_0, M_1)$ change coordinates so that the hyperplane $x_1 = 0$ appears in both \mathcal{A}_0 and \mathcal{A}_1 . These will be the hyperplanes that get identified in the parallel connection. Write

$$Q_1(y) = y_1 \hat{Q}_1(y_1, \dots, y_{r_1}).$$

With $(x_1, \dots, x_{r_0}, y_2, \dots, y_{r_1})$ as coordinates in $\mathbb{C}^{r_0+r_1-1}$, the parallel connection $P(\mathcal{A}_0, \mathcal{A}_1)$ is the arrangement in $\mathbb{C}^{r_0+r_1-1}$ defined by the polynomial

$$Q_0(x_1, \dots, x_{r_0}) \hat{Q}_1(x_1, y_2, \dots, y_{r_1})$$

We are now prepared to prove Theorem 1.3. Let \mathcal{S} denote the arrangement in \mathbb{C}^1 with defining polynomial x . So \mathcal{S} has as underlying matroid the isthmus S , and $C(\mathcal{S}) = \mathbb{C}^*$.

Proof of Theorem 1.3: Write $Q_0(x) = x_1 \hat{Q}_0(x_1, \dots, x_{r_0})$. Following the recipe given above for dehomogenizing an arrangement, and using the given defining polynomial for $P(\mathcal{A}_0, \mathcal{A}_1)$, we see that the affine arrangement $\overline{P(\mathcal{A}_0, \mathcal{A}_1)}$ has defining polynomial

$$\hat{Q}_0(1, x_2, \dots, x_{r_0}) \hat{Q}_1(1, y_2, \dots, y_{r_1}),$$

which is precisely the defining polynomial of $d\mathcal{A}_0 \oplus d\mathcal{A}_1$. By the preceding lemmas we have

$$\begin{aligned} C(\mathcal{S} \oplus P(\mathcal{A}_0, \mathcal{A}_1)) &\cong C(\mathcal{S}) \times C(P(\mathcal{A}_0, \mathcal{A}_1)) \\ &\cong \mathbb{C}^* \times \mathbb{C}^* \times C(\overline{P(\mathcal{A}_0, \mathcal{A}_1)}) \\ &\cong \mathbb{C}^* \times \mathbb{C}^* \times C(d\mathcal{A}_0) \times C(d\mathcal{A}_1). \end{aligned}$$

On the other hand,

$$C(\mathcal{A}_0 \oplus \mathcal{A}_1) \cong C(\mathcal{A}_0) \times C(\mathcal{A}_1) \cong \mathbb{C}^* \times C(d\mathcal{A}_0) \times \mathbb{C}^* \times C(d\mathcal{A}_1).$$

This proves the result. □

Returning to Theorem 1.1, if we assume the matroid M_0 is realizable over \mathbb{C} , we can take such a realization for \mathcal{A}_0 , and any general position arrangement of n hyperplanes in \mathbb{C}^{n-1} for \mathcal{A}_1 . Then Theorems 1.3 and 3.2 together imply that the OS algebras $A(M_n)$ and $A(M'_n)$ over \mathbb{C} are isomorphic.

The arrangements constructed in [5] are generic 3-dimensional sections of the arrangements of Theorem 1.3, with the seed arrangements \mathcal{A}_0 and \mathcal{A}_1 both of rank two. The fact that their fundamental groups are isomorphic is then an immediate consequence of the Lefschetz Hyperplane Theorem. This theorem does not imply that the sections are homotopy equivalent; this is proved in [5] by constructing an explicit isomorphism of canonical presentations of the fundamental groups, using Tietze transformations only of type I and II. We do not know if the diffeomorphic arrangements constructed in Theorem 1.3 will in general have homotopy equivalent generic 3-dimensional sections.

The question whether arrangements with different combinatorial structure could have homotopy equivalent complements was originally restricted to central arrangements because Lemma 5.2 provides trivial counter-examples in the affine case. The constructions presented in this paper are now seen from the proof of Theorem 1.3 to arise again from Lemma 5.2. So the examples of [5] also come about in some sense from Lemma 5.2. We feel compelled to again narrow the problem to rule out these other, not quite so trivial counter-examples.

Conjecture 5.4 *For central arrangements whose underlying matroid is connected and not erectible, the homotopy type of the complement determines the underlying matroid.*

Acknowledgments

This paper was completed during the second author's visit to the Mathematical Sciences Research Institute in Berkeley in the early spring of 1997. He is grateful to the staff and organizers for their support. He also thanks Richard Stanley for the last step in the proof of Corollary 4.7. We are both thankful to Terry Blows for his work in administering the REU program at Northern Arizona University during which this research was done. Finally we thank the referee for pointing us toward the reference [3].

References

1. A. Björner, "Topological methods," *Handbook of Combinatorics*, Elsevier, Amsterdam, 1995, pp. 1819–1872.
2. E. Brieskorn, *Sur les groupes de tresses*, volume 317 of *Lecture Notes in Mathematics*, Springer Verlag, Berlin, Heidelberg, New York, 1973, pp. 21–44.
3. T.H. Brylawski, "A combinatorial model for series-parallel networks," *Transactions of the American Mathematical Society* **154** (1971), 1–22.
4. M. Falk, "On the algebra associated with a geometric lattice," *Advances in Mathematics* **80** (1989), 152–163.
5. M. Falk, "Homotopy, types of line arrangements," *Inventiones Mathematicae* **111** (1993), 139–150.
6. M. Falk, "Arrangements and cohomology," *Annals of Combinatorics* **1** (1997), 135–157.
7. Tan Jiang and Stephen S.-T. Yau, "Topological invariance of intersection lattices of arrangements in \mathbf{CP}^2 ," *Bulletin of the American Mathematical Society* **29** (1993), 88–93.
8. P. Orlik and L. Solomon, "Topology and combinatorics of complements of hyperplanes," *Inventiones Mathematicae* **56** (1980), 167–189.
9. P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Springer Verlag, Berlin, Heidelberg, New York, 1992.
10. J. Oxley, *Matroid Theory*, Oxford University Press, Oxford, New York, Tokyo, 1992.
11. N. White (Ed.), *Theory of Matroids*, Cambridge University Press, Cambridge, 1986.