APPROXIMATING THE FIBONACCI SEQUENCE

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Abstract

We describe a new identity involving sums of powers of Fibonacci numbers and use this identity to prove that a certain family of combinatorial sequences converges, pointwise, to the Fibonacci sequence.

1. Introduction

We let F represent the Fibonacci sequence where $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1}+F_{n-2}$, and $F_{-n} = (-1)^n F_n$ for $n \in \mathbb{N}$. We then have $F_n = F_{n-1} + F_{n-2}$ for all $n \in \mathbb{Z}$. Our first main result is the following identity.

Theorem 1. *For all $m \in \mathbb{Z}$ and $k \in \mathbb{N}$,*

$$\sum_{i=1}^{k+1} \left(F_{m-1} + (-1)^{k+1-i} \cdot F_{m-k+i-3}\right) \cdot \left(\frac{F_{m+1}}{F_m}\right)^i = F_{m+1} \cdot (F_{k+3} - 1).$$

If we clear denominators, the identity becomes

$$\sum_{i=1}^{k+1} \left(F_{m-1} + (-1)^{k+1-i} \cdot F_{m-k+i-3}\right) \cdot \left(F_{m+1}^i F_m^{k+1-i}\right) = F_m^{k+1} F_{m+1} \cdot (F_{k+3} - 1).$$

We could not find a similar or related identity in the literature, so this appears to be new. The closest identity we could find is the amazing four-parameter identity

$$F_m^k F_n = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_r^h F_{r+m}^{k-h} F_{n+kr+hm},$$

which can be used to produce many interesting known identities (see [5]).

We discovered the identity in Theorem 1 while studying rational base representations of natural numbers (see [1], [8], [3], [4], [6] or [2] for instance), which explains why the identity involves powers of $\frac{F_{m+1}}{F_m}$. While these representations are quite complex from a language point of view, there is an elementary construction of an edge-labeled, infinite, rooted tree whose edge labels give the rational base representation of the integer associated to each vertex (see [1], [7] or [2]). It turns out that when using the rational base $\frac{F_{m+1}}{F_m}$, the number of nodes lying distance n from the root in the associated tree is given by the sequence A^m with $A_1^m = 1$ and

$$A_{n+1}^m = \left\lceil \frac{F_{m+1} - F_m}{F_m} \cdot \sum_{i=1}^{n} A_i^m \right\rceil = \left\lceil \frac{F_{m-1}}{F_m} \cdot \sum_{i=1}^{n} A_i^m \right\rceil \tag{1}$$

where $\lceil x \rceil$ represents the least integer larger than x (see [1] or [2]).

Interestingly, as m gets larger, the family of sequences $\{A^m\}$ converges pointwise to the Fibonacci sequence F. More precisely, we have the following theorem.

Theorem 2. *Let* $\{A^m \mid m \geq 1\}$ *be the family of sequences defined in* (1)*. For every* $n \in \mathbb{N}$ *with* $n \geq 1$*, there exists* $M \in \mathbb{N}$ *such that* $A_n^m = F_n$ *for all* $m \geq M$*.*

Thus, we have produced a family of sequences (with combinatorial interest) that can match the Fibonacci sequence for as many terms as we wish. Figure 1 shows the first 15 terms of the sequences A^m where $m \in \{1, \ldots, 10\}$. The numbers in blue represent coincidence with F. Note that A^{10} matches the Fibonacci sequence up to $n = 15$ (in fact $n = 19$ is the first index with $A_n^{10} \neq F_n$).

We also note that since $\frac{F_{m-1}}{F_m} \to \frac{1}{\phi}$ as $m \to \infty$ (where ϕ represents the golden

$A^m \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A^1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
A^2	1	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	8192
A^3	1	1	1	2	3	4	6	9	14	21	31	47	70	105	158
A^4	1	1	2	3	5	8	14	23	38	64	106	177	295	492	820
A^5	1	1	2	3	5	8	12	20	32	51	81	130	208	333	533
A^6	1	1	2	3	5	8	13	21	34	55	90	146	237	385	626
A^7	1	1	2	3	5	8	13	21	34	55	88	143	231	373	602
A^8	1	1	2	3	5	8	13	21	34	55	89	144	233	377	611
A^9	1	1	2	3	5	8	13	21	34	55	89	144	233	377	609
A^{10}	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

Figure 1: The first 15 terms of the sequences A^m for $m \in \{1, ..., 10\}$. For instance, see A000007, A011782, and A073941 in [9].

ratio, $\phi = \frac{1+\sqrt{5}}{2}$), Theorem 2 implies that

$$F_{n+1} = \left\lceil \frac{1}{\phi} \cdot \sum_{i=1}^{n} F_i \right\rceil \tag{2}$$

with $F_0 = 0$ and $F_1 = 1$. While we could not find a citation for this formula, it must be known as it follows from well known facts. We know that $F_{n+2} = \text{round}(\phi \cdot F_{n+1})$ so that $\frac{1}{\phi}F_{n+2} - \frac{1}{2\phi} < F_{n+1} < \frac{1}{\phi}F_{n+2} + \frac{1}{2\phi}$, which implies $\frac{1}{\phi}F_{n+2} < F_{n+1} + \frac{1}{2\phi}$ and $F_{n+1} - \frac{1}{2\phi} < \frac{1}{\phi}F_{n+2}$. Thus

$$F_{n+1} - 1 < F_{n+1} - \frac{3}{2\phi} < \frac{1}{\phi}F_{n+2} - \frac{1}{\phi} < F_{n+1} - \frac{1}{2\phi} < F_{n+1}$$

so that

$$\left\lceil \frac{1}{\phi} \cdot \sum_{i=1}^{n} F_i \right\rceil = \left\lceil \frac{1}{\phi} \cdot (F_{n+2} - 1) \right\rceil = F_{n+1}$$

where the first equality is the well known formula for the sum of the first n Fibonacci numbers.

This note is organized as follows. In Section 2, we prove Theorem 1 using elementary techniques. In Section 3, we introduce the terminology of $\frac{p}{q}$-representations of natural numbers and state results from [1] in order to prove Theorem 2.

2. Proof of Theorem 1

To prove Theorem 1, we let $m \in \mathbb{Z}$ and use induction on k. For ease of notation, we define $\mathbf{y}_m := \frac{F_{m+1}}{F_m}$. We can check that the identity holds for $k = 0$ and $k = 1$. Indeed, we have (since $F_3 - 1 = 2 - 1 = 1$):

$$\sum_{i=1}^{1} \left(F_{m-1} + (-1)^{1-i} \cdot F_{m+i-3}\right) \cdot \mathbf{y}_m^i = (F_{m-1} + F_{m-2})\mathbf{y}_m = F_{m+1} \cdot (F_3 - 1), \text{ and}$$

$$\begin{aligned}\sum_{i=1}^{2} \left(F_{m-1} + (-1)^{2-i} \cdot F_{m+i-4}\right) \cdot \mathbf{y}_m^i &= (F_{m-1} - F_{m-3})\mathbf{y}_m + (F_{m-1} + F_{m-2})\mathbf{y}_m^2 \\ &= F_{m-2} \cdot \mathbf{y}_m + F_m \cdot \mathbf{y}_m^2 \\ &= \frac{F_{m+1}(F_{m-2} + F_{m+1})}{F_m} \\ &= \frac{F_{m+1}(F_m - F_{m-1} + F_m + F_{m-1})}{F_m} \\ &= F_{m+1} \cdot 2 = F_{m+1} \cdot (F_4 - 1).\end{aligned}$$

Now, let $k \in \mathbb{N}$ with $k \geq 1$ and assume that the identity holds for $j \in \{k-1, k\}$.

Notice that

$$F_{m+1}(F_{k+4}-1) = F_{m+1} + \underbrace{F_{m+1}(F_{k+3}-1)}_{A} + \underbrace{F_{m+1}(F_{k+2}-1)}_{B}.$$

Applying the inductive hypothesis to the quantities A and B in the previous equality yields

$$A = \sum_{i=1}^{k+1}(F_{m-1} + (-1)^{k+1-i}F_{m-k+i-3})\mathbf{y}_m^i$$

$$B = \sum_{i=1}^{k}(F_{m-1} + (-1)^{k-i}F_{m-k+i-2})\mathbf{y}_m^i$$

so that

$$A+B = (F_{m-1}+F_{m-2})\cdot\mathbf{y}_m^{k+1} + \sum_{i=1}^{k}(2F_{m-1} + (-1)^{k-i}(F_{m-k+i-2} - F_{m-k+i-3}))\mathbf{y}_m^i.$$

Rearranging sums and applying the Fibonacci identity leaves us with

$$A+B = F_{m-2}\mathbf{y}_m^{k+1} + \underbrace{F_{m-1}\mathbf{y}_m^{k+1} + \sum_{i=1}^{k}F_{m-1}\mathbf{y}_m^i}_{C} + \underbrace{\sum_{i=1}^{k}(F_{m-1} + (-1)^{k-i}F_{m-k+i-4})\mathbf{y}_m^i}_{D}.$$

In the expression above, since $F_{m+1} - F_m = F_{m-1}$, we know that

$$\begin{aligned} C = F_{m-1}\sum_{i=1}^{k+1}\mathbf{y}_m^i = F_{m-1}\cdot\left(\frac{\mathbf{y}_m^{k+2}-1}{\mathbf{y}_m - 1} - 1\right) &= F_{m-1}\cdot\left(\frac{F_m\mathbf{y}_m^{k+2} - F_m}{F_{m+1}-F_m} - 1\right) \\ &= F_m\mathbf{y}_m^{k+2} - F_m - F_{m-1}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} F_{m+1}(F_{k+4}-1) &= F_{m+1} + F_{m-2}\mathbf{y}_m^{k+1} + F_m\mathbf{y}_m^{k+2} - F_m - F_{m-1} + D \\ &= F_{m-2}\mathbf{y}_m^{k+1} + F_m\mathbf{y}_m^{k+2} + D \end{aligned}$$

since $F_{m+1} - F_m - F_{m-1} = 0$. Next, $F_{m-2} = F_{m-1} - F_{m-3}$ and $F_m = F_{m-1} + F_{m-2}$, so that

$$\begin{aligned} F_{m+1}(F_{k+4}-1) &= (F_{m-1} - F_{m-3})\mathbf{y}_m^{k+1} + (F_{m-1} + F_{m-2})\mathbf{y}_m^{k+2} + D \\ &= \sum_{i=1}^{k+2}(F_{m-1} + (-1)^{k-i}F_{m-k+i-4})\mathbf{y}_m^i \\ &= \sum_{i=1}^{k+2}(F_{m-1} + (-1)^{k+2-i}F_{m-(k+1)+i-3})\mathbf{y}_m^i, \end{aligned}$$

as required. □

3. $\frac{p}{q}$-representations

For this section, we fix $p, q \in \mathbb{N}$ such that $p > q \geq 1$ and $\gcd(p, q) = 1$. For any $n \in \mathbb{N}$, we say $(n_0, n_1, \ldots, n_k)_{\frac{p}{q}}$ is a $\frac{p}{q}$*-representation* for n if $0 \leq n_i < p$ for all i and $n = \sum_{i=0}^{k} n_i \left(\frac{p}{q}\right)^i$; in this case we write $n = (n_0, n_1, \ldots, n_k)_{\frac{p}{q}}$. We note that, unlike base-b representations (with $b > 1$ an integer), not every string of digits, $(d_0, d_1, \ldots, d_k)_{\frac{p}{q}}$, yields a natural number. However, it is known from [1] (and earlier, see A024629 in [9] for instance) that every natural number n has a unique $\frac{p}{q}$-representation. Hence we can define $\mathrm{len}_{\frac{p}{q}}(n) = k+1$ when $n = (n_0, n_1, \ldots, n_k)_{\frac{p}{q}}$. If the length of $n+1$ is larger than the length of n, i.e., $n \in \mathbb{N}$ satisfies $\mathrm{len}_{\frac{p}{q}}(n+1) - \mathrm{len}_{\frac{p}{q}}(n) = 1$, we say $n+1$ is *new-length element* (or *nl-element* for short).

Many properties of $\frac{p}{q}$-representations (and related representations) are studied in [1] and [3], where the authors define an infinite, rooted tree, called $I_{p/q}$ that describes the $\frac{p}{q}$-representations. A combinatorial construction of this tree is also given in [2] or [7]. In that tree, the nl-elements correspond to the nodes with the least label of any fixed distance from the root; these lie on the left branch of the tree when drawn as in [1].

To prove Theorem 2, we need the following results about $\frac{p}{q}$-representations. We omit the proofs as these may be found in, or are straightforward consequences of, Proposition 21 and Corollary 23 in [1], though our terminology differs.

Lemma 1. *Let n be a natural number with $n = (n_0, n_1, \ldots, n_k)_{\frac{p}{q}}$. Then n is an nl-element if and only if $n = 1$ or $n_0 = 0$, $0 \leq n_i < q$ for $1 \leq i \leq k-1$ and $n_k = q$.*

Next, let $g : \mathbb{N} \to \mathbb{N}$ be defined by $g(n) = p \left\lceil \frac{n}{q} \right\rceil$.

Proposition 1. *The sequence $(\mathcal{K}_1, \mathcal{K}_2, \ldots)$ of nl-elements is given by $\mathcal{K}_1 = 1$ and $\mathcal{K}_i = g(\mathcal{K}_{i-1})$ for all $i > 1$.*

Corollary 1. *For $k > 1$, the number of natural numbers with $\frac{p}{q}$-representations of length k is given by $\mathcal{K}_{k+1} - \mathcal{K}_k$. There are $\mathcal{K}_2 = p$ such representations of length 1 (this includes the natural number 0).*

Corollary 2. *Let $(\mathcal{K}_1, \mathcal{K}_2, \ldots)$ be the sequence of nl-elements. Then for $k \geq 2$, $\mathcal{K}_{k+1} - \mathcal{K}_k = pa_k$ where $a_1 = 1$ and*

$$a_{n+1} = \left\lceil \frac{(p-q)}{q} \cdot \sum_{i=1}^{n} a_i \right\rceil .$$

3.1. Rational Fibonacci Representations

Fix $m > 1$. By definition, we have $F_{m+1} > F_m$, and it is well known that $\gcd(F_{m+1}, F_m) = 1$. Consequently, we can consider $\frac{p}{q}$-representations where $p =$

F_{m+1} and $q = F_m$. For the remainder of this section, we let $p = F_{m+1}$ and $q = F_m$ and call the associated $\frac{p}{q}$-representations simply $F_{\mathbf{m}}$-representations. The following lemma allows us to prove Theorem 2.

Lemma 2. *Let $m, k \in \mathbb{N}$ with $k < m-2$. The $F_{\mathbf{m}}$-representation of $F_{m+1}(F_{k+3}-1)$ is given by $(n_0, n_1, \ldots, n_{k+1})_{\frac{p}{q}}$ where $n_0 = 0$, and*

$$n_i := F_{m-1} + (-1)^{k+1-i} F_{m-k+i-3}$$

for each $1 \le i \le k+1$. Furthermore $\mathcal{K}_{k+2} = F_{m+1}(F_{k+3} - 1)$.

Proof. Let $n = (n_0, n_1, \ldots, n_{k+1})_{\frac{p}{q}}$. First, we note that $n_0 = 0$ and we check that $n_{k+1} = F_{m-1} + F_{m-2} = F_m$. Also, since $k < m - 2$ and $i \le k+1$ we have $0 \le F_{m-k+i-3} \le F_{m-2}$. Thus, we see that

$$0 \le F_{m-1} - F_{m-k+i-3} \le n_i \le F_{m-1} + F_{m-k+i-3} < F_{m-1} + F_{m-2} = F_m$$

for $1 \le i \le k$. According to Lemma 1, $n = \mathcal{K}_{k+2}$, and Theorem 2 implies that $n = F_{m+1}(F_{k+3} - 1)$. □

Proof of Theorem 2. Let $n \in \mathbb{N}$ with $n \ge 1$. Then, choose $M = n + 3$. Then for any $m \ge M$ consider the $F_{\mathbf{m}}$-representations of natural numbers and the associated sequence of nl-elements. According to Lemma 2, we have $\mathcal{K}_{k+2} = F_{m+1}(F_{k+3} - 1)$ for all $1 \le k \le n$. In particular, we have

$$\begin{aligned}\mathcal{K}_{n+1} - \mathcal{K}_n &= F_{m+1}(F_{n+2} - 1) - F_{m+1}(F_{n+1} - 1)\\ &= F_{m+1}(F_{n+2} - F_{n+1}) = F_{m+1}F_n.\end{aligned}$$

Furthermore, by Corollary 2, we have

$$\mathcal{K}_{n+1} - \mathcal{K}_n = F_{m+1}A_n^m$$

where A^m is defined in equation (1). Since $F_{m+1} \neq 0$, we have $F_n = A_n^m$. By definition $A_1^m = F_1$ and so the result holds. □

Therefore, the family of sequences $\{A^m\}$ converges pointwise to F. Moreover, by Corollary 1 and Corollary 2, we see that A_k^m counts the number of multiples of $p = F_{m+1}$ having $F_{\mathbf{m}}$-representations of length k, giving these sequences a combinatorial interpretation. Moreover, in terms of the tree $I_{p/q}$ defined in [1], the sequence A^m gives the number of vertices at fixed distances from the root. Finally, it can be checked (using methods similar to those describing equation 2 in the introduction) that $A^m \neq F$ for all m.

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