THE DIRECT AND INVERSE THEOREMS ON INTEGER SUBSEQUENCE SUMS REVISITED

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Abstract

Let $A = (\underbrace{a_0, \ldots, a_0}_{r \text{ copies}}, \underbrace{a_1, \ldots, a_1}_{r \text{ copies}}, \ldots, \underbrace{a_{k-1}, \ldots, a_{k-1}}_{r \text{ copies}})$ be a finite sequence of integers, where $a_0 < a_1 < \cdots < a_{k-1}$, $k \geq 1$ and $r \geq 1$. Given a subsequence, the sum of all the terms of the subsequence is called the *subsequence sum*. The set of all nontrivial subsequence sums of A is denoted by $S(r, \mathcal{A})$, where $\mathcal{A} = \{a_0, a_1, \ldots, a_{k-1}\}$ is the set of distinct terms of the sequence A, called the associated set of the sequence A. For $r = 1$, this sumset is the usual sumset $S(\mathcal{A})$ of nontrivial subet sums of $\mathcal{A}$. The direct problem for the sumset $S(r, \mathcal{A})$ is to find a lower bound for $|S(r, \mathcal{A})|$ in terms of $|\mathcal{A}|$ and r. The inverse problem for $S(r, \mathcal{A})$ is to determine the structure of the finite set $\mathcal{A}$ of integers for which $|S(r, \mathcal{A})|$ is minimal. In this paper, we give new proofs of existing direct and inverse theorems for $S(r, \mathcal{A})$ using the direct and inverse theorems of Nathanson for $S(\mathcal{A})$.

1. Introduction

Let $A = (\underbrace{a_0, \ldots, a_0}_{r \text{ copies}}, \underbrace{a_1, \ldots, a_1}_{r \text{ copies}}, \ldots, \underbrace{a_{k-1}, \ldots, a_{k-1}}_{r \text{ copies}})$ be a finite sequence of integers, where $a_0 < a_1 < \cdots < a_{k-1}$, $k \geq 1$ and $r \geq 1$. The set $\mathcal{A} = \{a_0, a_1, \ldots, a_{k-1}\}$ of all distinct terms of the sequence A is called the *associated set* of the sequence A. Since r is fixed, we shall always identify the sequence A by the associated set $\mathcal{A}$. Given a subsequence, the sum of all the terms of the subsequence is called the *subsequence*

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sum. The set of all nontrivial subsequence sums of A is denoted by $S(r,\mathcal{A})$. Thus

$$S(r,\mathcal{A}) := \left\{ \sum_{i=0}^{k-1} r_i a_i : 0 \le r_i \le r \text{ for } i = 0, 1, \ldots, k-1 \text{ and } \sum_{i=0}^{k-1} r_i \ge 1 \right\}.$$

The *direct problem* for the sumset $S(r,\mathcal{A})$ is to find a lower bound for $|S(r,\mathcal{A})|$ in terms of $|\mathcal{A}|$ and r. The *inverse problem* for $S(r,\mathcal{A})$ is to determine the structure of the finite set $\mathcal{A}$ of integers for which $|S(r,\mathcal{A})|$ is minimal. The case $r = 1$ corresponds to the usual sumset $S(\mathcal{A})$ of nontrivial subset sums. The direct and inverse problems for the sumset $S(r,\mathcal{A})$ have been studied by the authors (see [1]) using the notation $S(r,A)$ instead of $S(r,\mathcal{A})$. Those proofs are constructive. Here, we give small and new proofs of these results using the direct and inverse theorems for subset sums $S(\mathcal{A})$ due to Nathanson [2]. Some more direct and inverse theorems for the general sequence A may be found in [1].

In Section 2, we study the direct problem and in Section 3, we study the inverse problem. We agree with the convention that $\binom{a}{b} = 0$ if a and b are two positive integers with $a < b$.

2. Direct Problem

The following theorems are the direct theorems.

Theorem 1. *(See [1], Theorem 2.1.) Let $k \ge 1$ and $r \ge 1$. Let $\mathcal{A}$ be a set of k positive (negative) integers. Then*

$$|S(r,\mathcal{A})| \ge r\binom{k+1}{2}. \tag{1}$$

Let $\mathcal{A}$ be a set of k nonnegative (nonpositive) integers and $0 \in \mathcal{A}$. Then

$$|S(r,\mathcal{A})| \ge 1 + r\binom{k}{2}. \tag{2}$$

The lower bounds in (1) and (2) are best possible.

Theorem 2. *(See [1], Theorem 2.2.) Let $k \ge 2$ and $r \ge 1$. Let $\mathcal{A}$ be a set of k integers. If $0 \in \mathcal{A}$, then*

$$|S(r,\mathcal{A})| \ge \begin{cases} \frac{r(k^2-1)}{4} + 1 & \text{if } k \equiv 1 \pmod 2, \\ \frac{rk^2}{4} + 1 & \text{if } k \equiv 0 \pmod 2. \end{cases} \tag{3}$$

If $0 \notin \mathcal{A}$, then

$$|S(r,\mathcal{A})| \ge \begin{cases} r(\frac{k+1}{2})^2 + 1 & \text{if } k \equiv 1 \pmod 2, \\ r\frac{(k+1)^2-1}{4} + 1 & \text{if } k \equiv 0 \pmod 2. \end{cases} \tag{4}$$

The lower bounds in (3) and (4) are best possible.

For the proofs of these theorems, we need the following well-known results.

Theorem A. *(See [3], Theorem 1.4.) Let $h \geq 2$. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_h$ be nonempty finite sets of integers. Then*

$$|\mathcal{A}_1 + \mathcal{A}_2 + \cdots + \mathcal{A}_h| \geq |\mathcal{A}_1| + |\mathcal{A}_2| + \cdots + |\mathcal{A}_h| - h + 1.$$

Theorem B. *(See [2], Theorem 3.) Let $k \geq 2$. If $\mathcal{A}$ is a set of k positive (negative) integers, then*

$$|S(\mathcal{A})| \geq \binom{k+1}{2}. \tag{5}$$

If $\mathcal{A}$ is a set of k nonnegative (nonpositive) integers and $0 \in \mathcal{A}$, then

$$|S(\mathcal{A})| \geq 1 + \binom{k}{2}. \tag{6}$$

The lower bounds in (5) and (6) are best possible.

Theorem C. *(See [2], Theorem 4.) Let $k \geq 2$ and let $\mathcal{A}$ be a set of k integers. If $0 \in \mathcal{A}$, then*

$$|S(\mathcal{A})| \geq \begin{cases} \frac{k^2-1}{4} + 1 & \text{if } k \equiv 1 \pmod 2, \\ \frac{k^2}{4} + 1 & \text{if } k \equiv 0 \pmod 2. \end{cases} \tag{7}$$

If $0 \notin \mathcal{A}$, then

$$|S(\mathcal{A})| \geq \begin{cases} (\frac{k+1}{2})^2 + 1 & \text{if } k \equiv 1 \pmod 2, \\ \frac{(k+1)^2-1}{4} + 1 & \text{if } k \equiv 0 \pmod 2. \end{cases} \tag{8}$$

The lower bounds in (7) and (8) are best possible.

We also require the following simple lemma.

Lemma 1. *Let $k \geq 1$ and $r \geq 1$. If $\mathcal{A}$ is a set of k integers, then*

$$|S(r, \mathcal{A})| \geq r|S(\mathcal{A})| - r + 1. \tag{9}$$

If $\mathcal{A}$ is a set of k positive (negative) integers, then

$$|S(r, \mathcal{A})| \geq r|S(\mathcal{A})|. \tag{10}$$

Proof. For $r = 1$, inequality (9) is obvious. Now assume that $r \geq 2$. If $\mathcal{A}$ is a set of k integers, then clearly

$$S(r, \mathcal{A}) \supseteq rS(\mathcal{A}),$$

and so by Theorem A, we have

$$|S(r, \mathcal{A})| \geq r|S(\mathcal{A})| - r + 1,$$

which proves (9). Now, let $\mathcal{A}$ be a set of k positive integers (the proof is similar if $\mathcal{A}$ is a set of negative integers). Clearly, inequality (10) is true for $r = 1$. Let the inequality (10) be true for $r - 1$, where $r \geq 2$. Now

$$S(r, \mathcal{A}) \supseteq (S(r-1, \mathcal{A}) + S(\mathcal{A})) \cup \{a\},$$

where a is the smallest element of $\mathcal{A}$. Since $a \notin S(r-1, \mathcal{A}) + S(\mathcal{A})$ as $0 \notin S(r-1, \mathcal{A})$, we have

$$|S(r, \mathcal{A})| \geq |S(r-1, \mathcal{A}) + S(\mathcal{A})| + 1.$$

By Theorem A, we have

$$|S(r, \mathcal{A})| \geq |S(r-1, \mathcal{A})| + |S(\mathcal{A})| - 1 + 1,$$

and so by induction hypothesis,

$$|S(r, \mathcal{A})| \geq (r-1)|S(\mathcal{A})| + |S(\mathcal{A})| = r|S(\mathcal{A})|.$$

This completes the proof. □

For integers a and b, let $[a, b] = \{n \in \mathbb{Z} : a \leq n \leq b\}$.

Proof of Theorem 1. If $k = 1$ and $r \geq 1$, then $\mathcal{A} = \{a\}$ for some integer a. If $a \neq 0$, then $|S(r, \mathcal{A})| = r = r\binom{1+1}{2}$. If $a = 0$, then $|S(r, \mathcal{A})| = 1 = 1 + r\binom{1}{2}$. Therefore, the theorem is true for $k = 1$ and $r \geq 1$. For $k \geq 2$ and $r = 1$, the result follows from Theorem B. Hence we may assume that $k \geq 2$ and $r \geq 2$. First assume that $\mathcal{A}$ is a set of k positive (negative) integers. By inequality (10) of Lemma 1, we have

$$|S(r, \mathcal{A})| \geq r|S(\mathcal{A})|,$$

and so by Theorem B,

$$|S(r, \mathcal{A})| \geq r\binom{k+1}{2}.$$

Thus we have proved inequality (1). Now assume that $\mathcal{A}$ is a set of k nonnegative (nonpositive) integers and $0 \in \mathcal{A}$. Then by inequality (9) of Lemma 1, we have

$$|S(r, \mathcal{A})| \geq r|S(\mathcal{A})| - r + 1,$$

and so by Theorem B,

$$|S(r, \mathcal{A})| \geq 1 + r\binom{k}{2}.$$

Thus we have proved inequality (2). Next we show that the lower bounds in (1) and (2) are best possible. Let $k \geq 2$ and $r \geq 1$. Let $\mathcal{A}_0 = [1, k]$. Then the smallest and the largest integers in $S(r, \mathcal{A}_0)$ are 1 and $r\binom{k+1}{2}$, respectively. Therefore,

$$|S(r, \mathcal{A}_0)| \leq r\binom{k+1}{2}.$$

This inequality together with inequality (1) implies

$$|S(r, \mathcal{A}_0)| = r\binom{k+1}{2}.$$

Thus the lower bound in (1) is best possible. Similarly, by considering the set $\mathcal{A}_1 = [0, k-1]$, where $k \geq 2$ and $r \geq 1$, it can be verified that the lower bound in (2) is best possible. This completes the proof. $\square$

Proof of Theorem 2. For $r = 1$, the result follows from Theorem C. Let $r \geq 2$. First assume that $\mathcal{A}$ is a set of k integers such that $0 \in \mathcal{A}$. Then by inequality (9) of Lemma 1, we have

$$|S(r, \mathcal{A})| \geq r|S(\mathcal{A})| - r + 1,$$

and so by Theorem C, we have

$$|S(r, \mathcal{A})| \geq \begin{cases} \frac{r(k^2-1)}{4} + 1 & \text{if } k \equiv 1 \pmod{2}, \\ \frac{rk^2}{4} + 1 & \text{if } k \equiv 0 \pmod{2}. \end{cases}$$

Thus we have proved inequality (3). The proof is similar for the case $0 \notin \mathcal{A}$. If $k \equiv 1 \pmod{2}$, then by considering the sets $\mathcal{A}_0 = [-\frac{k-1}{2}, \frac{k-1}{2}]$ and $\mathcal{A}_1 = [-\frac{k-1}{2}, \frac{k+1}{2}] \setminus \{0\}$, it can be verified that the lower bounds in (3) and (4), respectively, are best possible. If $k \equiv 0 \pmod{2}$, then by considering the sets $\mathcal{A}_2 = [-\frac{k}{2}, \frac{k}{2} - 1]$ and $\mathcal{A}_3 = [-\frac{k}{2}, \frac{k}{2}] \setminus \{0\}$, it can be verified that the lower bounds in (3) and (4), respectively, are best possible. This completes the proof. $\square$

3. Inverse Problem

For a set $\mathcal{A} \subseteq \mathbb{Z}$ and for an integer c, let $c * \mathcal{A} = \{ca : a \in \mathcal{A}\}$. Following theorems are the inverse theorems.

Theorem 3. *(See [1], Theorem 2.3.) Let $k \geq 3$ and $r \geq 1$. If $\mathcal{A}$ is a set of k positive integers such that*

$$|S(r, \mathcal{A})| = r\binom{k+1}{2},$$

then $\mathcal{A} = d * [1, k]$ for some positive integer d.

If $\mathcal{A}$ is a set of k nonnegative integers such that $0 \in \mathcal{A}$ and

$$|S(r, \mathcal{A})| = 1 + r\binom{k}{2},$$

then $\mathcal{A} = d * [0, k-1]$ for some positive integer d.

Theorem 4. *(See [1], Theorem 2.4.) Let $k \geq 3$ and $r \geq 1$. Let $\mathcal{A}$ be a set of k integers. If $0 \in \mathcal{A}$ and*

$$|S(r, \mathcal{A})| = \begin{cases} \frac{r(k^2-1)}{4} + 1 & \text{if } k \equiv 1 \pmod 2, \\ \frac{rk^2}{4} + 1 & \text{if } k \equiv 0 \pmod 2, \end{cases}$$

then there is a nonzero integer d such that

$$\mathcal{A} = \begin{cases} d * \left[-\frac{k-1}{2}, \frac{k-1}{2}\right] & \text{if } k \equiv 1 \pmod 2, \\ d * \left[-\frac{k}{2}, \frac{k}{2} - 1\right] & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

If $0 \notin \mathcal{A}$ and

$$|S(r, \mathcal{A})| = \begin{cases} r(\frac{k+1}{2})^2 + 1 & \text{if } k \equiv 1 \pmod 2, \\ r\frac{(k+1)^2-1}{4} + 1 & \text{if } k \equiv 0 \pmod 2, \end{cases}$$

then there is a nonzero integer d such that

$$\mathcal{A} = \begin{cases} d * \left[-\frac{k-1}{2}, \frac{k+1}{2}\right] \setminus \{0\} & \text{if } k \equiv 1 \pmod 2, \\ d * \left[-\frac{k}{2}, \frac{k}{2}\right] \setminus \{0\} & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

For the proof of these theorems we need the following well-known results.

Theorem D. *(See [2], Theorem 5.) Let $k \geq 3$. If $\mathcal{A}$ is a set of k positive integers such that*

$$|S(\mathcal{A})| = \binom{k+1}{2},$$

then

$$\mathcal{A} = d * [1, k]$$

for some positive integer d. If $\mathcal{A}$ is a set of k nonnegative integers such that $0 \in \mathcal{A}$ and

$$|S(\mathcal{A})| = 1 + \binom{k}{2},$$

then

$$\mathcal{A} = d * [0, k-1]$$

for some positive integer d.

Theorem E. *(See [2], Theorem 6.) Let $k \geq 3$ and let $\mathcal{A}$ be a set of k integers. If $0 \in \mathcal{A}$ and*

$$|S(\mathcal{A})| = \begin{cases} \frac{k^2-1}{4} + 1 & \text{if } k \equiv 1 \pmod 2, \\ \frac{k^2}{4} + 1 & \text{if } k \equiv 0 \pmod 2, \end{cases}$$

then there is a nonzero integer d such that

$$\mathcal{A} = \begin{cases} d * \left[-\frac{k-1}{2}, \frac{k-1}{2}\right] & \text{if } k \equiv 1 \pmod 2, \\ d * \left[-\frac{k}{2}, \frac{k}{2} - 1\right] & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

If $0 \notin \mathcal{A}$ and

$$|S(\mathcal{A})| = \begin{cases} (\frac{k+1}{2})^2 + 1 & \text{if } k \equiv 1 \pmod 2, \\ \frac{(k+1)^2-1}{4} + 1 & \text{if } k \equiv 0 \pmod 2, \end{cases}$$

then there is a nonzero integer d such that

$$\mathcal{A} = \begin{cases} d * \left[-\frac{k-1}{2}, \frac{k+1}{2}\right] \setminus \{0\} & \text{if } k \equiv 1 \pmod 2, \\ d * \left[-\frac{k}{2}, \frac{k}{2}\right] \setminus \{0\} & \text{if } k \equiv 0 \pmod 2. \end{cases}$$

Proof of Theorem 3. For $r = 1$, the result follows from Theorem D. Let $r \geq 2$.

`Case 1.` $\mathcal{A}$ is a set of k positive integers such that $|S(r, \mathcal{A})| = r\binom{k+1}{2}$.
By Lemma 1, we have

$$r|S(\mathcal{A})| \leq |S(r, \mathcal{A})|,$$

and so by Theorem B, we have

$$r\binom{k+1}{2} \leq r|S(\mathcal{A})| \leq |S(r, \mathcal{A})| = r\binom{k+1}{2}.$$

Therefore,

$$r|S(\mathcal{A})| = r\binom{k+1}{2},$$

and so

$$|S(\mathcal{A})| = \binom{k+1}{2}.$$

Hence it follows by Theorem D that $\mathcal{A} = d * [1, k]$ for some positive integer d.

`Case 2.` $\mathcal{A}$ is a set of k nonnegative integers such that $0 \in \mathcal{A}$ and $|S(r, \mathcal{A})| = 1 + r\binom{k}{2}$.
By Lemma 1, we have

$$r|S(\mathcal{A})| - r + 1 \leq |S(r, \mathcal{A})|,$$

and so by Theorem B, we have

$$r\left(1 + \binom{k}{2}\right) - r + 1 \leq r|S(\mathcal{A})| - r + 1 \leq |S(r, \mathcal{A})| = 1 + r\binom{k}{2}.$$

Therefore,

$$r|S(\mathcal{A})| - r + 1 = 1 + r\binom{k}{2},$$

and so

$$|S(\mathcal{A})| = 1 + \binom{k}{2}.$$

Hence it follows by Theorem D that $\mathcal{A} = d * [0, k-1]$ for some positive integer d. This completes the proof. □

Proof of Theorem 4. The proof follows by similar arguments as in the above proof using Lemma 1 and Theorem E. □

References

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[3] M. B. Nathanson, *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, Springer, 1996.