RUEHR'S IDENTITIES WITH TWO ADDITIONAL PARAMETERS

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Abstract

We present generalizations of Ruehr's identities with two additional parameters. We prove the claimed results by two different proof methods, namely combinatorially and mechanically. Further, we derive recurrence relations for some special cases by using the Zeilberger algorithm.

1. Introduction and Results

Define the following four binomial sums

$$A_n = \sum_{j=0}^{n} \binom{3n-j}{2n} 3^j, \qquad B_n = \sum_{j=0}^{n} \binom{3n+1}{n-j} 2^j,$$
$$C_n = \sum_{j=0}^{2n} \binom{3n-j}{n} (-3)^j, \quad D_n = \sum_{j=0}^{2n} \binom{3n+1}{n+j+1} (-4)^j.$$

The study of two integral equations led Ruehr [3] to the identities for $n \geq 0$

$$A_n = C_n \text{ and } B_n = D_n.$$

Recently Meehan and at all [4] gave a combinatorial proof for the identity $A_n = B_n$, for $n \geq 0$. In particular, by using Zeilberger's algorithm (for more details see [6]) they showed that A_n, B_n, C_n and D_n satisfy the recursion formula, for $n \geq 0$

$$X_0 = 1, X_{n+1} = \frac{27}{4} X_n - \frac{3}{4(n+1)} \binom{3n+1}{n}.$$

Afterwards, Alzer and Prodinger [1] defined the following four polynomials to obtain polynomial generalizations of Ruehr's identities [3]:

$$A_n(x)=\sum_{j=0}^{n}\binom{3n-j}{2n}x^j,\quad B_n(x)=\sum_{j=0}^{n}\binom{3n+1}{n-j}x^j,$$
$$C_n(x)=\sum_{j=0}^{2n}\binom{3n-j}{n}x^j,\quad D_n(x)=\sum_{j=0}^{2n}\binom{3n+1}{n+j+1}x^j.$$

Note that $A_n(3)=A_n$, $B_n(2)=B_n$, $C_n(-3)=C_n$ and $D_n(-4)=D_n$. Moreover, they derived the following recursions for the above polynomials:

$$A_{n+1}(x)=\frac{x^3}{(x-1)^2}A_n(x)+\frac{(4n+2)x^2-(15n+10)x+9n+6}{2(n+1)(x-1)^2}\binom{3n+1}{n},$$
$$B_{n+1}(x)=\frac{(x+1)^3}{x^2}B_n(x)+\frac{(4n+2)x^2-(7n+6)x-2n-2}{2(n+1)x^2}\binom{3n+1}{n},$$
$$C_{n+1}(x)=\frac{x^3}{(x-1)^2}C_n(x)+\frac{(2n+2)x^2+(3n+2)x-9n-6}{2(n+1)(x-1)}\binom{3n+1}{n},$$
$$D_{n+1}(x)=\frac{(x+1)^3}{x^2}D_n(x)+\frac{(2n+2)x^2+(7n+6)x-4n-2}{2(n+1)x}\binom{3n+1}{n}.$$

From the recursions, they obtained the identities

$$A_n(x+1)=B_n(x)\quad\text{and}\quad C_n(x+1)=D_n(x).$$

In particular, from the recursion mentioned above, they also derived that $A_n=B_n=C_n=D_n$. As mentioned above, the results of Alzer and Prodinger are the polynomial generalizations of the results given in [4]. Note that the binomial coefficients of the sums considered in both works [1, 4] are the same.

In this paper, by adding two extra parameters to the binomial coefficients in the works [1, 4], we define new kinds of polynomials with two additional parameters m and t:

$$A_n(m,t;x)=\sum_{k=0}^{n}\binom{mn+t-k}{n-k}x^k,\qquad B_n(m,t;x)=\sum_{k=0}^{n}\binom{mn+t+1}{n-k}x^k,$$
$$C_n(m,t;x)=\sum_{k=0}^{(m-1)n}\binom{mn+t-k}{mn-k-n}x^k,\quad D_n(m,t;x)=\sum_{k=0}^{(m-1)n}\binom{mn+t+1}{mm-k-n}x^k.$$

Now we state our main results:

Theorem 1. *For any integers $n,m,t\geq 0$ and any real number x,*

$$A_n(m,t;x+1)=B_n(m,t;x).$$

Theorem 2. *For any integers $n,m,t\geq 0$ and any real number x,*

$$C_n(m,t;x+1)=D_n(m,t;x).$$

2. Proofs

In this section, we give two different proof methods to show the claimed results. The first one is a *combinatorial proof* and the later is a *mechanical proof.* In Section 3, we will give recursions for the special cases of the sums $A_n(m,t;x)$, $B_n(m,t;x)$, $C_n(m,t;x)$, $D_n(m,t;x)$ by using the Zeilberger algorithm.

Firstly, we will give a combinatorial proof of Theorem 1, which is inspired by the proof method given in [4]. Secondly, we will give a mechanical proof for Theorem 1. In order to prove Theorem 2, firstly we give a new combinatorial proof and secondly give a mechanical proof.

Assume that x is a positive integer such that $x > 1$. Let A_1 and A_2 be the sets of the words over the alphabets $\{1, 2, \ldots, x+1\}$ and $\{1, 2, \ldots, x\}$, respectively. The length of a word w is denoted by $|w|$. Let S be the set which consists of the pairs of words (w_1, w_2) satisfying the following conditions:

(i) $w_1 \in A_1$ and $w_2 \in A_2$,

(ii) $|w_1| + |w_2| = mn + t$,

(iii) $|w_1| + (\#\text{ of 1's in } w_2) = n$.

We count the number of elements of S in two different ways. Hence, we get two different expressions which are equal.

`First way:` We start with counting elements of S by considering the length of the word w_1. Let $|w_1| = k$, which implies $0 \le k \le n$ by the condition (iii). So there are $(x+1)^k$ different ways to choose the word w_1. From the condition (ii), the length of w_2 is equal $mn+t-k$, i.e. $|w_2| = mn+t-k$. In addition, the number of 1's in w_2 is $n-k$ by the condition (iii). Thus we are given $\binom{mn+t-k}{n-k}$ different places for 1's and $mn-k+t-n+k = (m-1)n+t$ places for other alphabets of the set A_2, by the condition (ii). There are $(x-1)^{(m-1)n+t}$ different ways to choose these alphabets. Finally, we count $(x+1)^k \binom{mn+t-k}{n-k} (x-1)^{(m-1)n+t}$ different ways to construct the words w_1 and w_2 for fixed k. Since $0 \le k \le n$, we get

$$\#S = \sum_{k=0}^{n} (x+1)^k \binom{mn+t-k}{n-k} (x-1)^{(m-1)n+t}. \tag{2.1}$$

`Second way:` Now we consider the number of the alphabets of $\{2, 3, \ldots, x+1\}$ in the word w_1. Let k be this number which implies $0 \le k \le n$ and $|w_1| = j$ which implies $k \le j \le n$ by the condition (iii). So there are $x^k \binom{j}{k}$ different ways to choose the word w_1. From the condition (ii), $|w_2| = mn+t-j$ and from the condition (iii), the number of 1's in w_2 is equal to $n-j$. Also there are $(m-1)\,n+t$ places for other alphabets of the set A_2. Thus there are $\binom{mn+t-j}{n-j}(x-1)^{(m-1)n+t}$ different ways to construct the word w_2. Consequently,

$$\begin{aligned}\#S &= \sum_{k=0}^{n}\sum_{j=k}^{n}\binom{j}{k}\binom{mn+t-j}{n-j}x^k(x-1)^{(m-1)n+t}\\ &= (x-1)^{(m-1)n+t}\sum_{k=0}^{n}x^k\sum_{j=k}^{n}\binom{j}{k}\binom{mn+t-j}{n-j},\end{aligned}$$

which, by the Vandermonde convolution [2], equals

$$(x-1)^{(m-1)n+t}\sum_{k=0}^{n}\binom{mn+t+1}{n-k}x^k. \tag{2.2}$$

Finally, from the equations (2.1) and (2.2), Theorem 1 holds for any positive integer $x>1$ since $(x-1)^{(m-1)n+t}$ is a constant factor. The case $x=1$ can be easily checked. Now consider the polynomial of degree n

$$f(x)=\sum_{k=0}^{n}\binom{mn+t-k}{n-k}(x+1)^k-\sum_{k=0}^{n}\binom{mn+t+1}{n-k}x^k,$$

which has infinitely many roots because of our combinatorial proof. Since it may have at most n roots, the polynomial $f(x)$ must be identically equal to 0. Consequently, Theorem 1 holds for any real number x, which completes the proof.

Note that this polynomial argument may apply on m and t, as well, by using the fact that the binomial coefficient $\binom{n}{k}$ can be considered as a kth-degree polynomial in n.

Now we give an alternative proof which is an application of the Binomial Theorem and Vandermonde's convolution. So we write

$$\begin{aligned}\sum_{k=0}^{n}\binom{mn+t-k}{n-k}(x+1)^k &= \sum_{k=0}^{n}\binom{mn+t-k}{n-k}\sum_{j=0}^{k}\binom{k}{j}x^j\\ &= \sum_{j=0}^{n}x^j\sum_{k=j}^{n}\binom{mn+t-k}{n-k}\binom{k}{j}\\ &= \sum_{j=0}^{n}x^j\binom{mn+t+1}{mn-n+t+j+1}\\ &= \sum_{j=0}^{n}x^j\binom{mn+t+1}{n-j},\end{aligned}$$

as claimed.

For the combinatorial proof of Theorem 2, let us construct a set, similar to the combinatorial proof of Theorem 1. For an integer x such that $x > 1$, let B_1 and B_2 be the sets of the words over the alphabets $\{1, 2, \ldots, x+1\}$ and $\{1, 2, \ldots, x\}$, respectively. Then let S' be the set which consists of the pairs of the words (w_1, w_2) with the following conditions:

(i) $w_1 \in B_1$ and $w_2 \in B_2$,

(ii) $|w_1| + |w_2| = mn + t$,

(iii) $|w_1| + (\#\text{ of } x\text{'s in } w_2) = (m-1)\,n$.

Again, we present two different ways to count the number of elements of the set S'.

`First way:` Let $|w_1| = k$. Then we have $0 \leq k \leq (m-1)\,n$ by the condition (iii). So there are $(x+1)^k$ possibilities to choose the word w_1. By the condition (ii), $|w_2| = mn + t - k$. Besides, the number of x's in w_2 is equal to $(m-1)\,n - k$ by (iii) and there are $n+t$ places for the other alphabets. Thus we have $\binom{mn+t-k}{(m-1)\,n-k}(x-1)^{n+t}$ different ways to select the word w_2. Finally, we obtain $(x+1)^k\binom{mn+t-k}{(m-1)\,n-k}(x-1)^{n+t}$ ways to choose tuples $(w_1, w_2) \in S'$ for the fixed k such that $0 \leq k \leq (m-1)\,n$. So

$$\#S' = \sum_{k=0}^{(m-1)n} \binom{mn+t-k}{mn-n-k}(x+1)^k\,(x-1)^{n+t}. \tag{2.3}$$

`Second way:` In this case, let k be the number of the alphabets of $\{1, 2, \ldots, x\}$ in the word w_1 which implies $0 \leq k \leq (m-1)\,n$ and $|w_1| = j$ which implies $k \leq j \leq (m-1)\,n$ by the condition (iii). So there are $x^k\binom{j}{k}$ possible ways to choose w_1. In addition, from (ii) $|w_2| = mn + t - j$ and from condition (iii), the number of x's in w_2 equals to $(m-1)\,n - j$. Thus there are $\binom{mn+t+1}{(m-1)\,n-j}(x-1)^{n+t}$ possibilities. Since $0 \leq k \leq (m-1)\,n$ and $k \leq j \leq (m-1)\,n$, we have

$$\#S' = \sum_{k=0}^{(m-1)n}\sum_{j=k}^{(m-1)n}\binom{j}{k}\binom{mn+t+1}{mn-n-j}x^k\,(x-1)^{n+t}.$$

By Vandermonde's convolution formula, we have that

$$\#S' = (x-1)^{n+t}\sum_{k=0}^{(m-1)n}\binom{mn+t+1}{mn-n-k}x^k. \tag{2.4}$$

From (2.3) and (2.4), the proof is completed for any integer x such that $x > 1$. Finally, by using a similar polynomial argument in the combinatorial proof of Theorem 1, the proof of Theorem 2 is completed.

Similar to the proof of Theorem 1, as an alternative proof we have

$$\begin{aligned}\sum_{k=0}^{(m-1)n}\binom{mn+t-k}{mn-n-k}(x+1)^k &= \sum_{k=0}^{(m-1)n}\binom{mn+t-k}{mn-n-k}\sum_{j=0}^{k}\binom{k}{j}x^j \\ &= \sum_{j=0}^{n}x^j\sum_{k=0}^{(m-1)n}\binom{mn+t-k}{mn-n-k}\binom{k}{j} \\ &= \sum_{j=0}^{(m-1)n}\binom{mn+c+1}{mn-n-j}x^j.\end{aligned}$$

Note that when $m = 3$, $t = 0$ and $x = 2$ in Theorem 1, and, when $m = 3$, $t = 0$ and $x = -4$ in Theorem 2, we reobtain the results given in [4]. Also when $m = 3$ and $t = 0$ in both Theorems 1 and 2, we reobtain the results given in [1].

3. Recursions for Some Special Cases

In this section, we will give the recurrence relations for the sums $A_n(m,t;x)$, $B_n(m,t;x)$, $C_n(m,t;x)$ and $D_n(m,t;x)$ for some special values of m and t by using the Zeilberger algorithm with Mathematica implementation (for more details, see [5]). Note that for the case $m = 3$ and $t = 0$, the corresponding recursions of the four sums were given in [1].

Before giving further recurrence relations for various special cases, we note that Zeilberger's algorithm calculates recurrences for $A_n(m,t;x)$, $B_n(m,t;x)$, $C_n(m,t;x)$ and $D_n(m,t;x)$ with respect to n only if one specializes m to specific integers.

For the case $m = 2$ and $t = 0$, note that $A_n(2,0;x) = C_n(2,0;x)$ and $B_n(2,0;x) = D_n(2,0;x)$. By the Zeilberger algorithm for $A_n(2,0;x)$, we obtain

$$x^2F(k,n) + (1-x)F(k,n+1) = G(k+1,n) - G(k,n),$$

where $F(k,n)$ is the summand term and

$$G(k,n) = \frac{(nx+x+k-2n-2)}{n+1}x^k\binom{2n-k+1}{n-k+1}.$$

Summing both sides on k, we have

$$x^2A_n(2,0;x) + (1-x)A_{n+1}(2,0;x) = (x-2)\binom{2n+1}{n+1}.$$

Similarly, by the Zeilberger algorithm for $B_n(2,0;x)$ and after some arrangements, we get the recurrence

$$(x+1)^2 B_n(2,0;x) - xB_{n+1}(2,0;x) = (x-1)\binom{2n+1}{n}.$$

It is easy to see that $A_n(2,0;x+1) = B_n(2,0;x)$ since $A_0(2,0;x+1) = B_0(2,0;x) = 1$.

Now we consider the case $m = 4$ and $t = 0$. We omit all details here. We give just the recurrence relations generated by the algorithm:

$$\begin{aligned}&x^4 A_n(4,0;x) - (x-1)^3 A_{n+1}(4,0;x) = \frac{1}{3(4n+2)(4n+3)}\binom{4n+3}{n+1}\\ &\times 6(-4+11x-9x^2+x^3) + n(-80+220x-182x^2+27x^3)\\ &+ n^2(-64+176x-148x^2+27x^3),\end{aligned}$$

$$\begin{aligned}&(x+1)^4 B_n(4,0;x) - x^3 B_{n+1}(4,0;x) = \frac{1}{3(4n+2)(4n+3)}\binom{4n+3}{n+1}\\ &\times 6(-1-4x-6x^2+x^3) + n(-15-63x-101x^2+27x^3)\\ &+ n^2(-9-39x-67x^2+27x^3),\end{aligned}$$

$$\begin{aligned}&x^4 C_n(4,0;x) + (1-x)C_{n+1}(4,0;x) = \frac{1}{3(4n+2)(4n+3)}\binom{4n+3}{n+1}\\ &\times 6(-4+x+x^2+x^3) + n(-80+20x+18x^2+15x^3)\\ &+ n^2(-64+16x+12x^2+9x^3)\end{aligned}$$

and

$$\begin{aligned}&(1+x)^4 D_n(4,0;x) - xD_{n+1}(4,0;x) = \frac{1}{3(4n+2)(4n+3)}\binom{4n+3}{n+1}\\ &\times 6(-1+6x+4x^2+x^3) + n(-27+101x+63x^2+15x^3)\\ &+ n^2(-27+67x+39x^2+9x^3).\end{aligned}$$

From their recurrence relations, it is seen that $A_n(4,0;x+1) = B_n(4,0;x)$ and $C_n(4,0;x+1) = D_n(4,0;x)$. For general m, it is hard to determine recurrence relations for these sums because their nonhomogeneous parts are very complicated.

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