A NOTE ON A METHOD OF ERDŐS AND THE STANLEY-ELDER THEOREMS

George E. Andrews
Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania
gea1@psu.edu

Emeric Deutsch
Polytechnic Institute of New York University, Brooklyn, New York
emericdeutsch@msn.com

Received: 12/30/15, Revised: 2/10/16, Accepted: 4/14/16, Published: 4/22/16

Abstract

An enumeration method of Erdős is applied to provide a generalization of the theorems of Stanley and Elder on integer partitions.

1. Introduction

In [4], Erdős proved the asymptotics of the partition function $p(n)$ by elementary means. His starting point was the identity of Ford [7] (probably going back to Euler):

$$np(n) = \sum_{j=1}^{n} p(n-j)\sigma(j), \tag{1}$$

where $\sigma(j)$ is the sum of divisors of j. The standard proof of (1) is by logarithmic differentiation of

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \tag{2}$$

([7], also [1, p.98]). However, Erdős wanted to avoid even this amount of analysis. So he rewrote (1) as follows

$$np(n) = \sum_{v\geq 1}\sum_{k\geq 1} vp(n-kv), \tag{3}$$

and then he remarked: "We easily obtain (3) by adding up all the partitions of n, and noting that v occurs in $p(n-v)$ partitions." We assume he is telegraphing that v appears twice in $p(n-2v)$ partitions, etc.

This same counting method makes transparent a very general theorem in partitions.

Definition 1. A *partition configuration*, A, is a non-decreasing sequence of non-negative integers $(a_1, \ldots, a_k)$ with *length* k and weight $w(A) = a_1 + a_2 + \cdots + a_k$.

Definition 2. A partition, $\lambda : \lambda_1 + \lambda_2 + \cdots + \lambda_m$ $(1 \leq \lambda_1 \leq \lambda_2 \cdots \leq \lambda_m)$ is said to have a partition configuration A if there is a subset of parts of λ of the form $a_1 + j, a_2 + j, \ldots, a_k + j$ for some $j \geq 1$.

For example, the partition $(2 + 4 + 4 + 5 + 8 + 9)$ contains an instance of $A = (0, 3, 6, 7)$ because the parts $2, 5, 8, 9$ exceed by 2 the successive entries of A.

Theorem 1. *Given a partition configuration A, in each partition of n we count the number of distinct configurations A and then sum over all partitions of n. Call this sum $p_A(n)$. Then*

$$p_A(n) = p(k; n - w(A)), \tag{4}$$

where $p(k; n)$ is the total number of appearances of k in the partitions of n.

As an example of Theorem 1, we take $A = (0, 1, 2)$ (having length $k = 3$ and weight $w(A) = 3$) and $n = 10$. The partitions of 10 containing the partition configuration A are $1+1+1+1+1+2+3$, $1+1+1+2+2+3$, $1+2+2+2+3$, $1+1+2+3+3$ and $1+2+3+4$ which contain A $1+1+1+1+2 = 6$ times. So $p_A(10) = 6$. As for $p(3; 10-3) = p(3; 7)$ we see that the partitions of 7 containing 3's are 1+1+1+1+3, 1+1+2+3, 2+2+3, 1+3+3, 3+4. So $p(3; 7) = 1+1+1+2+1 = 6$, the total number of 3's in the partitions of 7.

In Section 2, we use the Erdős method to provide a short proof of Theorem 1 together with the theorems of Elder and Stanley (see Corollaries 2 and 3). We refer the reader to [8] for an extensive account of the Elder and Stanley theorems. In Section 3, we extend these ideas to a question concerning divisibility restrictions on parts. We conclude with some general observations.

2. Proof of Theorem 1

We remark, following Erdős, that to obtain $p_A(n)$ there must be $p(n - ((a_1 + j) + \cdots + (a_k + j)))$ partitions which contain the partition configuration A in the form $(a_1 + j) + (a_2 + j) + \cdots + (a_k + j)$.

Hence

$$\begin{aligned}\sum_{n\geq 0} p_A(n)q^n &= \sum_{j=1}^{\infty} \frac{q^{(j+a_1)+(j+a_2)+\cdots+(j+a_k)}}{\prod_{n=1}^{\infty}(1-q^n)} = \frac{q^{w(A)}\sum_{j=1}^{\infty} q^{kj}}{\prod_{n=1}^{\infty}(1-q^n)} \\ &= \frac{q^{w(A)+k}}{(1-q^k)^2\prod_{\substack{n=1\\ n\neq k}}^{\infty}(1-q^n)} \\ &= q^{w(A)}\left(q^k+2q^{2k}+3q^{3k}+\cdots\right)\prod_{\substack{n=1\\ n\neq k}}^{\infty}(1+q^n+q^{2n}+q^{3n}+\cdots) \\ &= q^{w(A)}\sum_{n\geq 0} p(k,n)q^n,\end{aligned} \tag{5}$$

and Theorem 1 follows by comparing coefficients of q^n in the extremes of (5). □

Corollary 1 (Stanley's Theorem [2],[8]). *The number of 1's in the partitions of n is equal to the number of parts that appear at least once in a given partition of n, summed over all partitions of n.*

Proof. Take $A = (0)$ in Theorem 1. □

A more general theorem is attributed to Paul Elder.

Corollary 2 (Elder's Theorem [2][8]). *The number of j's appearing in the partitions of n is equal to the number of parts that appear at least j times in a given partition of n, summed over all partitions of n.*

Proof. Take $A = (0, 0, \ldots, 0)$ of length j in Theorem 1. □

Corollary 3. *In each partition of n count the number of sequences of consecutive integers of length k. Then sum these numbers over all partitions of n. This equals the number of appearances of k in the partitions of $n - k(k-1)/2$.*

This result is originally due to Knopfmacher and Munagi and occurs as Theorem 5 in [9].

Proof. In Theorem 1 take $A = (0, 1, \ldots, k-1)$. □

3. Divisibility of Parts

The method of Erdős can be further extended in many ways.

Theorem 2. *Given $k \geq 1$, in each partition of n we count the number of times a part divisible by k appears uniquely (i.e., is not a repeated part); then sum these numbers over all the partitions of n. The result is equal to the number of appearances of $2k$ in the partitions of $n + k$.*

Example. $k = 1$, $n = 5$. There are eight singletons in the partitions of 5: 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, $1+1+1+1+1$. There are eight 2's in the partitions of 6: $4+2$, $3+2+1$, $2+2+2$, $2+2+1+1$, $2+1+1+1+1$.

Remark. The case $k = 1$ was published as a problem in [3].

Proof. The generating function for multiples of k being unique parts is

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{q^{kj}}{\prod_{\substack{n=1\\ n\neq kj}}^{\infty}(1-q^n)} &= \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{j=1}^{\infty} q^{kj}(1-q^{kj}) \\
&= \frac{1}{\prod_{n=1}^{\infty}(1-q^n)} \left(\frac{q^k}{1-q^k} - \frac{q^{2k}}{1-q^{2k}} \right) \\
&= \frac{q^k}{(1-q^{2k})} \cdot \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \\
&= q^{-k}\left(q^{2k} + 2q^{2\cdot 2k} + 3q^{3\cdot 2k} + \cdots\right) \prod_{\substack{n=1\\ n\neq 2k}}^{\infty} \frac{1}{1-q^n},
\end{aligned}$$

and this last expression is the generating function for the number of appearances of $2k$ in the partitions of $n + k$. $\square$

4. Conclusion

It is clear that the scope of Theorem 1 could be generalized to account for results like Theorem 5. We should also note that Dastidar and Gupta [2] have generalized the Stanley and Elder theorems where they add what they term "packets" of size k to partitions, and this count equals the number of appearances of k in the partitions of $n + k$.

Finally, we note the charming survey "A Fine Rediscovery" by R. Gilbert [8], which provides a detailed history of the Stanley and Elder theorems and points out that N. J. Fine was the original discoverer of both theorems [5],[6].

References

[1] G.E. Andrews, *The Theory of Partitions*, Addison-Wesley,Reading, 1976 (Reissued: Cambridge University Press, Cambridge, 1984).

[2] M. G. Dastidar and S. S. Gupta, Generalization of a few results in integer partitions, arXiv: 1111.0094v1 [csDM].

[3] E. Deutsch, Problem 11237, *Amer. Math. Monthly* **115** (2006), 666-667.

[4] P. Erdős, On an elementary proof of some asymptotic formulas in the theory of partitions, *Annals of Math.* **43** (1942), 437-450.

[5] N. J. Fine, Sums over partitions, in Report of the Institute in the Theory of Numbers, University of Colorado, Boulder, Colorado, June 21-July 17, 1959. 86-94.

[6] N. J. Fine, *Basic Hypergeometric Series and Applications.* Mathematical Surveys and Monographs, Vol. 27, Amer. Math. Soc. Providence, 1988.

[7] W. B. Ford, Two theorems on the partitions of numbers, *Amer. Math. Monthly* **38** (1931), 183-184.

[8] R. A. Gilbert, A Fine rediscovery, *Amer. Math. Monthly* **122** (2015), 322-331.

[9] A. Knopfmacher and A. O. Munagi, Successions in integer partitions, *Ramanujan J.* **18** (2009), 239-255.