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IMPARTIAL CHOCOLATE BAR GAMES

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Abstract

Chocolate bar games are variants of the CHOMP game in which the goal is to make your opponent eat a poisoned square of chocolate. The rectangular chocolate bar is a thinly disguised form of the NIM game. In this paper, we investigate chocolate bars whose widths are proportional to the distance from the poisoned square. We find the nim-values when the constant of proportionality is even, and present some conjectures for other cases.

1. Introduction

The original chocolate bar game [7] consists of a rectangular bar of chocolate with one poisoned corner. Each player takes it in turn to break the bar in a straight line along the grooves, and eats the piece that is broken off. The player who breaks the chocolate bar so as to leave his opponent with the single poisoned block (black block) is the winner. Since the horizontal and vertical grooves are independent, an $m \times n$ bar (of squares) is equivalent to a game of NIM with up to four heap sizes equal to the number of grooves above, below, to the left, and to the right of the poisoned square.

In this paper, we consider bars of the shapes shown in Figures 1.2–1.4, where the gray blocks are sweet chocolate that can be eaten and the black block is the poisoned square. In these cases, a vertical break can reduce the number of horizontal breaks. We can still think of the game as being played with heaps, but now a move may change more than one heap.

There are other types of chocolate bar games, and one of the most well known is CHOMP. CHOMP uses a rectangular chocolate bar. The players take turns to choose one block, and eat it together with those blocks below it and to its right. The top left block is poisoned, and the players cannot eat this block. Although many people have studied this game, the winning strategy is yet to be discovered. For an overview of research on CHOMP, see [8].

Example 1.1. Examples of chocolate bar games.

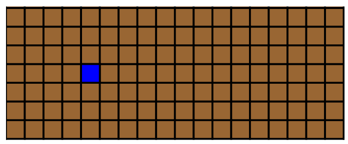


Figure 1.1.

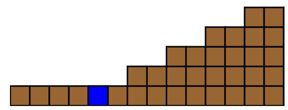


Figure 1.2.

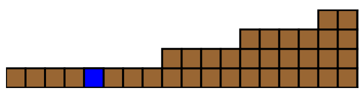


Figure 1.3.

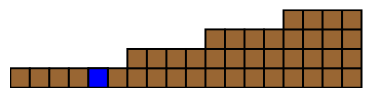


Figure 1.4.

For completeness, we briefly review some necessary concepts in game theory; see [1] for more details. Since chocolate bar games are impartial games that cannot end in a draw, there will only be two outcome classes: first player wins and second player wins, also called $\mathcal{N}$-positions and $\mathcal{P}$-positions.

The *disjunctive sum* of two games, denoted $G+H$, is a super-game in which a player may move either in G or in H, but not both. In Figures 1.2–1.4, each game is the disjunctive sum of the chocolate bar to the left and the chocolate bar to the right of the poisoned square.

For any game G, there is a set of states (games) that can be reached by making precisely one move in G, which we will denote by $move(G)$. The *minimum excluded value* (*mex*) of a set S of non-negative integers is the least non-negative integer that is not in S. Each impartial game G also has an associated nim-value, sometimes called the Grundy value, denoted by $\mathcal{G}$. The nim-value is found recursively: $\mathcal{G}(G) = mex\{\mathcal{G}(H) : H \in \text{move}(G)\}$.

Let x, y be non-negative integers, and write them in base 2, so that $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$ with $x_i, y_i \in \{0,1\}$. We define the nim-sum $x \oplus y = \sum_{i=0}^{n} z_i 2^i$, where $z_i = x_i + y_i (mod\ 2)$. The power of the Sprague–Grundy theory for impartial games is contained in the next result.

Theorem 1.1. *Let G and H be impartial games. Then,*

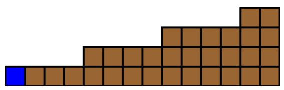
- $\mathcal{G}(G) = 0$ *if and only if G is a $\mathcal{P}$-position;*
- $\mathcal{G}(G+H) = \mathcal{G}(G) \oplus \mathcal{G}(H)$.

For a proof of this theorem, see [1].

In this paper, the authors present the nim-values of chocolate bar games. For a general bar, the strategies seem complicated. We focus on bars that grow regularly in height.

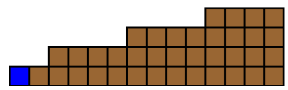
Definition 1.1. Fix a natural number k and a non-negative integer h. For non-negative integers y and z such that $y \leq \lfloor \frac{z+h}{k} \rfloor$, the chocolate bar will consist of $z+1$ columns, where the 0-th column is the poisoned square and the height of the i-th column is $t(i) = \min(y, \lfloor \frac{i+h}{k} \rfloor) + 1$. We will denote these by $CB(h, k, y, z)$.

Example 1.2. Examples of chocolate bar games $CB(h, k, y, z)$.



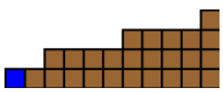
$CB(0, 4, 3, 13)$

Figure 1.5.



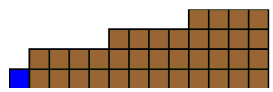
$CB(0, 2, 3, 13)$

Figure 1.6.



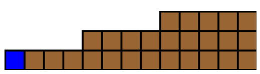
$CB(2, 4, 3, 10)$

Figure 1.7.



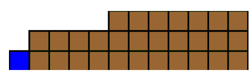
$CB(3, 4, 3, 12)$

Figure 1.8.



$CB(0, 4, 2, 12)$

Figure 1.9.



$CB(3, 4, 2, 11)$

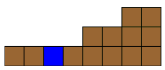
Figure 1.10.

In this paper, we derive the nim-values for $CB(0, k, y, z)$ where k is an even number in Theorem 2.1, and for $CB(h, k, y, z)$ where k is an even number and $h \in \{0, 1, 2, ..., k-1\}$ or $h = k2^t + m2^{t+1}$ for non-negative integers t, m with $m < k/2$ in Theorem 3.2. Finally, we give several conjectures for $CB(0, 1, y, z)$ based on computational results.

In our proofs, it will be useful to know the disjunctive sum of the chocolate to the right of the poisoned square and a single strip of chocolate to the left, as in Figures 1.2, 1.3, and 1.4. We will denote such a position by $\{x, y, z\}$, where x is the number of possible moves in the strip, y is the number of vertical moves in the bar, and z is the number of horizontal moves. Figures 1.11, 1.12, 1.13, 1.14, and 1.15 give some examples of the coordinate system.

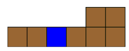
For the Chocolate Bar $CB(0, k, y, z)$ in which k is an even number, we will show that the $\mathcal{P}$-positions are when $x \oplus y \oplus z = 0$, so that the nim-value of $CB(0, k, y, z)$ to the right is $x = y \oplus z$. Similarly, for the Chocolate Bar $CB(h, k, y, z)$ in which k is an even number and $h > 0$, we will show that the $\mathcal{P}$-positions are when $(x+h) \oplus y \oplus (z+h) = 0$, so that the nim-value of $CB(h, k, y, z)$ to the right is $x = (x+h)-h$ $= (y \oplus (z+h)) - h$.

Example 1.3. Here, we present some examples of the states of chocolate bars with their coordinates.



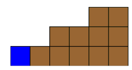
$\{2, 2, 5\}$

Figure 1.11.



$\{2, 1, 3\}$

Figure 1.12.

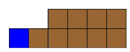


$\{0, 2, 5\}$

Figure 1.13.

$\{2, 0, 5\}$

Figure 1.14.



$\{0, 1, 5\}$

Figure 1.15.

2. Nim-values of the Chocolate Bar CB(0,k,y,z) for an Even Number *k*

In this section, we analyze the Chocolate Bar $CB(0, k, y, z)$ for a fixed even number k.

Definition 2.1. Let $A_k = \{\{x, y, z\} : x, y, z \in Z_{\geq 0}, y \leq \left\lfloor \frac{z}{k} \right\rfloor$ and $x \oplus y \oplus z = 0\}$, $B_k = \{\{x, y, z\} : x, y, z \in Z_{\geq 0}, y \leq \left\lfloor \frac{z}{k} \right\rfloor$, and $x \oplus y \oplus z \neq 0\}$.

Theorem 2.1. *The nim-value of the Chocolate Bar $CB(0, k, y, z)$ is $y \oplus z$ when k is an even number.*

We now prove Theorem 2.1 for an arbitrary even number k. First, we need several facts about the relations between numbers in base 2, the nim-sum of numbers, and the floor function.

Lemma 2.1. *If k and h are even numbers, then $\lfloor (h+1)/k \rfloor = \lfloor h/k \rfloor$.*

Proof. Let $h = k \times p + q$ for integers p, q with $0 \leq q < k$. If k and h are even numbers, then q is an even number. Therefore, $q + 1 < k$, and the conclusion of this lemma follows directly from $h + 1 = k \times p + q + 1$. $\square$

Lemma 2.2. *Suppose that $x \oplus y \oplus z = 0$. Then, we have the following:*
(1) $y = \lfloor z/k \rfloor$ *if and only if* $x_n = z_n = 1, y_n = 0$ *and, for* $i = 0, 1, 2, ..., n-1$,

$$y_i = \lfloor \left(\sum_{j=1}^{n-i} 2^j z_{i+j} - \left(\sum_{j=1}^{n-i-1} 2^j y_{i+j} \right) k \right) /k \rfloor \tag{1}$$

and

$$z_i = x_i + y_i \ (mod\ 2). \tag{2}$$

(2) $y < \lfloor z/k \rfloor$ *if and only if the following conditions hold:*
(a) $x_n = z_n = 1$ *and* $y_n = 0$.
(b) *Equation* (2) *is true for* $i = 0, 1, 2, ..., n-1$.
(c) *There exists some* m *such that Equation* (1) *is true for* $i = m+1, m+2, ..., n-1$.
(d)

$$y_m < \lfloor \left(\sum_{j=1}^{n-m} 2^j z_{m+j} - \left(\sum_{j=1}^{n-m-1} 2^j y_{m+j} \right) k \right) /k \rfloor.$$

Proof. We first prove (1). Suppose that $y = \lfloor z/k \rfloor$. Then, we have

$$\begin{aligned} &2^n y_n + 2^{n-1} y_{n-1} + 2^{n-2} y_{n-2} + ... + 2^0 y_0 \\ &= \lfloor \frac{2^n z_n + 2^{n-1} z_{n-1} + 2^{n-2} z_{n-2} + ... + 2^0 z_0}{k} \rfloor. \end{aligned} \tag{3}$$

By comparing $2^n y_n + ...$ and $\lfloor \frac{2^n z_n + ...}{k} \rfloor$, we have $y_n = \lfloor \frac{z_n}{k} \rfloor = 0$. By comparing $2^n y_n + 2^{n-1} y_{n-1} + ...$ and $\lfloor \frac{2^n z_n + 2^{n-1} z_{n-1} + ...}{k} \rfloor$, we have $y_{n-1} = 2y_n + y_{n-1}$ $= \lfloor \frac{2z_n + z_{n-1}}{k} \rfloor = \lfloor \frac{2z_n}{k} \rfloor$, where the last equation follows directly from Lemma 2.1. By comparing $2^n y_n + 2^{n-1} y_{n-1} + 2^{n-2} y_{n-2} + ...$ and $\lfloor \frac{2^n z_n + 2^{n-1} z_{n-1} + 2^{n-2} z_{n-2} + ...}{k} \rfloor$, we have $2y_{n-1} + y_{n-2} = 2^2 y_n + 2y_{n-1} + y_{n-2} = \lfloor \frac{2^2 z_n + 2z_{n-1} + z_{n-2}}{k} \rfloor$, and hence we have

$$\begin{aligned} y_{n-2} &= \lfloor (2^2 z_n + 2z_{n-1} + z_{n-2} - 2k y_{n-1})/k \rfloor \\ &= \lfloor (2^2 z_n + 2z_{n-1} - 2k y_{n-1})/k \rfloor, \end{aligned} \tag{4}$$

where the last equation follows directly from Lemma 2.1. Similarly, we have

$$\begin{aligned} y_{n-3} &= \lfloor (2^3 z_n + 2^2 z_{n-1} + 2z_{n-2} + z_{n-3} - (2^2 y_{n-1} + 2y_{n-2})k)/k \rfloor \\ &= \lfloor (2^3 z_n + 2^2 z_{n-1} + 2z_{n-2} - (2^2 y_{n-1} + 2y_{n-2})k)/k \rfloor, \end{aligned}$$

where the last equation follows directly from Lemma 2.1. In general, for $i = 0, 1, 2, ..., n-1$,

$$y_i = \left\lfloor (\sum_{j=1}^{n-i} 2^j z_{i+j} - (\sum_{j=1}^{n-i-1} 2^j y_{i+j})k)/k \right\rfloor.$$

Therefore, we have Equation (1) for $i = 0, 1, 2, ..., n-1$.

Conversely, if we have Equation (1) for $i = 0, 1, 2, ..., n-1$, then we have Equation (3). This is equivalent to $y = \lfloor z/k \rfloor$. Equation (2) follows directly from $x \oplus y \oplus z = 0$.

Next, we prove statement (2) of this lemma. By the result of (1), Equation (1) is true for $i = 0, 1, ..., n-1$ if and only if $y = \lfloor z/k \rfloor$. Therefore, $y < \lfloor z/k \rfloor$ if and only if there exists some i such that $0 \leq i < n$ and Equation (1) is not true for i. Let m be the largest integer that does not satisfy Equation (1). Then, we have statement (2). □

Remark 2.1. Suppose that $x \oplus y \oplus z = 0$ and $y = \lfloor z/k \rfloor$. Then, by Lemma 2.2, we have $y_n = 0, z_n = x_n = 1$ and $y_{n-1} = \lfloor 2z_n/k \rfloor$, and hence y_{n-1} can be expressed with k and x_n. We express this fact as

$$y_{n-1} = f(k, x_n). \tag{5}$$

We have

$$z_n = x_n, z_{n-1} = x_{n-1} + y_{n-1} \ (mod\ 2),$$

and

$$y_{n-2} = \left\lfloor (2^2 z_n + 2z_{n-1} - 2ky_{n-1})/k \right\rfloor.$$

Hence, by Equation (5), we can express y_{n-2} in terms of k, x_n, and x_{n-1}. We express this fact as

$$y_{n-2} = f(k, x_n, x_{n-1}). \tag{6}$$

In this way, y_i can be expressed in terms of $k, x_n, x_{n-1}, ..., x_{i+1}$, and we express this as

$$y_i = f(k, x_n, x_{n-1}, ..., x_{i+1}). \tag{7}$$

If $y < \lfloor z/k \rfloor$, there exists $m \in Z_{>0}$ such that Equation (7) is true for $i = m+1, m+2, m+3, ..., n-1$ and Equation (7) is not true for $i = m$.

If $y = \lfloor z/k \rfloor$, Equation (7) is true for $i = 0, 1, 2, ..., n-1$.

The situation changes considerably when k is an odd number. In Equation (4), we have

$$\begin{aligned} y_{n-2} &= \left\lfloor (2^2 z_n + 2z_{n-1} + z_{n-2} - 2ky_{n-1})/k \right\rfloor \\ &= \left\lfloor (2^2 z_n + 2z_{n-1} - 2ky_{n-1})/k \right\rfloor, \end{aligned}$$

but the last equation is not true if k is an odd number. Note that we cannot use Lemma 2.1 for an odd number k.

Lemma 2.3. *For any $x \in Z_{\geq 0}$, there exist unique y, z such that $x \oplus y \oplus z = 0$ and $y = \lfloor z/k \rfloor$.*

Proof. By Lemma 2.2 $x \oplus y \oplus z = 0$ and $y = \lfloor z/k \rfloor$ if and only if Equations (1) and (2) are true for $i = 0, 1, ..., n-1$, $x_n = z_n = 1$, and $y_n = 0$. Therefore, for any $x \in Z_{\geq 0}$, there exist y, z such that $x \oplus y \oplus z = 0$ and $y = \lfloor z/k \rfloor$.

Next, we prove that these y, z are uniquely determined by x. Suppose that $x \oplus y \oplus z = 0$ and $y = \lfloor z/k \rfloor$. Then, from Remark 2.1, we have $x_n = z_n = 1$, $y_n = 0$ and $y_i = f(k, x_n, x_{n-1}, ..., x_{i+1})$ for $i = 0, 1, ..., n-1$. Therefore, y is determined by x. Since $x \oplus y \oplus z = 0$, z is also determined by x. In this way y, z are uniquely determined by x. □

Example 2.1. Lemma 2.3 is not true when k is odd.
We present two counterexamples.
(1) Let $k = 3$ and $x = 7$. Then, there are no y, z that satisfy

$$7 \oplus y \oplus z = 0 \tag{8}$$

and

$$y = \lfloor z/3 \rfloor . \tag{9}$$

We prove this by contradiction. Suppose that there exist y, z that satisfy Equations (8) and (9). Then, we have $y, z \leq 7$. Let $y = \sum_{i=1}^{2} y_i 2^i$ and $z = \sum_{i=1}^{2} z_i 2^i$. Since $z \leq 7$, Equation (9) implies that $y = 2, 1, 0$. If $y = 2$, then Equation (8) gives $z = 5$. If $y = 1$, then Equation (8) gives $z = 6$. If $y = 0$, then Equation (8) gives $z = 7$. It is clear that none of $\{y, z\} = \{2, 5\}, \{1, 6\}, \{0, 7\}$ satisfy Equation (9), which leads to a contradiction.
(2) Let $k = 3$ and $x = 5$. Then, there exist more than one pair of y, z that satisfy Equations $x \oplus y \oplus z = 0$ and $y = \lfloor z/k \rfloor$. For example, $\{x, y, z\} = \{5, 2, 7\}$ and $\{x, y, z\} = \{5, 1, 4\}$ satisfy these equations.

Lemma 2.4. *For any $x, y \in Z_{\geq 0}$, there exists some z that satisfies one of the following two conditions:*
(i) $\{x, y, z\}$ satisfies $x \oplus y \oplus z = 0$ and $y \leq \lfloor z/k \rfloor$;
(ii) $x \oplus \lfloor z/k \rfloor \oplus z = 0$ and $\lfloor z/k \rfloor < y$.

Proof. Let $u_i = x_i + y_i \ (mod\ 2)$ for $i = 0, 1, 2, ..., n$. We consider two cases.

`Case (1).` Suppose that $y_n = 0$. Here, we consider four subcases.

`Subcase (1.1).` We assume that, for $i = 0, 1, 2, ..., n-1$,

$$y_i = \lfloor \left(\sum_{j=1}^{n-i} 2^j u_{i+j} - \left(\sum_{j=1}^{n-i-1} 2^j y_{i+j} \right) k \right) /k \rfloor. \tag{10}$$

Then, let $z = u = \sum_{i=0}^{n} u_i 2^i$. By (1) of Lemma 2.2, we have $y = \lfloor z/k \rfloor$, and hence we have statement (i) of this lemma.

`Subcase (1.2).` Suppose that there exists some m such that Equation (10) is true for $i = m+1, m+2, ..., n-1$ and

$$y_m < \lfloor \left(\sum_{j=1}^{n-m} 2^j u_{m+j} - \left(\sum_{j=1}^{n-m-1} 2^j y_{m+j} \right) k \right) /k \rfloor. \tag{11}$$

Then, let $z = u = \sum_{i=0}^{n} u_i 2^i$. By (2) of Lemma 2.2, we have $y < \lfloor z/k \rfloor$, and hence we have statement (i) of this lemma.

`Subcase (1.3).` Suppose that there exists some m such that Equation (10) is true for $i = m+1, m+2, ..., n-1$ and

$$y_m = 1 > 0 = \lfloor \left(\sum_{j=1}^{n-m} 2^j u_{m+j} - \left(\sum_{j=1}^{n-m-1} 2^j y_{m+j} \right) k \right) /k \rfloor.$$

Let $y'_i = y_i$ and $z_i = u_i$ for $i = m+1, m+2, ..., n$. We let $y'_m = 0$, $z_m = x_m + y'_m$, and we also let

$$y'_i = \lfloor \left(\sum_{j=1}^{n-i} 2^j z_{i+j} - \left(\sum_{j=1}^{n-i-1} 2^j y'_{i+j} \right) k \right) /k \rfloor$$

and

$$z_i = x_i + y'_i \ (mod\ 2)$$

for $i = 0, 1, 2, ..., m-1$.

Let $z = \sum_{i=0}^{n} z_i 2^i$ and $y' = \sum_{i=0}^{n} y'_i 2^i$. Clearly, $y' < y$. By (1) of Lemma 2.2, we have $y' = \lfloor z/k \rfloor$, and hence we have statement (ii) of this lemma.

`Subcase (1.4).` Suppose that $y_{n-1} = 1 > 0 = u_n$. Then, by a similar method to that used in Subcase (1.3), we get statement (ii) of this lemma.

`Case (2).` Suppose that $y_n = 1$. Then, let $y'_n = 0$, $z_n = x_n + y'_n \ (mod\ 2)$,

$$y'_i = \lfloor \left(\sum_{j=1}^{n-i} 2^j z_{i+j} - \left(\sum_{j=1}^{n-i-1} 2^j y'_{i+j} \right) k \right) /k \rfloor,$$

and $z_i = y'_i + x_i$ for $i = 0, 1, 2, ..., n-1$.

Let $z = \sum_{i=0}^{n} z_i 2^i$ and $y' = \sum_{i=0}^{n} y'_i 2^i$. Clearly, $y' = \lfloor z/k \rfloor$ and $y' < y$. Then, we have statement (ii) of this lemma. □

Lemma 2.5. *Suppose that*

$$x \oplus y \oplus z = 0 \text{ and } y = \lfloor z/k \rfloor. \tag{12}$$

If there exist $v, w \in Z_{\geq 0}$ such that $x \oplus v \oplus w = 0$ and $v < \lfloor w/k \rfloor$, then $v < y$.

Proof. By Lemma 2.2, we have $x_n = z_n = 1$, $y_n = 0$,

$$y_i = \left\lfloor (\sum_{j=1}^{n-i} 2^j z_{i+j} - (\sum_{j=1}^{n-i-1} 2^j y_{i+j})k)/k \right\rfloor, \tag{13}$$

and

$$z_i = x_i + y_i \ (mod\ 2) \tag{14}$$

for $i = 0, 1, 2, ..., n-1$. Suppose that there exist $v, w \in Z_{\geq 0}$ such that

$$x \oplus v \oplus w = 0 \tag{15}$$

and

$$v < \lfloor w/k \rfloor. \tag{16}$$

From Equations (15) and (16), and using Remark 2.1, there exists $m \in Z_{\geq 0}$ such that, for $i = m+1, ..., n-1$,
$v_i = f(k, x_n, x_{n-1}, ..., x_{i+1})$ and $v_m < f(k, x_n, x_{n-1}, ..., x_{m+1})$.
Using Equation (12) and Remark 2.1, $y_i = f(k, x_n, x_{n-1}, ..., x_{i+1})$ for $i = 1, ..., n-1$. Therefore, we have that $v_i = y_i$ for each $i = m+1, m+2, ..., n$ and $v_m < y_m$. Hence, $v < y$. □

Theorem 2.2. *If $x \oplus y \oplus z = 0$ and $y \leq \lfloor z/k \rfloor$, then the following hold:*
(1) $u \oplus y \oplus z \neq 0$ *for any* $u \in Z_{\geq 0}$ *such that* $u < x$.
(2) $x \oplus v \oplus z \neq 0$ *for any* $v \in Z_{\geq 0}$ *such that* $v < y$.
(3) $x \oplus y \oplus w \neq 0$ *for any* $w \in Z_{\geq 0}$ *such that* $w < z$.
(4) $x \oplus v \oplus w \neq 0$ *for any* $v, w \in Z_{\geq 0}$ *such that* $v < y, w < z$ *and* $v = \lfloor w/k \rfloor$.

Proof. Statements (1), (2), and (3) of this lemma follow directly from the definition of the nim-sum. We now prove statement (4). Let $x \oplus v \oplus w = 0$ and $v = \lfloor w/k \rfloor$ for some $w \in Z_{\geq 0}$ such that $v < y, w < z$. If $y < \lfloor z/k \rfloor$, then Lemma 2.5 implies that $y < v$, which contradicts $v < y$. If $y = \lfloor z/k \rfloor$, then Lemma 2.3 implies $y = v$, which contradicts $v < y$. Therefore, $x \oplus v \oplus w \neq 0$. □

Theorem 2.3. *Suppose that $x \oplus y \oplus z \neq 0$ and $y \leq \lfloor z/k \rfloor$.*
Then, at least one of the following statements is true.
(1) $u \oplus y \oplus z = 0$ *for some* $u \in Z_{\geq 0}$ *such that* $u < x$.
(2) $x \oplus v \oplus z = 0$ *for some* $v \in Z_{\geq 0}$ *such that* $v < y$.
(3) $x \oplus y \oplus w = 0$ *for some* $w \in Z_{\geq 0}$ *such that* $w < z$.
(4) $x \oplus v \oplus w = 0$ *for some* $v, w \in Z_{\geq 0}$ *such that* $v < y, w < z$ *and* $v = \lfloor w/k \rfloor$.

Proof. Suppose that $x_m + y_m + z_m \neq 0$ $(mod\ 2)$ and

$$x_i + y_i + z_i = 0 \ (mod\ 2) \tag{17}$$

for $i = m+1, m+2, ..., n$. We consider three cases.

Case (1). If $x_m = 1$, we define $u = \sum_{j=i+1}^{n} u_i 2^i$ as $u_i = x_i$ for $i = m+1, m+2, ..., n, u_m = 0 < x_m$ and $u_i = y_i + z_i$ for $i = 0, 1, ..., m-1$. Then, we have $u \oplus y \oplus z = 0$ and $u < x$. Therefore, we have statement (1) of this theorem.

Case (2). If $y_m = 1$, then the method employed above for Case (1) can be used to prove that $x \oplus v \oplus z = 0$ for some $v \in Z_{\geq 0}$ such that $v < y$. Therefore, we have statement (2) of this theorem.

Case (3). Next, we suppose that $x_m = y_m = 0$ and $z_m = 1$. By Lemma 2.4, we have either (18) or (19):

$$x \oplus y \oplus w = 0 \text{ and } y \leq \lfloor w/k \rfloor, \tag{18}$$

$$x \oplus \lfloor w/k \rfloor \oplus w = 0 \text{ and } \lfloor w/k \rfloor < y. \tag{19}$$

Here, we consider two subcases.

Subcase (3.1). Suppose that (18) holds. Then, we have $w_i = z_i$ for $i = m+1, m+2, ..., n$, since $x_i + y_i + z_i = 0$ $(mod\ 2)$ for $i = m+1, m+2, ..., n$. Therefore, because $w_m = x_m + y_m = 0 < 1 = z_m$, we have $w < z$, which gives statement (3) of this theorem.

Subcase (3.2). Suppose that (19) holds. Since $x_i + y_i + z_i = 0$ $(mod\ 2)$ for $i = m+1, m+2, ..., n$, we have $w_i = z_i$ for $i = m+1, m+2, ..., n$. By $w_m = x_m + y_m = 0 < 1 = z_m$, we have $w < z$. Letting $v = \lfloor w/k \rfloor$ gives statement (4) of this theorem. □

We now consider the disjunctive sum of the chocolate to the right of the poisoned square and a single strip of chocolate to the left, as in Figures 1.2, 1.3, and 1.4. Hence, we have the state of a chocolate bar with three coordinates $\{x, y, z\}$, where x is the number of possible moves in the strip, y is the number of vertical moves in the bar, and z is the number of horizontal moves. Figures 1.11–1.15 show examples of these coordinates.

Next we define the function $move(\{x, y, z\})$ for each state $\{x, y, z\}$ whose coordinates satisfy the inequality $y \leq \lfloor z/k \rfloor$. The function $move(\{x, y, z\})$ is the set of all states that can be reached directly (i.e., in one step) from state $\{x, y, z\}$.

Definition 2.2. For $x, y, z \in Z_{\geq 0}$, we define $move(\{x, y, z\}) = \{\{u, y, z\} : u < x\} \cup \{\{x, v, z\} : v < y\} \cup \{\{x, y, w\} : w < z\} \cup \{\{x, \min(y, \lfloor w/k \rfloor), w\} : w < z\}$, where $u, v, w \in Z_{\geq 0}$.

Next, we prove that if we start with an element of A_k, then any move leads to an element of B_k.

Lemma 2.6. *For any $\{x, y, z\} \in A_k$, we have $move(\{x, y, z\}) \subset B_k$.*

Proof. Let $\{x, y, z\} \in A_k$. Then, we have $x \oplus y \oplus z = 0$ and $y \leq \lfloor z/k \rfloor$. Let $\{p, q, r\} \in move(\{x, y, z\})$. Next, we prove that $\{p, q, r\} \in B_k$. Since $move(\{x, y, z\}) = \{\{u, y, z\} : u < x\} \cup \{\{x, y, z\} : v < y\}\} \cup \{\{x, y, w\} : w < z\} \cup \{\{x, \min(y, \lfloor w/k \rfloor), w\} : w < z\}$, Theorem 2.2 gives that $p \oplus q \oplus r \neq 0$. Therefore, we have $\{p, q, r\} \in B_k$. $\square$

Next, we prove that if we start with an element of B_k, then there is a proper move that leads to an element of A_k.

Lemma 2.7. *Let $\{x, y, z\} \in B_k$. Then, $move(\{x, y, z\}) \cap A_k \neq \phi$.*

Proof. Let $\{x, y, z\} \in B_k$. Then, we have $x \oplus y \oplus z \neq 0$ and $y \leq \lfloor z/k \rfloor$. Since $move(\{x, y, z\}) = \{\{u, y, z\} : u < x\} \cup \{\{x, v, z\} : v < y\} \cup \{\{x, y, w\} : w < z\} \cup \{\{x, \min(y, \lfloor w/k \rfloor), w\} : w < z\}$, Theorem 2.3 implies that there exists $\{p, q, r\}$ in $move(\{x, y, z\})$ such that $p \oplus q \oplus r = 0$. Therefore, $\{p, q, r\} \in move(\{x, y, z\}) \cap A_k$. $\square$

Lemma 2.8. *Let A_k and B_k be the sets defined in Definition 2.1. A_k is the set of $\mathcal{P}$-positions and B_k is the set of $\mathcal{N}$-positions.*

Proof. If we start the game from a state $\{x, y, z\} \in A_k$, then Lemma 2.6 indicates that any option we take leads to a state $\{p, q, r\}$ in B_k. From this state $\{p, q, r\}$, Lemma 2.7 implies that our opponent can choose a proper option that leads to a state in A_k. Note that any option reduces some of the numbers in the coordinates. In this way, our opponent can always reach a state in A_k, and will finally win by reaching $\{0, 0, 0\} \in A_k$. Therefore, A_k is the set of $\mathcal{P}$-positions.

If we start the game from a state $\{x, y, z\} \in B_k$, then Lemma 2.7 means we can choose a proper option that leads to a state $\{p, q, r\}$ in A_k. From $\{p, q, r\}$, any option taken by our opponent leads to a state in B_k. In this way, we win the game by reaching $\{0, 0, 0\}$. Therefore, B_k is the set of $\mathcal{N}$-positions. $\square$

By Lemma 2.8, the state with coordinates $\{x, y, z\}$ is a $\mathcal{P}$-position when $x \oplus y \oplus z = 0$. Therefore, the nim-value of the chocolate bar to the right is $x = y \oplus z$, which completes the proof of Theorem 2.1.

3. Nim-values of CB(h,k,y,z) When *h* is a Natural Number

In this section, we study the chocolate bar games $CB(h, k, y, z)$ with $h > 0$. There is some overlap with the proofs of the previous section, but $h > 0$ is more involved.

Throughout this section, we assume that k is an even number such that $k \geq 2$. We also assume that p, q, r, h are non-negative integers.
Let $p = \sum_{i=0}^{n} p_i 2^i$, $q = \sum_{i=0}^{n} q_i 2^i$ and $r = \sum_{i=0}^{n} r_i 2^i$ such that $p_i, q_i, r_i \in \{0, 1\}$.

Lemma 3.1. *If $x \oplus y \oplus z = 0$, then $x + y \geq z$, $x + z \geq y$, and $y + z \geq x$.*

Proof. This follows directly from the definition of the nim-sum. □

Lemma 3.2. *Suppose that $p \oplus q \oplus r = 0$ such that $q \leq \lfloor r/k \rfloor$ and $0 \leq h \leq k - 1$. Then, $p \geq h$ if and only if $r \geq h$.*

Proof. We consider two cases.

`Case (1).` If $r < k$, then $q = 0$. From $p \oplus 0 \oplus r = 0$, we have $p = r$. Therefore, $p \geq h$ if and only if $r \geq h$.

`Case (2).` If $r \geq k > h$, then Lemma 3.1 implies that $r \leq p+q$. Therefore, we have $kr \leq kp + kq \leq kp + r$, and hence $(k - 1)r \leq kp$. It follows that $\frac{k-1}{k} h < \frac{k-1}{k} r \leq p$, and hence $h - \frac{h}{k} = \frac{k-1}{k} h < p$. Since $h \leq k - 1$ and h, k, p are integers, we have $h \leq p$. Therefore, we have completed the proof of this lemma. □

Lemma 3.3. *We assume that*

$$p \oplus q \oplus r = 0, \tag{20}$$

$q \leq \lfloor r/k \rfloor$, and $h = k2^t + m2^{t+1}$ for non-negative integers t, m such that $m = 0, 1, 2, ..., \frac{k}{2} - 1$.
Then,

$$p \geq h \tag{21}$$

if and only if

$$r \geq h. \tag{22}$$

Proof. We can assume that $p_n = r_n = 1$ and $q_n = 0$. Let $s = \lfloor \log_2 q \rfloor$. We consider two cases.

`Case (1).` We assume that the inequality in (21) holds. Here, we consider two subcases.

`Subcase (1.1).` Suppose that $s \geq t + 1$. Then, we have $r \geq kq \geq k2^s \geq k2^{t+1} > k2^t + m2^{t+1} = h$, where $m = 0, 1, ..., \frac{k}{2} - 1$.

`Subcase (1.2).` Assume that

$$s \leq t. \tag{23}$$

Since k is even, (21) implies that

$$\sum_{i=t+1}^{n} p_i 2^i \geq k2^t + m2^{t+1}. \tag{24}$$

Inequality (23) means that $q_i = 0$ for $i = t+1, t+2, ..., n$, and hence, by Equation (20), $r_i = p_i$ for $i = t+1, t+2, ..., n$. Therefore, the inequality in (24) implies that $r \geq \sum\limits_{i=t+1}^{n} r_i 2^i = \sum\limits_{i=t+1}^{n} p_i 2^i \geq k2^t + m2^{t+1} = h$.

`Case (2).` Assume that the inequality in (22) is true. Here, we consider two subcases.

`Subcase (2.1).` Suppose that $s \geq t+1$. Then,
$r \geq kq \geq k2^s = \frac{k}{2}2^{s+1}$. Note that k is an even number. Therefore, we have

$$\sum_{i=s+1}^{n} r_i 2^i \geq \frac{k}{2}2^{s+1}. \tag{25}$$

By the definition of s, we have $q_i = 0$ for $i = s+1, s+2, ..., n$, and hence Equation (20) gives that $p_i = r_i$ for $i = s+1, s+2, ..., n$. Therefore, under the inequality in (25), $p \geq \sum\limits_{i=s+1}^{n} p_i 2^i = \sum\limits_{i=s+1}^{n} r_i 2^i \geq \frac{k}{2}2^{s+1} = k2^s \geq k2^{t+1} > k2^t + m2^{t+1} = h$.

`Subcase (2.2).` Suppose that $s \leq t$. Then, we have $q_i = 0$ for $i = t+1, ..., n$, and hence $p_i = r_i$ for $i = t+1, ..., n$. Therefore, from the inequality in (22), $p \geq \sum\limits_{i=t+1}^{n} p_i 2^i = \sum\limits_{i=t+1}^{n} r_i 2^i \geq k2^t + m2^{t+1} = h$. □

Lemmas 3.3 and 3.2 are not true if h does not satisfy their respective conditions. We now present a counterexample.

Example 3.1. Let k be an even number, and let $h = k2^t + (2s+1)2^t$, $p = k2^t + s2^{t+1}$, $q = 2^t$, and $r = k2^t + (2s+1)2^t$ for some non-negative integers t, s such that $(2s+1) < k$. Then, we have that $p \oplus q \oplus r = 0$, $q \leq \lfloor r/k \rfloor$ and $r \geq h$, but we also have $p < h$.

It is useful to know the disjunctive sum of the chocolate to the right of the poisoned square and for a single strip of chocolate to the left. For example, if we make a disjunctive sum of a single strip of chocolate and the chocolate bar in Figure 1.6, then we have the chocolate in Figure 1.4. We denote the state of the disjunctive sum as $\{x, y, z\}$, where x is the number of possible moves in the strip, y is the number of vertical moves in the bar, and z is the number of horizontal moves.

We define the function $moveh(\{x, y, z\})$ for each state $\{x, y, z\}$ whose coordinates satisfy $y \leq \lfloor (z+h)/k \rfloor$. The function $moveh(\{x, y, z\})$ is the set of states that can be reached directly from $\{x, y, z\}$.

Definition 3.1. For $x, y, z \in Z_{\geq 0}$, we define $moveh(\{x, y, z\}) = \{\{u, y, z\} : u < x\} \cup \{\{x, v, z\} : v < y\} \cup \{\{x, y, w\} : w < z\} \cup \{\{x, \min(y, \lfloor (z+h)/k \rfloor), w\} : w < z\}$, where $u, v, w \in Z_{\geq 0}$.

Definition 3.2. Let $A_{k,h} = \{\{x, y, z\} : x, y, z \in Z_{\geq 0}, y \leq \lfloor (z+h)/k \rfloor$ and $(x+h) \oplus y \oplus (z+h) = 0\}$, $B_{k,h} = \{\{x, y, z\} : x, y, z \in Z_{\geq 0}, y \leq \lfloor (z+h)/k \rfloor$, and $(x+h) \oplus y \oplus (z+h) \neq 0\}$.

Lemma 3.4. *For any $y, z \in Z_{\geq 0}$, we have*

$$y \leq \lceil z/k \rceil \quad \textit{if and only if} \quad y \leq \lfloor (z+k-1)/k \rfloor$$

and

$$y = \lceil z/k \rceil \quad \textit{if and only if} \quad y = \lfloor (z+k-1)/k \rfloor,$$

where $\lceil \quad \rceil$ is the ceiling function.

Proof. This result follows directly from the definitions of the floor function and the ceiling function. □

Lemma 3.5. *For $x, y, z \in Z_{\geq 0}$, we have:*
(1) $\{x, y, z\} \in A_{k,h}$ *if and only if* $\{x+h, y, z+h\} \in A_k$.
(2) $\{x, y, z\} \in B_{k,h}$ *if and only if* $\{x+h, y, z+h\} \in B_k$.

Proof. These results follow directly from Definition 3.2. □

Lemma 3.6. *We have $moveh(\{x, y, z\}) \subset B_{k,h}$ for any $\{x, y, z\} \in A_{k,h}$.*

Proof. Let $\{x, y, z\} \in A_{k,h}$. Then, by Lemma 3.5, we have

$$\{x+h, y, z+h\} \in A_k. \tag{26}$$

We consider two cases.

`Case (1).` Let $\{u, y, z\}$, $\{x, v, z\}$, $\{x, y, w\} \in moveh(\{x, y, z\})$ such that $u < x$, $v < y$, and $w < z$. Then, using Definitions 2.2 and 3.1 with Lemma 3.5, we have

$$\{u+h, y, z+h\}, \{x+h, v, z+h\}, \{x+h, y, w+h\} \in move(\{x+h, y, z+h\}). \tag{27}$$

From relations (26), (27), and Lemma 2.6, we have that $\{u+h, y, z+h\}$, $\{x+h, v, z+h\}$, $\{x+h, y, w+h\} \in B_k$. Hence, by Lemma 3.5, we have $\{u, y, z\}$,$\{x, v, z\}$,$\{x, y, w\} \in B_{k,h}$.

`Case (2).` Let $\{x, \min(y, \lfloor (w+h)/k \rfloor), w\} \in moveh(\{x, y, z\})$ such that $w < z$. Then, by Definitions 2.2 and 3.1 with Lemma 3.5, we have

$$\{x+h, \min(y, \lfloor (w+h)/k \rfloor), w+h\} \in move(\{x+h, y, z+h\}). \tag{28}$$

From relations (26), (28), and Lemma 2.6, we have that $\{x+h, \min(y, \lfloor (w+h)/k \rfloor), w+h\} \in B_k$, and hence Lemma 3.5 implies that $\{x, \min(y, \lfloor (w+h)/k \rfloor), w\} \in B_{k,h}$. □

Lemma 3.7. *Let h satisfy one of the following conditions:*
(1) *h can be written in the form $h = k2^t + m2^{t+1}$ for non-negative integers t, m such that $m = 0, 1, 2, ..., \frac{k}{2} - 1$.*
(2) *$h \in \{1, 2, ..., k-1\}$.*
Then, for each $\{x, y, z\} \in B_{k,h}$, we have $moveh(\{x, y, z\}) \cap A_{k,h} \neq \phi$.

Proof. Let $\{x, y, z\} \in B_{k,h}$. Then, by Lemma 3.5, we have $\{x+h, y, z+h\} \in B_k$, and hence

$$y \leq \lfloor (z+h)/k \rfloor. \tag{29}$$

By Theorem 2.3, Lemma 2.7, and Definition 2.2, at least one of the following cases holds:

`Case (1).` There exists $u < x+h$ such that $\{u, y, z+h\} \in A_k$, and hence $u \oplus y \oplus (z+h) = 0$.
Since $z+h \geq h$, the inequality in (29) and Lemmas 3.3 and 3.2 gives $u \geq h$.
Let $u' + h = u$. Then, $0 \leq u' < x$. $\{u'+h, y, z+h\}= \{u, y, z+h\} \in A_k$, and hence, by Lemma 3.5, we have $\{u', y, z\} \in A_{k,h}$. Clearly, $\{u', y, z\} \in moveh(\{x, y, z\})$.

`Case (2).` There exists $v < y$ such that $\{x+h, v, z+h\} \in A_k$, and hence, by Lemma 3.5, we have $\{x, v, z\} \in A_{k,h}$. Clearly, $\{x, v, z\} \in moveh(\{x, y, z\})$.

`Case (3).` There exists $w < z+h$ such that $\{x+h, y, w\} \in A_k$, and hence $(x+h) \oplus y \oplus w = 0$ and

$$y \leq \lfloor w/k \rfloor. \tag{30}$$

Since $x+h \geq h$, the inequality in (30) can be combined with Lemmas 3.3 and 3.2 to give $h \leq w$.
Let $w' + h = w$. Then, $0 \leq w' < z$, $\{x+h, y, w'+h\}= \{x+h, y, w\} \in A_k$, and hence, by Lemma 3.5, we have $\{x, y, w'\} \in A_{k,h}$. Clearly, $\{x, y, w'\} \in moveh(\{x, y, z\})$.

`Case (4).` There exists $w < z+h$ such that

$$v = \lfloor w/k \rfloor \tag{31}$$

and $\{x+h, v, w\} \in A_k$. Hence, $(x+h) \oplus v \oplus w = 0$. Since $x+h \geq h$, Equation (31) with Lemmas 3.3 and 3.2 imply that $w \geq h$. Let $w' + h = w$. Then, $0 \leq w' < z$, $\{x+h, v, w'+h\}= \{x+h, v, w\} \in A_k$, and hence, by Lemma 3.5, we have $\{x, v, w'\} \in A_{k,h}$. Clearly, $\{x, v, w'\} \in moveh(\{x, y, z\})$. □

Theorem 3.1. *Let h satisfy condition (1) or (2) in Lemma 3.7. Then, $A_{k,h}$ is the set of $\mathcal{P}$-positions and $B_{k,h}$ is the set of $\mathcal{N}$-positions of the game. (Note that, throughout this section, we assume that k is an even number.)*

Proof. Using the same method as in Lemma 2.8, this is clear from Lemmas 3.6 and 3.7. □

Theorem 3.2. *Let h satisfy one of the following conditions:*
(1) *h can be written in the form $h = k2^t + m2^{t+1}$ for non-negative integers t, m such that $m = 0, 1, 2, ..., \frac{k}{2} - 1$, or*
(2) *$h \in \{1, 2, ..., k-1\}$.*
Then, the nim-value of $CB(h, k, y, z)$ is $(y \oplus (z + h)) - h$. (We again assume that k is an even number.)

Proof. By Theorem 3.1, a state $\{x, y, z\}$ of the disjunctive sum of the chocolate to the right of the poisoned square and a single strip of chocolate to the left is a $\mathcal{P}$-position when $(x+h) \oplus y \oplus (z+h) = 0$. Thus, the nim-value of the chocolate bar to the right is $x = (x + h) - h = (y \oplus (z + h)) - h$. Therefore, we have completed the proof. $\square$

Lemma 3.7 and Theorem 3.1 do not hold if h satisfies neither (1) nor (2) of Lemma 3.7.

Example 3.2. Suppose that h does not satisfy (1) or (2) of Lemma 3.7. Then, $h = k2^t + (2s + 1)2^t$ for some non-negative integers t, s such that $(2s + 1) < k$. We have $\{k2^{t+1}, 2^t, 0\} \in B_{k,h}$, since $2^t \leq \lfloor (0 + h)/k \rfloor$ and $(k2^{t+1} + h) \oplus 2^t \oplus h \neq 0$. For $\{k2^{t+1}, 2^t, 0\}$, there is no option that leads to an element of $A_{k,h}$. Note that $\{k2^{t+1} + h, 2^t, h\} \in B_k$ and $\{k2^t + s2^{t+1}, 2^t, h\} \in A_k \cap move(\{k2^{t+1} + h, 2^t, h\})$, but because $k2^t + s2^{t+1} < h$, we have $moveh(\{k2^{t+1}, 2^t, 0\}) \cap A_{k,h} = \emptyset$.

4. Chocolates Without Simple Formulas for $\mathcal{P}$-positions

In this section, we study the nim-values of the Chocolate Bar $CB(0, 1, y, z)$. Figure 4.1 is an example of such a bar. The mathematical structure of this chocolate bar is interesting when compared to that of $CB(0, k, y, z)$ for an even number k. The nim-value of this chocolate bar has a complicated mathematical structure.

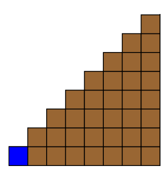


Figure 4.1.

4.1. Structure of Each Row of the chart of Nim-values

If we let $G_{1,0}(\{y, z\})$ be the nim-value for the Chocolate Bar $CB(0, 1, y, z)$, we can form a chart of the nim-values. In this section, we present some conjectures about the nim-value $G_{1,0}$ using the computer algebra system Mathematica.

Figure 4.2 shows the nim-values $G_{1,0}(\{y,z\})$. Note that, in this figure, the horizontal direction shows the y-coordinate, and the vertical direction gives the z-coordinate. For example, $G_{1,0}(\{2,3\}) = 1$ and $G_{1,0}(\{5,9\}) = 12$.

Example 4.1. The following Mathematica program calculates $G_{1,0}(\{y,z\})$ for any $y, z \in Z_{\geq 0}$ such that $y \leq z$. In this program, "allcases" is the set of all states $\{a,b\}$ of the chocolate for $a, b = 0, 1, 2, ..., 30$ and $a \leq b$. $Gr(\{a,b\})$ is the nim-value.

```
k = 1;
ss=30;al = Flatten[Table[{a,b},{a,0,ss},{b,0,ss}],1];
allcases = Select[al,(1/k)(#[[2]]) >= #[[1]]&];
move[z_]:= Block[{p},p = z;
Union[Table[{Min[Floor[(1/k)(t2)],p[[1]]],t2},
{t2,0,p[[2]] - 1}],
Table[{t1,p[[2]]},{t1,0,p[[1]]-1}]]];
Mex[L_]:= Min[Complement[Range[0,Length[L]],L]];
Gr[pos_]:= Gr[pos] = Mex[Map[Gr,move[pos]]];
pposition = Select[allcases,Gr[#] == 0 &];
```

Z \ Y	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0															
1	1	2														
2	2	1	3													
3	3	4	1	5												
4	4	3	5	1	6											
5	5	6	4	7	1	8										
6	6	5	7	4	8	1	9									
7	7	8	6	9	4	10	1	11								
8	8	7	9	6	10	4	11	1	12							
9	9	10	8	11	7	12	4	13	1	14						
10	10	9	11	8	12	7	13	4	14	1	15					
11	11	12	10	13	9	14	7	15	4	16	1	17				
12	12	11	13	10	14	9	15	7	16	4	17	1	18			
13	13	14	12	15	11	16	10	17	7	18	4	19	1	20		
14	14	13	15	12	16	11	17	10	18	7	19	4	20	1	21	
15	15	16	14	17	13	18	12	19	10	20	7	21	4	22	1	23

Figure 4.2.

9	9	10	8	11	7	12	4	13	1	14

Figure 4.3.

From Figure 4.3, we arrive at Conjecture 4.1.

Conjecture 4.1. Suppose that $z = 4m+1$ for some non-negative integer m. Then,
(1) $G_{1,0}(\{2i-1, 4m+1\}) = 4m+1+i$ for $i = 1, 2, ..., 2m+1$.
(2) $G_{1,0}(\{2i, 4m+1\}) = 4m+1-i$ for $i = 0, 1, 2, ..., m$.
(3) $G_{1,0}(\{2i, 4m+1\}) = 6m+1-3i$ for $i = m+1, m+2, ..., 2m$.

10	10	9	11	8	12	7	13	4	14	1	15

Figure 4.4.

Figure 4.4 then leads to Conjecture 4.2.

Conjecture 4.2. Suppose that $z = 4m+2$ for some non-negative integer m. Then,
(1) $G_{1,0}(\{2i, 4m + 2\}) = 4m + 2 + i$ for $i = 0, 1, 2, ..., 2m + 1$.
(2) $G_{1,0}(\{2i - 1, 4m + 2\}) = 4m + 2 - i$ for $i = 1, 2, ..., m + 1$.
(3) $G_{1,0}(\{2i - 1, 4m + 2\}) = 6m + 4 - 3i$ for $i = m + 2, m + 2, ..., 2m + 1$.

11	11	12	10	13	9	14	7	15	4	16	1	17

Figure 4.5.

From Figure 4.5, we have Conjecture 4.3.

Conjecture 4.3. Suppose that $z = 4m+3$ for some non-negative integer m. Then,
(1) $G_{1,0}(\{2i - 1, 4m + 3\}) = 4m + 3 + i$ for $i = 1, 2, ..., 2m + 2$.
(2) $G_{1,0}(\{2i, 4m + 3\}) = 4m + 3 - i$ for $i = 0, 1, 2, ..., m$.
(3) $G_{1,0}(\{2i, 4m + 3\}) = 6m + 4 - 3i$ for $i = m + 1, m + 2, ..., 2m + 1$.

12	12	11	13	10	14	9	15	7	16	4	17	1	18

Figure 4.6.

Finally, from Figure 4.6, we can state Conjecture 4.4.

Conjecture 4.4. Suppose that $z = 4m+4$ for some non-negative integer m. Then,
(1) $G_{1,0}(\{2i, 4m + 4\}) = 4m + 4 + i$ for $i = 0, 1, 2, ..., 2m + 2$.
(2) $G_{1,0}(\{2i - 1, 4m + 4\}) = 4m + 4 - i$ for $i = 1, 2, ..., m + 1$.
(3) $G_{1,0}(\{2i - 1, 4m + 4\}) = 6m + 7 - 3i$ for $i = m + 2, ..., 2m + 2$.

The authors have attempted to prove these conjectures using mathematical induction, but have not thus far succeeded. The difficulty lies in the fact that, using mathematical induction, there are too many cases to cover.

Although these conjectures have not been proved, the patterns of nim-values show that the mathematical structure of this chocolate game is very different from that of the chocolate games treated in previous sections.

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