

**PARTIZAN KAYLES AND MISÈRE INVERTIBILITY****Rebecca Milley***Computational Mathematics, Grenfell Campus, Memorial University of  
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The impartial combinatorial game KAYLES is played on a row of pins, with players taking turns removing either a single pin or two adjacent pins. A natural partizan variation is to allow one player to remove only a single pin and the other only a pair of pins. This paper develops a complete solution for PARTIZAN KAYLES under misère play, including the misère monoid of all possible sums of positions, and discusses its significance in the context of misère invertibility: the universe of PARTIZAN KAYLES contains a position whose additive inverse is not its negative, and moreover, this position is an example of a right-win game whose inverse is previous-win.

**1. Introduction**

In the game of KAYLES, two players take turns throwing a bowling ball at a row of pins. A player either hits dead-on and knocks down a single pin, or hits in-between and knocks down a pair of adjacent pins. This game has been analyzed for both normal play (under which the player who knocks down the last pin wins) and misère play (the player who knocks down the last pin loses) [3, 13, 9]. Since both players have the same legal moves, KAYLES is an *impartial* game. Although there are several natural non-impartial or *partizan* variations, in this paper the rule set of PARTIZAN KAYLES is as follows: the female player ‘Left’ can only knock down a single pin and the male player ‘Right’ can only knock down a pair of adjacent pins. This game can be seen as a one-dimensional variant of DOMINEERING, played on strips of squares (representing the rows of pins), with Left placing the bottom half of her vertical dominoes and Right placing his horizontal dominoes as usual. For notational purposes, we will play ‘domineering-style’, on  $1 \times n$  strips denoted  $S_n$ , with Left placing squares and Right placing dominoes, as illustrated in Figure 1.

This paper develops a complete solution for PARTIZAN KAYLES under misère play. We will see that the set (*universe*) of PARTIZAN KAYLES positions is remarkable for

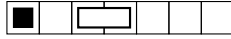


Figure 1: A game of PARTIZAN KAYLES: after one Left move and one Right move, the original position  $S_6$  becomes  $S_1 + S_3$ .

its unusual examples of invertibility.

It is assumed that the reader is familiar with basic normal-play combinatorial game theory<sup>1</sup>. A brief review of the necessary misère background is given in Section 1.1; a more detailed overview can be found in [8]. Section 2 establishes domination of options among PARTIZAN KAYLES positions, and Section 3 uses this to show how every strip of length at least three reduces to a disjunctive sum of single squares and strips of length two. Section 4 then gives the outcome and strategy for a general sum, including the *misère monoid* of the universe of PARTIZAN KAYLES positions. Finally, Section 5 discusses the significance of this universe in the context of misère invertibility.

### 1.1. Misère Prerequisites

A game or *position* is defined in terms of its options:  $G = \{G^L \mid G^R\}$ , where  $G^L$  is the set of positions  $G^L$  to which Left can move in one turn, and similarly for  $G^R$ . The simplest game is the zero game,  $0 = \{\cdot \mid \cdot\}$ , where the dot indicates an empty set of options. The outcome function  $o^-(G)$  gives the misère outcome of a game  $G$ . If  $o^-(G) = \mathcal{P}$  we write  $G \in \mathcal{P}^-$ , where again the superscript indicates that this is the outcome under misère play. Thus, for example,  $0 \in \mathcal{N}^-$ . Under either ending condition, the outcome classes are partially ordered as shown in Figure 2.

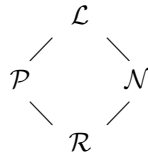


Figure 2: The partial order of outcome classes.

Many definitions from normal-play theory are used without modification for misère games, including disjunctive sum, equality, and inequality, as well as domination and reversibility. However, the normal-play *negative* of  $G$ ,  $-G = \{-G^R \mid -G^L\}$ , is instead called the *conjugate* of  $G$ , denoted  $\overline{G}$ , since in general we do not have

<sup>1</sup>A complete overview of normal-play combinatorial game theory can be found in [1].

$G + \overline{G} = 0$  in misère play [5]. Thus

$$\overline{G} = \{\overline{G^{\mathcal{R}} | G^{\mathcal{L}}}\}.$$

In addition to the well-studied *normal-play canonical form*, every position has a unique *misère canonical form* obtained by eliminating dominated options and bypassing reversible ones [14]; note that the definitions of domination and reversibility are indeed dependant on the ending condition, since the definition of inequality is dependant on the ending condition.

Misère games are much more difficult to analyze than normal-play games (see [11], for example). No position besides  $\{\cdot | \cdot\}$  is equal to 0 [5]; thus, no non-zero position has an additive inverse, and there is no easy test for equality or inequality of games. This in turn means that instances of domination and reversibility are rare and hard to establish, so that we cannot take advantage of canonical forms in misère play as we do in normal play.

Some of these problems are mitigated by considering restricted versions of equality and inequality [10, 12]. Let  $\mathcal{U}$  be a set of games closed under disjunctive sum and followers (but not necessarily under conjugation). *Indistinguishability* or *equivalence (modulo  $\mathcal{U}$ )* is defined by

$$G \equiv H \pmod{\mathcal{U}} \text{ if } o^-(G + X) = o^-(H + X) \text{ for all games } X \in \mathcal{U},$$

while *inequality (modulo  $\mathcal{U}$ )* is defined by

$$G \geq H \pmod{\mathcal{U}} \text{ if } o^-(G + X) \geq o^-(H + X) \text{ for all games } X \in \mathcal{U}.$$

The set  $\mathcal{U}$  is called the *universe*. If  $G \not\equiv H \pmod{\mathcal{U}}$  then  $G$  and  $H$  are said to be *distinguishable modulo  $\mathcal{U}$* , and in this case there must be a game  $X \in \mathcal{U}$  such that  $o^-(G + X) \neq o^-(H + X)$ . If  $G \not\geq H \pmod{\mathcal{U}}$  and  $G \not\leq H \pmod{\mathcal{U}}$  then  $G$  and  $H$  are *incomparable* in  $\mathcal{U}$ . The symbol  $\not\geq$  is used to indicate strict modular inequality. Thus, the symbols  $\equiv$ ,  $\geq$ , and  $\not\geq \pmod{\mathcal{U}}$  correspond to, but are differentiated from, the symbols  $=$ ,  $\geq$ , and  $>$ , respectively, which are used in non-restricted misere play. In this paper it is assumed that both  $G$  and  $H$  are contained in  $\mathcal{U}$  when we compare them modulo  $\mathcal{U}$ .

Note that indistinguishability is a congruence relation. Given a universe  $\mathcal{U}$ , we can determine the equivalence classes under indistinguishability modulo  $\mathcal{U}$ . Since we may still not have inverses for every element, the classes form a quotient monoid. Together with the tetra-partition of elements into the sets  $\mathcal{P}^-$ ,  $\mathcal{N}^-$ ,  $\mathcal{R}^-$ , and  $\mathcal{L}^-$ , this quotient is called the *misère monoid* of the set  $\mathcal{U}$ , denoted  $\mathcal{M}_{\mathcal{U}}$  [10].

Indistinguishability and misère monoids have been successfully used to analyze various impartial [12] and partizan games [2, 7, 4]. This paper, which summarizes a section of the author’s PhD thesis [6], develops the monoid for the universe of PARTIZAN KAYLES positions, and discusses its relevance to ‘restricted’ (modulo  $\mathcal{U}$ )

misère invertibility. The game of PARTIZAN KAYLES is a *placement game*, in that players move by putting pieces on a board, and is thus also *dead-ending*, meaning that once a player has no current move, that player will never have another move. The universe of dead-ending games is introduced and explored in [8], and both this and the subuniverse of placement games are exciting areas of current misère research.

## 2. Domination

The goal of this section is to establish domination of moves in PARTIZAN KAYLES, with the concluding and most important result (Corollary 3) being that  $S_2 \not\geq S_1 + S_1$  modulo this universe. Recall that  $S_n$  denotes a strip of length  $n$ . The disjunctive sum of  $k$  copies of  $G$  is denoted  $kG$ , so that, for example,  $S_1 + S_1 = 2S_1$ . Let  $\mathcal{K}$  be the universe of PARTIZAN KAYLES positions; that is,  $\mathcal{K}$  is the set of all possible sums of positions of the form  $S_n$ . Note that  $S_0 = \{\cdot | \cdot\} = 0$  and  $S_1 = \{0 | \cdot\} = 1$  (the normal-play canonical-form integers), but this is not the case for higher values of  $n$ ; for example,  $S_2 = \{1 | 0\} \neq 2$ .

It should be immediately apparent to any player of misère games that this version of KAYLES is heavily biased in favour of Right: Left can always move, if the position is non-zero, while Right cannot move on any sum of single squares. It is therefore not surprising that there are no left-win positions in this universe, as demonstrated in Lemma 1. As a consequence, we have Corollary 1: if Left can win playing first in a PARTIZAN KAYLES position under misère play, then the position must be in  $\mathcal{N}^-$ , and similarly if Left can win playing second then the position is in  $\mathcal{P}^-$ .

**Lemma 1.** *If  $G \in \mathcal{K}$  then  $G \notin \mathcal{L}^-$ .*

*Proof.* If  $G \in \mathcal{K}$  then  $G$  is a sum of positions of the form  $S_n$ . Let  $m$  be the total number of squares in  $G$ . Note that each of Right's turns reduces the total number of free squares by 2 and each of Left's moves reduces the number by 1.

If the total number  $m$  is a multiple of 3 and Right plays first, then Left begins each turn with  $3k + 1$  free squares (for some  $k \in \mathbb{N}$ ); in particular Left never begins a turn with zero free squares, and so can never run out of moves before Right. This shows Right wins playing first, so  $G \in \mathcal{R}^- \cup \mathcal{N}^-$ .

If  $m \equiv 1 \pmod{3}$  then Left playing first begins each turn with  $3k + 1$  free squares and Left playing second begins each turn with  $3k + 2$  free squares; in either case Left cannot run out of moves before Right by the same argument as above. Here Right wins playing first or second so  $G \in \mathcal{R}^-$ .

Finally, if  $m \equiv 2 \pmod{3}$ , then Left playing first necessarily moves the game to one in which the total number of squares is congruent to 1 modulo 3, and as shown

above this is a right-win position. Thus Left loses playing first and the game is in  $\mathcal{R}^-$  or  $\mathcal{P}^-$ .  $\square$

**Corollary 1.** *If Left wins playing first on  $G \in \mathcal{K}$ , then  $G \in \mathcal{N}^-$ , and if Left wins playing second on  $G$ , then  $G \in \mathcal{P}^-$ .*

Since single squares are so detrimental for Left, we might naively suspect that Left should get rid of them as quickly as she can<sup>2</sup>. That is, given a position that contains an  $S_1$ , Left should do at least as well by playing in the  $S_1$  as playing anywhere else. This is indeed the case, as established in Corollary 2. The bulk of the work is done in Lemma 2. Lemma 1 (that is, the non-existence of left-win positions in  $\mathcal{K}$ ) is used repeatedly without reference in the following proof.

**Lemma 2.** *If  $G \in \mathcal{K}$  and  $G^L$  is any left option of  $G$  then  $G \geq G^L + S_1 \pmod{\mathcal{K}}$ .*

*Proof.* We must show that  $o^-(G + X) \geq o^-(G^L + S_1 + X)$ , for any  $X \in \mathcal{K}$ , where  $G^L$  is any Left option of  $G$ . Since  $G$  is already an arbitrary game in  $\mathcal{K}$ , it suffices to show  $o^-(G) \geq o^-(G^L + S_1)$ . To do so we will show that when  $G^L + S_1$  is in  $\mathcal{N}^-$ ,  $G$  is also in  $\mathcal{N}^-$ , and that when  $G^L + S_1$  is in  $\mathcal{P}^-$ ,  $G$  is also in  $\mathcal{P}^-$ . If  $G^L + S_1$  is in  $\mathcal{R}^-$ , then we trivially have  $o^-(G) \geq o^-(G^L + S_1)$ .

Suppose  $G^L + S_1 \in \mathcal{N}^-$ , so that Left has a good first move in  $G^L + S_1$ . If the good move is to  $G^L + 0 = G^L$  then Left has the same good first move in  $G$ , and so  $G \in \mathcal{N}^-$ . Otherwise the good move is to  $G^{LL} + S_1 \in \mathcal{P}^-$ , for some Left option  $G^{LL}$  of  $G^L$ ; but then by induction  $o^-(G^L) \geq o^-(G^{LL} + S_1)$  and so  $G^L \in \mathcal{P}^-$  and  $G \in \mathcal{N}^-$ .

Now suppose  $G^L + S_1 \in \mathcal{P}^-$ . We must show that  $G \in \mathcal{P}^-$ . Since Right has no good first move in  $G^L + S_1$ , we have  $G^{LR} + S_1 \in \mathcal{N}^-$  for every right option  $G^{LR}$  of  $G^L$ . So Left has a good first move in  $G^{LR} + S_1$ ; by induction the move to  $G^{LR}$  is at least as good as any other, and so  $G^{LR} \in \mathcal{P}^-$  for every right option  $G^{LR}$  of  $G^L$ . We will see that there exists a previous-win Left response  $G^{RL}$  to every first Right move  $G^R$ , by finding a  $G^{RL}$  that is equal to some  $G^{LR} \in \mathcal{P}^-$ . This will give a winning strategy for Left playing second in  $G$ , proving  $G \in \mathcal{P}^-$ .

Let  $G^R$  be any right option. If the domino placed by Right to move from  $G$  to  $G^R$  would not overlap the square placed by Left to move from  $G$  to  $G^L$ , then Left can place that square now, achieving a position  $G^{RL}$  equal to some  $G^{LR}$ , which we know to be in  $\mathcal{P}^-$ . This is a good second move for Left in  $G$ , so  $G \in \mathcal{P}^-$ . If Right's move from  $G$  to  $G^R$  *does* interfere with Left's move from  $G$  to  $G^L$ , then we will see that Left can still move  $G^R$  to a position equal to some  $G^{LR}$ . If there are

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<sup>2</sup>In normal play, there is a principle of *number avoidance*, which says that players should save (avoid) positions in which the opponent can never move. A naive approach to misère play is to assume that the opposite strategy should be optimal, but in fact this does not always (or even usually) work: for example, the game  $S_1 = \{0 \mid \cdot\}$  is incomparable with the zero game in general misère play, and so in general it is not true that Left would rather have nothing than have a single Left move.

free squares adjacent to both sides of the domino Right places for  $G^R$ , then Left can respond to  $G^R$  by playing in one of those squares, so that the resulting position  $G^{RL}$  is equal to a position  $G^{LR} \in \mathcal{P}^-$ . This is illustrated in Figure 3.

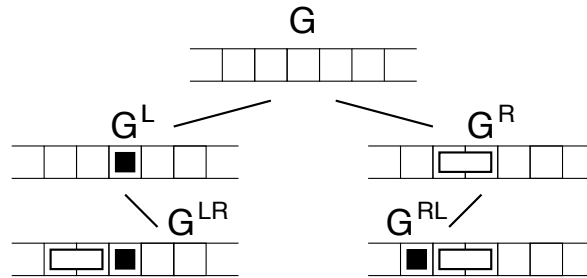


Figure 3: Although the pieces for  $G^L$  and  $G^R$  overlap, Left still has an option of  $G^R$  that is equal to a right option of  $G^L$ .

If Left cannot so easily obtain such a position — if there is not a free square on both sides of Right’s domino — then we have several cases to consider.

**Case 1:** The domino placed by Right to move from  $G$  to  $G^R$  is at the end of a component  $S_n$ ,  $n \geq 4$ . So  $G = S_n + G'$ , for some position  $G'$  (perhaps equal to 0), and  $G^R = S_{n-2} + G'$ . Since this domino interferes with Left’s move from  $G$  to  $G^L$ , Left’s move to  $G^L$  must be to place a square at the end or one away from the end of  $S_n$ :  $G^L = S_{n-1} + G'$  or  $G^L = S_1 + S_{n-2} + G'$ . If the former, then as above Left responds to  $G^R$  by playing adjacent to Right, obtaining  $G^{RL} = S_{n-3} + G'$ , which is in  $\mathcal{P}^-$  because it is a Right option of  $G^L = S_{n-1} + G'$ . If the latter, then Left responds to  $G^R$  by playing one away from the end of  $S_{n-2}$ , leaving the position  $S_1 + S_{n-4} + G'$ , which is a Right option of  $G^L = S_1 + S_{n-2} + G'$ , and is therefore in  $\mathcal{P}^-$ .

**Case 2:** The domino placed by Right to move from  $G$  to  $G^R$  is at the end of a component  $S_3$ . So  $G = S_3 + G'$  and  $G^R = S_1 + G'$ . If Left’s move to  $G^L$  is to play at the end of this  $S_3$ , then she simply plays in the  $S_1$  now to obtain  $G^{RL} = G'$ , which is equal to a  $G^{LR}$  and so is in  $\mathcal{P}^-$ . If Left’s move to  $G^L$  is to play in the middle of  $S_3$ , then  $G^L = S_1 + S_1 + G'$ . It cannot be that  $G'$  is a sum of all  $S_1$  positions, else  $G^L + S_1$  is right-win, and we are assuming it is previous-win. So there is at least one component  $S_n$  in  $G'$  with  $n \geq 2$ . Thus we can write  $G' = S_n + G''$ . Left should respond to  $G^R = S_1 + S_n + G''$  by moving one away from the end of the  $S_n$ , to obtain  $G^{RL} = S_1 + S_1 + S_{n-2} + G''$ , which is in  $\mathcal{P}^-$  because it is a right option of  $G^L = S_1 + S_1 + G' = S_1 + S_1 + S_n + G''$ .

**Case 3:** The domino placed by Right to move from  $G$  to  $G^R$  is in an  $S_2$ . So

$G = S_2 + G'$  and  $G^R = G'$ . Since this domino interferes with Left's move to  $G^L$ , we must have  $G^L = S_1 + G'$ . Again, it cannot be that  $G'$  is a sum of all  $S_1$  positions, since  $G^L + S_1 \in \mathcal{P}^-$ . So there is at least one component  $S_n$  in  $G'$  with  $n \geq 2$ . Left should respond to  $G^R = G' = S_n + G''$  by playing one away from the end of  $S_n$ , so that she obtains the position  $S_1 + S_{n-2} + G''$ , which is a Right option from  $G^L = S_1 + G' = S_1 + S_n + G''$ .

In every case Left has a good second move in  $G$  ( $G^{RL} \in \mathcal{P}^-$ ) and so  $G \in \mathcal{P}^-$ , as required.  $\square$

As corollaries of this lemma we obtain both a general strategy for Left in PARTIZAN KAYLES as well as the inequality  $S_2 \succeq 2S_1 \pmod{\mathcal{K}}$ .

**Corollary 2.** *For any position  $G \in \mathcal{K}$ , if Left can win  $G + S_1$  then Left can win by moving to  $G$ .*

*Proof.* Any other option of  $G + S_1$  is of the form  $G^L + S_1$ , and by Lemma 2,  $G^L + S_1$  is dominated by  $G$ .  $\square$

**Corollary 3.**  $S_2 \succeq 2S_1 \pmod{\mathcal{K}}$ .

*Proof.*  $S_2 \succeq 2S_1$  follows directly from Lemma 2 with  $G = S_2$ , since the only left option  $G^L$  is  $S_1$ . The inequality is strict because  $S_2 \in \mathcal{P}^-$  while  $2S_1 \in \mathcal{R}^-$ .  $\square$

### 3. Reduction

Corollary 3 is the key to the solution of PARTIZAN KAYLES: it allows us, by establishing domination of options, to show that every strip  $S_n$  'splits' into a sum of  $S_1$  positions (single squares) and  $S_2$  positions (dominoes). Theorem 1 demonstrates the reduction. Let us work through a few reductions by hand to gain some insight into this process. These reductions are illustrated in Figure 4. All instances of domination in this discussion are modulo  $\mathcal{K}$ .

Trivially,  $S_1$  and  $S_2$  are already sums of single squares and dominoes. In a strip of length 3, Left has options to  $S_2$  (playing at either end) and  $S_1 + S_1$  (playing in the middle). Right has only one option, to  $S_1$ . These are precisely the options of  $S_1 + S_2$ ; both games are equal to  $\{S_2, 2S_1 \mid S_1\}$  (which, in canonical form modulo  $\mathcal{K}$ , is the game  $\{S_2 \mid S_1\}$ , by Corollary 3). Thus,  $S_3 = S_1 + S_2$ .

In a strip of length 4, Left's options are to  $S_3$  or  $S_1 + S_2$ ; as just established, these are equivalent. Right's options are to  $S_2$  or  $S_1 + S_1$ , and the second dominates the first by Corollary 3. So  $S_4 \equiv \{S_1 + S_2 \mid 2S_1\}$ . Compare this to the position  $2S_1 + S_2 = \{S_1 + S_2, 3S_1 \mid 2S_1\}$ ; they are equivalent because the first left option dominates the second. Thus,  $S_4 \equiv 2S_1 + S_2$ .

Lastly, consider a strip of length 5. Left's options are  $S_4 \equiv 2S_1 + S_2$ ,  $S_1 + S_3 \equiv 2S_1 + S_2$ , and  $S_2 + S_2$ , which dominates the others since  $S_2 \not\geq 2S_1$ . Right's options are  $S_3$  and  $S_1 + S_2$ , which are equivalent. So  $S_5 \equiv \{2S_2 \mid S_1 + S_2\}$ . This is the same as the position  $S_1 + 2S_2$ , as Left's move to  $2S_2$  dominates here and Right's only move is to  $S_1 + S_2$ . That is,  $S_5 \equiv S_1 + 2S_2$ .

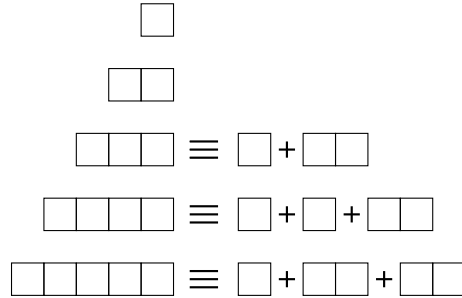


Figure 4: Reduction of  $S_n$  into a sum of  $S_1$  and  $S_2$  positions, for  $n = 1, \dots, 5$ .

If we were to continue with  $S_6$ ,  $S_7$ , and  $S_8$ , we would observe a pattern based on the congruency of  $n$  modulo 3. The reductions for longer strips use the same logic as the cases for  $n = 1, \dots, 5$ , and indeed the general inductive proof follows a similar method, of considering the possible options and removing those dominated via Corollary 3. We now begin the general argument for reducing any  $S_n$ . Lemma 3 serves to tidy up the proof of Theorem 1.

**Lemma 3.** *If  $k, j \in \mathbb{N}$  then  $kS_1 + jS_2 \equiv \{(k-1)S_2 + jS_2 \mid kS_1 + (j-1)S_2\} \pmod{\mathcal{K}}$ .*

*Proof.* Left's only moves in  $kS_1 + jS_2$  are to bring an  $S_1$  to zero or an  $S_2$  to an  $S_1$ . These moves give the options  $(k-1)S_1 + jS_2$  and  $(k+1)S_1 + (j-1)S_2$ , respectively, and the second is dominated by the first because  $S_2 \not\geq 2S_1$ . Right has only one move up to symmetry — play in an  $S_2$  — and so  $kS_1 + jS_2 \equiv \{(k-1)S_2 + jS_2 \mid kS_1 + (j-1)S_2\} \pmod{\mathcal{K}}$ , as claimed.  $\square$

**Theorem 1.** *If  $n \geq 3$  then*

$$S_n \equiv \begin{cases} kS_1 + kS_2 \pmod{\mathcal{K}}, & \text{if } n = 3k, \\ (k+1)S_1 + kS_2 \pmod{\mathcal{K}}, & \text{if } n = 3k + 1, \\ kS_1 + (k+1)S_2 \pmod{\mathcal{K}}, & \text{if } n = 3k + 2. \end{cases}$$

*Proof.* By Lemma 3, it suffices to show that  $S_n \equiv \{(k-1)S_1 + kS_2 \mid kS_1 + (k-1)S_2\}$  when  $n = 3k$ , that  $S_n \equiv \{kS_1 + kS_2 \mid \{(k+1)S_1 + (k-1)S_2\}$  when  $n = 3k + 1$ , and that  $S_n \equiv \{(k-1)S_1 + (k+1)S_2 \mid kS_1 + kS_2\}$  when  $n = 3k + 2$ . The proof is broken into these three cases. Note that any left option of  $S_n$  is of the form  $S_i + S_{n-1-i}$ ,



with  $0 \leq i \leq n - 1$ . Similarly, any right option of  $S_n$  is of the form  $S_i + S_{n-2-i}$ , with  $0 \leq i \leq n - 2$ .

**Case 1:  $n = 3k$ :**

If  $i = 3j$  then  $n - 1 - i = 3k - 3j - 1 = 3(k - j - 1) + 2$ , and  $n - 2 - i = 3k - 3j - 2 = 3(k - j - 1) + 1$ . By induction this gives left and right options

$$\begin{aligned} G^{L_1} &= S_i + S_{n-1-i} = jS_1 + jS_2 + (k - j - 1)S_1 + (k - j)S_2 \\ &= (k - 1)S_1 + kS_2; \\ G^{R_1} &= S_i + S_{n-2-i} = jS_1 + jS_2 + (k - j)S_1 + (k - j - 1)S_2 \\ &= kS_1 + (k - 1)S_2. \end{aligned}$$

If  $i = 3j + 1$  then  $n - 1 - i = 3k - 3j - 2 = 3(k - j - 1) + 1$  and  $n - 2 - i = 3k - 3j - 3 = 3(k - j - 1)$ . By induction, this gives left and right options

$$\begin{aligned} G^{L_2} &= S_i + S_{n-1-i} = (j + 1)S_1 + jS_2 + (k - j)S_1 + (k - j - 1)S_2 \\ &= (k + 1)S_1 + (k - 1)S_2; \\ G^{R_2} &= S_i + S_{n-2-i} = (j + 1)S_1 + jS_2 + (k - j - 1)S_1 + (k - j - 1)S_2 \\ &= kS_1 + (k - 1)S_2. \end{aligned}$$

If  $i = 3j + 2$  then  $n - 1 - i = 3k - 3j - 3 = 3(k - j - 1)$  and  $n - 2 - i = 3k - 3j - 4 = 3(k - j - 2) + 2$ , so by induction we have left and right options

$$\begin{aligned} G^{L_3} &= S_i + S_{n-1-i} = jS_1 + (j + 1)S_2 + (k - j - 1)S_1 + (k - j - 1)S_2 \\ &= (k - 1)S_1 + kS_2; \\ G^{R_3} &= S_i + S_{n-1-i} = jS_1 + (j + 1)S_2 + (k - j - 2)S_1 + (k - j - 1)S_2 \\ &= (k - 2)S_1 + kS_2. \end{aligned}$$

Left has only two distinct options: either  $G^{L_1} = G^{L_3} = (k - 1)S_1 + kS_2$  (obtained by moving to  $S_i + S_{n-1-i}$  with any  $i \equiv 0, 2 \pmod{3}$ ), or  $G^{L_2} = (k + 1)S_1 + (k - 1)S_2$  (obtained by moving to  $S_i + S_{n-1-i}$  with any  $i \equiv 1 \pmod{3}$ ). We can write  $G^{L_1} = G' + S_2$  and  $G^{L_2} = G' + 2S_1$  (where  $G' = (k - 1)S_1 + (k - 1)S_2$ ), and then we see that  $G^{L_2}$  is dominated by  $G^{L_1}$ , because  $2S_1$  is dominated by  $S_2$  (Corollary 3). Similarly, Right's options are  $G^{R_1} = G^{R_2} = kS_1 + (k - 1)S_2$  or  $G^{R_3} = (k - 2)S_1 + kS_2$ , and the latter is dominated by the former. With Lemma 3, we conclude that, when  $n = 3k$ , we have  $S_n \equiv \{(k - 1)S_1 + kS_2 \mid kS_1 + (k - 1)S_2\} \equiv kS_1 + kS_2 \pmod{\mathcal{K}}$ .

**Case 2:  $n = 3k + 1$ :**

In this case, by similar arguments and computations, we find Left's only move is to

$kS_1 + kS_2$ , while Right has an option to  $(k - 1)S_1 + kS_2$  dominated by an option to  $(k + 1)S_1 + (k - 1)S_2$ . Thus, if  $n = 3k + 1$  then

$$S_n \equiv \{kS_1 + kS_2 \mid (k + 1)S_1 + (k - 1)S_2\} \equiv (k + 1)S_1 + kS_2 \pmod{\mathcal{K}}.$$

**Case 3:**  $n = 3k + 2$ :

In this case, Left has a move to  $(k + 1)S_1 + kS_2$  that is dominated by a move to  $(k - 1)S_1 + (k + 1)S_2$ , while Right's only option is  $kS_1 + kS_2$ . Thus, if  $n = 3k + 2$  then  $S_n \equiv \{(k - 1)S_1 + (k + 1)S_2 \mid kS_1 + kS_2\} \equiv kS_1 + (k + 1)S_2 \pmod{\mathcal{K}}$ .  $\square$

#### 4. Outcome and Strategy

We have shown that every strip  $S_n$  splits into a sum of single squares and dominoes. This makes analysis of the PARTIZAN KAYLES universe much more manageable; we need only determine the outcome of a sum of any number of single squares and dominoes. One trivial observation is that if there are more single squares than dominoes, then Left will not be able to win, as Right can eliminate all of 'his' pieces before Left can run out of single squares. That is, if  $k > j$  then  $(kS_1 + jS_2) \in \mathcal{R}^-$ . Another immediate result is the outcome when there are exactly as many single squares as dominoes: the players are forced into a Tweedledum-Tweedledee situation, because Left will always choose to play in an  $S_1$  over an  $S_2$ , by Corollary 2, and so the the first player will run out of moves first. Thus, if  $k = j$  then  $(kS_1 + jS_2) \in \mathcal{N}^-$ . The outcome in the remaining case, when  $k < j$ , turns out to be dependant on the congruence of the total number of (not necessarily single) squares, modulo 3; that is, it depends on the value of  $k + 2j \pmod{3}$ .

**Theorem 2.** *For positive integers  $k$  and  $j$ ,*

$$kS_1 + jS_2 \in \begin{cases} \mathcal{N}^-, & \text{if } k = j, \text{ or if } k < j \text{ and } k + 2j \equiv 0 \pmod{3}, \\ \mathcal{R}^-, & \text{if } k > j, \text{ or if } k < j \text{ and } k + 2j \equiv 1 \pmod{3}, \\ \mathcal{P}^-, & \text{if } k < j \text{ and } k + 2j \equiv 2 \pmod{3}. \end{cases}$$

*Proof.* Lemma 3 states that

$$kS_1 + jS_2 \equiv \{(k - 1)S_1 + jS_2 \mid kS_1 + (j - 1)S_2\} \pmod{\mathcal{K}}.$$

Let  $G = kS_1 + jS_2$ . We can prove each case by applying induction to  $G^L = (k - 1)S_1 + jS_2$  and  $G^R = kS_1 + (j - 1)S_2$ .

If  $k = j$  then Left's option is in  $\mathcal{P}^-$  since  $(k - 1) + 2k = 3k - 1$ , and Right's option is in  $\mathcal{R}^-$  since  $k > k - 1$ . So  $G \in \mathcal{N}^-$ .

If  $k > j$  then  $G^L \in \mathcal{N}^- \cup \mathcal{R}^-$  and  $G^R \in \mathcal{R}^-$ , so  $G \in \mathcal{R}^-$ .

If  $k < j$  and  $k + 2j \equiv 0 \pmod{3}$  then  $G^L \in \mathcal{P}^-$  because  $k - 1 + 2j \equiv 2 \pmod{3}$ , while  $G^R \in \mathcal{R}^-$  because  $k + 2j - 2 \equiv 1 \pmod{3}$ . Thus  $G \in \mathcal{N}^-$ .

If  $k < j$  and  $k + 2j \equiv 1 \pmod{3}$  then  $G^L \in \mathcal{N}^-$  because  $k - 1 + 2j \equiv 0 \pmod{3}$ , and  $G^R \in \mathcal{P}^-$  because  $k + 2j - 2 \equiv 2 \pmod{3}$ . This confirms  $G \in \mathcal{R}^-$ .

Finally, if  $k < j$  and  $k + 2j \equiv 2 \pmod{3}$  then  $G^L \in \mathcal{R}^-$  because  $k - 1 + 2j \equiv 1 \pmod{3}$ , and  $G^R \in \mathcal{N}^-$  because  $k + 2j - 2 \equiv 0 \pmod{3}$ . Thus  $G \in \mathcal{P}^-$ .  $\square$

As an immediate corollary we can prove what might be intuitively guessed about this universe: a single square and a single domino ‘cancel each other out’. Essentially, we can think of a single square as one move for Left and a single domino as one move for Right. Things are more complicated when only dominoes are present, because Left must then play in a domino, but this way of thinking works when at least one of each exists. Corollary 4 has a very nice obvious consequence, which is given as Corollary 5: any strip of length a multiple of 3 is equivalent to zero.

**Corollary 4.**  $S_1 + S_2 \equiv 0 \pmod{\mathcal{K}}$ .

*Proof.* Let  $X \equiv kS_1 + jS_2$  be any Kayles sum. By Theorem 2,

$$o^-(X + S_1 + S_2) = o^-[(k + 1)S_1 + (j + 1)S_2] = o^-(kS_1 + jS_2),$$

since  $k + 1$  is equal to (respectively, less than, greater than)  $j + 1$  when  $k$  is equal to (respectively, less than, greater than)  $j$ , and  $k + 2j \equiv (k + 1) + 2(j + 1) \pmod{3}$ .  $\square$

**Corollary 5.** *If  $n \equiv 0 \pmod{3}$  then  $S_n \equiv 0 \pmod{\mathcal{K}}$ .*

*Proof.* This is clear from Theorem 1 and the previous corollary, since if  $n = 3k$  then  $S_n$  reduces to  $kS_1 + kS_2 = k(S_1 + S_2) \equiv 0 \pmod{\mathcal{K}}$ .  $\square$

With Theorem 2 its corollaries, we can identify the misère monoid of PARTIZAN KAYLES. Since  $S_1 + S_2 \equiv 0$ , the monoid is a group, isomorphic to the integers under addition. Given a general Kayles position  $kS_1 + jS_2$ , we can cancel the inverse pairs  $S_1, S_2$  until we are left with only  $S_1$  positions or only  $S_2$  positions. Then, by Theorem 2, the outcome class  $\mathcal{N}^-$  is composed of positions  $jS_2$  where  $j \equiv 0 \pmod{3}$ . The outcome class  $\mathcal{R}^-$  contains all positions  $kS_1$  as well as positions  $jS_2$  with  $2j \equiv 1 \pmod{3}$  (i.e.,  $j \equiv 2 \pmod{3}$ ). Finally,  $\mathcal{P}^-$  contains all positions  $jS_2$  with  $2j \equiv 2 \pmod{3}$ , or in other words  $j \equiv 1 \pmod{3}$ .

Thus, we have

$$\mathcal{M}_{\mathcal{K}} = \langle 0, S_1, S_2 \mid S_1 + S_2 = 0 \rangle,$$

with outcome partition

$$\begin{aligned} \mathcal{N}^- &= \{kS_2 \mid k \geq 0, k \equiv 0 \pmod{3}\} \\ \mathcal{P}^- &= \{kS_2 \mid k > 0, k \equiv 1 \pmod{3}\} \\ \mathcal{R}^- &= \{kS_1, jS_2 \mid k > 0, j > 0, j \equiv 2 \pmod{3}\} \\ \mathcal{L}^- &= \emptyset. \end{aligned}$$

It would be nice to go a few steps further and answer the following questions.

1. Can we ‘look’ at a general sum of strips and determine the outcome, without having to first reduce the position to single squares and dominoes?
2. Can we determine the optimal move for a player when he or she has a winning strategy?

The next theorem precisely answers question 1, by describing the outcome of a general Kayles position *without* directly computing its reduction into  $S_1$  and  $S_2$  pieces. We must simply compare the number of pieces of length congruent to 1 modulo 3 to the number of those congruent to 2 modulo 3. In fact, there is no new argument here: this is a compression of the two steps already discussed — the reduction into  $S_1$  and  $S_2$  pieces (Theorem 1) and the outcome of  $kS_1 + jS_2$  (Theorem 2). Note that if  $G$  is a piece of length 0 modulo 3, then  $G$  is equivalent to zero modulo  $\mathcal{K}$ . Thus in any Kayles position, we can ignore any components of length 0 modulo 3.

**Theorem 3.** *If  $G$  is a PARTIZAN KAYLES position with  $x$  pieces of length 1 modulo 3 and  $y$  pieces of length 2 modulo 3, then*

$$G \in \begin{cases} \mathcal{N}^-, & \text{if } x = y, \\ & \text{or if } x < y \text{ and } x + 2y \equiv 0 \pmod{3}; \\ \mathcal{R}^-, & \text{if } x > y, \\ & \text{or if } x < y \text{ and } x + 2y \equiv 1 \pmod{3}; \\ \mathcal{P}^-, & \text{if } x < y \text{ and } x + 2y \equiv 2 \pmod{3}. \end{cases}$$

Finally, Theorem 4 answers our second question, of most interest to anyone actually playing PARTIZAN KAYLES: *how* do you win a general non-reduced PARTIZAN KAYLES position, when you can? The winning moves described below can be confirmed using Theorem 3.

**Theorem 4.** *If Left can win a PARTIZAN KAYLES position, then she can win playing at the end of a strip of length 1 modulo 3, when possible, or the end of a strip of length 2 modulo 3, otherwise. If Right can win a PARTIZAN KAYLES position, then he can win playing at the end of a strip of length 2 modulo 3, when possible, or one away from the end of a strip of length 1 modulo 3, otherwise.*

### 5. Discussion: Misère Invertibility

Within a universe  $\mathcal{U}$ , a game  $G$  may satisfy  $G + \overline{G} \equiv 0 \pmod{\mathcal{U}}$ , and then  $G$  is said to be *invertible* modulo  $\mathcal{U}$ . For example, normal-play canonical-form numbers are invertible modulo the universe of all such positions [8]. If  $G + \overline{G} \not\equiv 0 \pmod{\mathcal{U}}$ , it is tempting to say that  $G$  is *not invertible* modulo  $\mathcal{U}$  — but once again misère

games surprise us! It is possible for some other position  $H \not\equiv \overline{G} \pmod{\mathcal{U}}$  to satisfy  $G+H \equiv 0 \pmod{\mathcal{U}}$ ; that is,  $G$  may have an additive inverse that is not its conjugate. The universe of PARTIZAN KAYLES is the only known partizan example of such a situation: here we have  $S_1 + S_2 \equiv 0 \pmod{\mathcal{K}}$ , with  $S_1 \not\equiv \overline{S_2}$  and  $S_2 \not\equiv \overline{S_1} \pmod{\mathcal{K}}$ .

In fact, these comparisons are not even defined, as  $\overline{S_1}$  and  $\overline{S_2}$  do not occur in the universe  $\mathcal{K}$ . The position  $\overline{S_1}$  would have no move for Left and one move for Right (that is,  $\overline{S_1} = \{\cdot \mid 0\} = -1$ ), and  $\overline{S_2}$  would be a position in which Left can move to 0 and Right can move to  $-1$ . Even if we generalize the definition of equivalence to allow  $G$  and  $H$  to be compared modulo  $\mathcal{U}$  without requiring that both are in  $\mathcal{U}$ , we do not obtain  $S_1 \equiv \overline{S_2}$ ; there is actually no universe<sup>3</sup> in which  $S_1$  and  $\overline{S_2}$  are equivalent, since  $S_1 \in \mathcal{R}^-$  and  $\overline{S_2} \in \mathcal{P}^-$ . This brings us to another oddity of PARTIZAN KAYLES: there is a position in  $\mathcal{R}^-$  whose additive inverse is in  $\mathcal{P}^-$ . There is no other known instance of an inverse pair having ‘asymmetric’ outcomes in this way. It is likely a symptom of the fact that  $\mathcal{K}$  is not closed under conjugation.

In [6], it was conjectured that  $G + H \equiv 0 \pmod{\mathcal{U}}$  implies  $H \equiv \overline{G} \pmod{\mathcal{U}}$  whenever  $\mathcal{U}$  is closed under conjugation. This, however, is false, as an impartial counterexample appears in [12] (appendix A.6). It was already known in [6] (by the results presented here) that the stronger statement, removing the closure condition, is false. The question now is whether a still weaker statement can be shown true: is there some condition on the universe  $\mathcal{U}$  so that  $G$  being invertible implies  $G + \overline{G} \equiv 0 \pmod{\mathcal{U}}$ ? Is there a condition on the specific game  $G$  that guarantees the invertibility of  $G$ ? Can we find more counterexamples (so far there is only one) to the original conjecture of [3]?

Without Plambeck’s theory of indistinguishability (equivalence), no non-zero game is invertible under misère play. We now have a meaningful concept of additive inverses in restricted universes, but as PARTIZAN KAYLES shows, invertibility for misère games is still strikingly different — more subtle and less intuitive — than invertibility for normal games. A better understanding of misère invertibility is a significant open problem in the growing theory of restricted misère play.

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<sup>3</sup>Assuming that any universe must contain zero,  $S_1$  and  $\overline{S_2}$  are always distinguishable.

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