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AN IMPROVED UPPER BOUND FOR RAMANUJAN PRIMES

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Abstract

For $n \geq 1$, the n^{th} Ramanujan prime is defined as the least positive integer R_n such that for all $x \geq R_n$, the interval $(\frac{x}{2}, x]$ has at least n primes. If $\alpha = 2n\left(1 + \dfrac{3}{\log n + \log_2 n - 4}\right)$, then we show that $R_n < p_{[\alpha]}$ for all $n > 241$, where p_i is the i^{th} prime. This bound improves upon all previous bounds for large n.

1. Introduction

For $n \geq 1$, the n^{th} Ramanujan prime is defined as the least positive integer R_n, such that for all $x \geq R_n$, the interval $(\frac{x}{2}, x]$ has at least n primes. Note that by the minimality condition, R_n is prime and the interval $(\frac{R_n}{2}, R_n]$ contains exactly n primes. Let p_n denote the n^{th} prime. Sondow [3] showed that $p_{2n} < R_n < p_{4n}$ for all n and conjectured that $R_n < p_{3n}$ for all n. This conjecture was proved by Laishram [2] and subsequently Sondow, Nicholson and Noe [4] improved Laishram's result by showing that $R_n < \frac{41}{47}p_{3n}$. Axler [1, Proposition 3.24] showed that for $t > \frac{48}{19}$ we have $R_n \leq p_{[tn]}$ for all n, where $[x]$ denotes the integer part of x. In [5] it was shown that for every $\epsilon > 0$, there exists an integer N such that $R_n < p_{[2n(1+\epsilon)]}$ for all $n > N$. Our main result below gives a new upper bound that, for large n, is better than all previous bounds.

Theorem 1. *Let $R_n = p_s$ be the n^{th} Ramanujan prime, where p_s is the s^{th} prime. Then $s < 2n\left(1 + \frac{3}{\log n + \log(\log n) - 4}\right)$ for all $n > 241$.*

Note that while all previous upper bounds for R_n are of the form of $p_{[2nc]}$, where $c > 1$ is a constant, the bound given in the theorem above is of the form $p_{[2nf(n)]}$, where $f(n) > 1$ and $\lim_{n\to\infty} f(n) = 1$. Hence for large n our bound is smaller and thus better than $p_{[2nc]}$ for any fixed c.

2. Proof of Theorem 1

The proof follows closely the proof of the main theorem in [5], where the following function was defined. Let

$$F(x) = x(\log x + \log\log x) - 2(x-n)(\log(x-n) + \log\log(x-n) - 1),$$

where n is a fixed positive integer. Henceforth we denote $\log(\log n)$ by $\log_2 n$. Also, let $\pi(x)$ denote the number of primes less than or equal to x.

In [5] we showed that for a Ramanujan prime $R_n = p_s$ we have $p_{s-n} < \frac{p_s}{2}$. A similar result, with the inequality reversed, holds for indices greater than s. We include both of these results in the following lemma, which while not used directly in the proofs here, is of independent interest and relevant to the current topic. The result in the second part of the lemma has been dubbed as the "Ramanujan prime corollary" by the second author (see the first three sequences and last sequence given in Table 1 of the Appendix).

Lemma 1. *Let $R_n = p_s$ be the n^{th} Ramanujan prime, where p_s is the s^{th} prime. Then the following hold:*

(i) $p_{s-n} < \frac{p_s}{2}$ *for all* $n \geq 2$. [5, Lemma 2.1].

(ii) $p_{s+k} < 2p_{s+k-n}$ *for all positive integers* k.

Proof. Let $i = s+k$. Note that by definition of R_n we have

$$\pi(p_i - 1) - \pi\left(\frac{p_i}{2}\right) = \pi(p_i - 1) - \pi\left(\frac{p_i - 1}{2}\right) \geq n.$$

Therefore, $p_i - 1 \geq p_{i-1} > p_{i-2} > p_{i-3} > \ldots > p_{i-n} > \frac{p_i}{2}$, and hence $p_{i-n} > \frac{p_i}{2}$ and the second part of the lemma follows noting that $i = s+k$. □

Lemma 2. *Let $R_n = p_s$ be the n^{th} Ramanujan prime, where p_s is the s^{th} prime. Then the following hold:*

(i) $2n < s < 3n$ ([3, 2]).

(ii) $F(s) > 0$ *and* $F(x)$ *is a decreasing function for all* $x \geq 2n$ [5, Proof of Theorem 1.1].

Lemma 3. *Let* $g = g(n) = \dfrac{\log n + \log_2 n - 4}{3}$. *Then for all* $n > 241$ *we have* $F\left(2n\left(1 + \frac{1}{g(n)}\right)\right) < 0$.

Proof. We first observe that for $n > 241$ we have $g > 1$. Let

$$\phi = \log\left(2n\left(1+\frac{1}{g}\right)\right) + \log_2\left(2n\left(1+\frac{1}{g}\right)\right)$$

and
$$\psi = \log\left(n\left(1+\frac{2}{g}\right)\right) + \log_2\left(n\left(1+\frac{2}{g}\right)\right) - 1.$$
Observe that
$$\phi - \psi = 1 + \log 2 + \log\left(\frac{1+\frac{1}{g}}{1+\frac{2}{g}}\right) + \log\left(\frac{\log\left(2n+\frac{2n}{g}\right)}{\log\left(n+\frac{2n}{g}\right)}\right)$$
$$< 2 + \log 2 < 3,$$
as $\log\left(\frac{1+\frac{1}{g}}{1+\frac{2}{g}}\right) < 0$ and $0 < \log\left(\frac{\log\left(2n+\frac{2n}{g}\right)}{\log\left(n+\frac{2n}{g}\right)}\right) < 1$. Therefore, $2\psi - \phi = \psi - (\phi - \psi) > \psi - 3$ and hence
$$\frac{2\psi - \phi}{\phi - \psi} > \frac{\psi - 3}{3} > \frac{\log n + \log_2 n - 4}{3} = g,$$
which gives
$$\left(1+\frac{1}{g}\right)\phi < \left(1+\frac{2}{g}\right)\psi,$$
and the lemma follows as $F\left(2n\left(1+\frac{1}{g}\right)\right) = 2n\left(1+\frac{1}{g}\right)\phi - 2n\left(1+\frac{2}{g}\right)\psi$. □

Proof of Theorem 1 Let $R_n = p_s$. Then by Lemma 2, part (i), we have $s > 2n$ and $F(s) > 0$. Moreover, by Lemma 2, part (ii), $F(x)$ is a decreasing function for $x \geq 2n$. Let $g = g(n) = \frac{\log n + \log_2 n - 4}{3}$ and $\alpha = 2n(1+\frac{1}{g})$. Then by Lemma 3 we have $F(\alpha) < 0$ for all $n > 241$. As F is a decreasing function for $x \geq 2n$ and $F(s) > 0$, we have $s \leq 2n(1+\frac{1}{g})$ for all $n > 241$. □

References

[1] C. Axler, On generalized Ramanujan primes, Arxiv:1401.7179v1.

[2] S. Laishram, On a conjecture on Ramanujan primes, *Int. J. Number Theory* **6**, (2010), 1869–1873.

[3] J. Sondow, Ramanujan primes and Bertrand's postulate, *Amer. Math. Monthly* **116**, (2009), 630–635.

[4] J. Sondow, J. W. Nicholson, T. D. Noe, Ramanujan primes: bounds, runs, twins, and gaps, *J. Integer Sequences* **14**, (2011), Article 11.6.2.

[5] A. Srinivasan, An upper bound for Ramanujan primes, *Integers* **14** (2014).

Appendix: Tables of Sequences from the OEIS at `http://www.oeis.org/`

Sequence	Title
A165959	Size of the range of the Ramanujan Prime Corollary, 2***A168421**(n) - **A104272**(n).
A168421	Small Associated Ramanujan Prime, p_{i-n}.
A168425	Large Associated Ramanujan Prime, p_i.
A174602	Smallest prime that begins a run of n Ramanujan primes that are consecutive primes.
A174635	Prime numbers that are not Ramanujan primes.
A174641	Smallest prime that begins a run of n consecutive primes that are not Ramanujan primes.
A179196	Number of primes up to the n^{th} Ramanujan prime: **A000720**(**A104272**(n)), $\pi(R_n)$.
A190124	Decimal expansion of Ramanujan prime constant: $\sum_{n=1}^{\infty} 1/R_n^2$, where R_n is the n^{th} Ramanujan prime, **A104272**(n).
A190303	Decimal expansion of sum of alternating series of reciprocals of Ramanujan primes, $\sum_{n=1}^{\infty}(1/R_n)(-1)^{n-1}$, where R_n is the n^{th} Ramanujan prime, **A104272**(n).
A190501	Number of Ramanujan primes R_k such that $2^{(n-1)} < R_k \leq 2^n$.
A190502	Number of Ramanujan primes $\leq 2^n$.
A190874	First differences of **A179196**, $\pi(R_{(n+1)}) - \pi(R_n)$ where R_n is **A104272**(n).
A191225	Number of Ramanujan primes R_k between triangular numbers $T(n-1) < R_k \leq T(n)$.
A191226	First occurrence of number n of Ramanujan primes in **A191225.**
A191227	Last known occurrence of number n of Ramanujan primes in **A191225.**
A191228	Greatest Ramanujan prime index less than x.
A214756	$a(n)$ = largest Ramanujan prime R_k in **A104272** that is $\leq$ **A002386**(n).
A214757	$a(n)$ = smallest Ramanujan prime R_k in **A104272** that is $\geq$ **A000101**(n).
A214926	Difference **A214925**(n) - **A214924**(n), prime count between Ramanujan primes bounding maximal gap primes.
A214934	Numbers R(k) such that $R(k) \geq 2k \log R(k)$, where $R(k)$ = **A104272**(k) is the k^{th} Ramanujan prime.
A233739	$R(n) - p(2n)$, where $R(n)$ is the n^{th} Ramanujan prime and $p(n)$ is the n^{th} prime. .
A234298	Ramanujan prime R_k such that $\pi(R_{k+1}) - \pi(R_k)$ are record values: record Ramanujan prime **A190874**(k).

Table 1: Sequences authored by John W. Nicholson

Sequence	Title	Comments
A000101	Increasing gaps between primes (upper end) (compare with **A002386**, which gives lower ends of these gaps).	Except for a(1)=3 and a(2)=5, a(n) = **A168421**(k). Primes 3 and 5 are special in that they are the only primes which do not have a Ramanujan prime between them and their double, ≤ 6 and 10 respectively. Because of the large size of a gap, there are many repeats of the prime number in **A168421**. - John W. Nicholson, Dec 10 2013
A005382	Primes p such that $2p-1$ is also prime.	If a(n) is in **A168421** then **A005383**(n) is a twin prime with a Ramanujan prime, **A005383**(n) - 2. If this sequence has an infinite number of terms in **A168421**, then the twin prime conjecture can be proved. - John W. Nicholson, Dec 05 2013
A104272	Ramanujan primes R_n: $a(n)$ is the smallest number such that if $x \leq a(n)$, then $\pi(x) - \pi(x/2) \leq n$, where $\pi(x)$ is the number of primes $\leq x$.	For some n and k, we see that **A168421**(k) = a(n) so as to form a chain of primes similar to a Cunningham chain. For example (and the first example), **A168421**(2) = 7, links a(2) = 11 = **A168421**(3), links a(3) = 17 = **A168421**(4), links a(4) = 29 = **A168421**(6), links a(6) = 47. Note that the links do not have to be of a form like q = 2*p+1 or q = 2*p-1. - John W. Nicholson, Feb 22 2015
A190661	Least number $a(n)$ such that there are at least n primes in the range $(T(k-1), T(k)]$ for all $k \geq a(n)$, where T(k) is the k-th triangular number.	With R_n the n-th Ramanujan prime (**A104272**), it is conjectured that for every $n \geq 0$, $(1/2)R_n \leq a(n) < (20/13)R_n$. These bounds have been verified for all n up to 8000. For most $n \leq 8000$, we have $a(n) > R_n$, with exceptions listed in **A190881**.

Table 2: Observations (four sequences)

Sequence	Title
A204814	Number of decompositions of 2n into an unordered sum of two Ramanujan primes.
A205616	Even numbers that are not the sum of two non-Ramanujan primes (**A174635**). COMMENTS: No other terms $< 2 * 10^8$. Conjectured to be complete.
A205617	Number of decompositions of 2n into an unordered sum of two non-Ramanujan primes (**A174635).**
A205618	Last occurrence of n partitions in **A205617.**

Table 3: Authored by (or suggested by) John W. Nicholson and Donovan Johnson (4 sequences). Note: All are related to Goldbach conjecture.