NOTE ON A RESULT OF CHUNG ON WEIL TYPE SUMS

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Abstract

Following previous works of Chung we are interested in Vinogradov's type inequalities for some multivariate character sums. Using Johnsen's bound on a complete unidimensional character sum we obtain a sharper result in several ranges of parameters.

– Dedicated to Endre Szemerédi on his 75^{th} birthday

1. Introduction

Let $\mathbb{F}_p$ be the prime field with p elements and $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$ its multiplicative group. We write $e(x)$ for the primitive additive character $\exp(2\pi ix/p)$ on $\mathbb{F}_p$. The following estimate for the double exponential sum, which has many applications in additive combinatorics, is a well-known result of Vinogradov.

Theorem 1.1. *Let $A, B \subseteq \mathbb{F}_p$. Then for $r \in \mathbb{F}_p^*$*

$$\Big| \sum_{(a,b)\in A\times B} e(rab) \Big| < \sqrt{p|A||B|}. \tag{1}$$

The analogous result for a nontrivial multiplicative character sum appears first in [3]:

Theorem 1.2. *Let $A, B \subseteq \mathbb{F}_p^*$ and let $\chi \neq \chi_0$ be a nontrivial multiplicative character modulo p. Then*

$$\Big| \sum_{(a,b)\in A\times B} \chi(a+b) \Big| \leq \sqrt{p|A||B|}.$$

To go further we observe that the above bounds (for both multiplicative and additive characters) are nontrivial only when $|A||B| > p$. In [5, Chapter 8, Problem 9] the author obtained a substantially better bound than that given in Theorem 1.2 when $|A||B|$ is small (see also [6] for related questions). The main ingredient in his proof is Hölder's inequality and the famous deep result due to Weil on exponential sums. Karatsuba's result reads as follows:

Theorem 1.3. *Let $A, B \subseteq \mathbb{F}_p^*$ and let $\chi \neq \chi_0$ be a nontrivial multiplicative character. Then for any positive integer n*

$$\Big| \sum_{(a,b)\in A\times B} \chi(a+b)\Big| \ll |A|^{1-\frac{1}{2n}}\big(p^{\frac{1}{2n}}|B|^{\frac{1}{2}} + p^{\frac{1}{4n}}|B|\big).$$

When $n = 1$ we obtain Theorem 1.2 (apart from the multiplicative constant implied in Vinogradov's symbol). There are many applications of this type of bound. We mention just one here. Let $N(A, B)$ be the number of pairs $(a, b) \in A \times B$ such that $a + b$ is a square in $\mathbb{F}_p^*$. Then

$$N(A,B) = \frac{|A||B| - |A\cap(-B)|}{2} + \frac{1}{2}\sum_{(a,b)\in A\times B}\left(\frac{a+b}{p}\right),$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

For additive characters and smaller subsets A, B of $\mathbb{F}_p$, Bourgain and Garaev proved the following theorem (see [1]):

Theorem 1.4. *Let $A, B \subseteq \mathbb{F}_p$, and for $r \in \mathbb{F}_p^*$, let*

$$S(r) = \sum_{(a,b)\in A\times B} e(rab).$$

Then

$$|S(r)| \le |A|^{1/2}|B|^{1/2}\Big(pE_2^+(A)E_2^+(B)\Big)^{1/8},$$

where $E_2^+(A)$ is the additive energy of the set A defined by

$$E_2^+(A) = |\{(a_1,a_2,a_3,a_4)\in A^4 \mid a_1+a_2 = a_3+a_4\}|.$$

Since clearly $E_2^+(A) < |A|^3$, we have

$$|S(r)| \le |A|^{7/8}|B|^{7/8}p^{1/8},$$

which gives

$$|S(r)| \le |A|^{7/8}|B|^{7/8}p^{1/8} < \sqrt{p|A||B|},$$

provided that $|A||B| < p$. Under the above condition, Theorem 1.4 provides, more generally, a considerably better estimate than Theorem 1.1 whenever there is a nontrivial bound for $E_2^+(A)$, e.g. $E_2^+(A) < |A|^{3-c}$ with some appropriate $c > 0$.

In [2] the author discussed a certain family of incomplete character sums restricted to triples (a,b,c) belonging to a set W which is not necessarily the hypercube $A\times B\times C$. We infer from Theorem 1.2 that when $W=A\times B\times C$ with $|A|\le|B|\le|C|$, the bound

$$\Big|\sum_{(a,b,c)\in A\times B\times C}\chi(a+b+c)\Big|\le\sqrt{p}|A||B|^{1/2}|C|^{1/2} \tag{2}$$

holds. It is nontrivial whenever

$$|B||C|>p. \tag{3}$$

More precisely, Chung considered the more general situation where the subset W of ${\mathbb{F}_p}^3$ could be described by its projections on the hyperplanes $\mathbb{F}_p\times\mathbb{F}_p\times\{0\}$, $\mathbb{F}_p\times\{0\}\times\mathbb{F}_p$ and $\{0\}\times\mathbb{F}_p\times\mathbb{F}_p$. We denote by p_{12}, p_{23} and p_{31}, the projections from ${\mathbb{F}_p}^3$ onto ${\mathbb{F}_p}^2$ mapping (a,b,c) on (a,b), (b,c) and (c,a), respectively. For $W\subset{\mathbb{F}_p}^3$, we let $X=p_{12}(W)$, $Y=p_{23}(W)$, $Z=p_{31}(W)$ and

$$W_{X,Y,Z}:=\{(a,b,c)\in{\mathbb{F}_p}^3:(a,b)\in X;(b,c)\in Y;(c,a)\in Z\}.$$

Then $W\subset W_{X,Y,Z}$. But the reverse inclusion is not true in general. We restrict our attention to the case where $W=W_{X,Y,Z}$. We first observe that we might have $W=W_{X',Y',Z'}$ for other subsets X',Y',Z' containing X,Y,Z, respectively; but in the sequel, we deal only with the strict images X,Y,Z of W by these three projections. Under these assumptions on W, we obtain

$$|W|=\sum_{(a,b)\in X}\sum_{\substack{c\in\mathbb{F}_p\\(b,c)\in Y\\(c,a)\in Z}}1=\sum_{(a,b)\in X}|Y_b\cap Z^a|\le p|X|,$$

where

$$Z_c:=\{a\in\mathbb{F}_p\ \mid\ (c,a)\in Z\},\quad Z^a:=\{c\in\mathbb{F}_p\ \mid\ (c,a)\in Z\}$$

for $a,b,c\in\mathbb{F}_p$ and X_a, X^b, Y_b and Y^c are defined similarly.
It follows that

$$|W|\le p\cdot\min(|X|,|Y|,|Z|).$$

By the Cauchy inequality we also have the upper bound

$$|W|\le|X|^{1/2}\Big(\sum_{(a,b)\in X}|Y_b\cap Z^a|^2\Big)^{1/2}\le|X|^{1/2}\Big(\sum_{a\in\mathbb{F}_p}|Z^a|\sum_{b\in\mathbb{F}_p}|Y_b|\Big)^{1/2}=\sqrt{|X||Y||Z|}.$$

Hence if $|X|=\min(|X|,|Y|,|Z|)$, then

$$|W|\le\min(p|X|,\sqrt{|X||Y||Z|}). \tag{4}$$

Moreover, for any $(a,b) \in X$ the intersection $Y_b \cap Z^a$ cannot be empty since there necessarily exists at least one element c such that $(a,b,c) \in W$. Thus the assumptions $(b,c) \in Y$ and $(c,a) \in Z$ imply $c \in Y_b \cap Z^a$. By symmetry this yields

$$|W| \geq \max(|X|, |Y|, |Z|).$$

Consequently we can assume $|Y| \geq |Z| \geq |X|$, and by (4) we obtain

$$|Y| \leq \min(|Z|, p)|X|. \tag{5}$$

Now Theorem 3.1 in [2] can be stated as follows:

Theorem 1.5. *Let χ denote a nontrivial multiplicative character on $\mathbb{F}_p$. Let $W \subset {\mathbb{F}_p}^3$, $X = p_{12}(W)$, $Y = p_{23}(W)$, $Z = p_{31}(W)$ and assume that $W = W_{X,Y,Z}$. Then we have*

$$S_2 := \Big| \sum_{(a,b,c)\in W} \chi(a+b+c) \Big| \leq \sqrt{2} p^{3/4} |X|^{1/2} |Y|^{1/4} |Z|^{1/4}. \tag{6}$$

Since clearly S_2 is at most $|W|$ by (4) and (6) we thus need $p^{3/4}|X|^{1/2}|Y|^{1/4}|Z|^{1/4} \ll \min(p|X|, |X|^{1/2}|Y|^{1/2}|Z|^{1/2})$. Consequently, this bound is nontrivial when

$$p|X|^2 \gg |Y||Z| \gg p^3. \tag{7}$$

Writing $|X| = p^x$, $|Y| = p^y$ and $|Z| = p^z$ and using (5), Theorem 1.5 is nontrivial when

$$2 \geq y \geq z \geq x > 0, \quad 1 + 2x \geq y + z \geq 3, \quad 1 + x \geq y, \quad x + z \geq y.$$

In the particular case when $W = A \times B \times C$ – so that $X = A \times B$, $Y = B \times C$ and $Z = C \times A$ – the bound (6) is nontrivial whenever

$$|A||B||C|^2 \gg p^3, \tag{8}$$

which is a stronger condition than (3), thus one can expect (6) is weaker than (2).

We now consider a directed graph on ${\mathbb{F}_p}^2$: the set W contains all triples (a,b,c) such that (a,b), (b,c) and (c,a) are in X, Y, Z respectively and in which two pairs (x,y) and (z,t) are connected if $y = z$. Observe that the trivial bound yields $S_2 \leq |W| \leq p \cdot \min(|X|, |Y|, |Z|)$. Hence Theorem 1.5 provides a possible gain of $p^{1/4}$ when the sizes of X, Y, Z are close to p^2. At this point we should mention that it is conjectured in [2] that the expected gain on the trivial bound p^3 should be $p^{1/2}$, namely $S_2 \ll p^{5/2}$, while Theorem 1.5 yields only $S_2 \ll p^{11/4}$.

2. Statement of the Results

We can prove the following result which is similar to Theorem 1.3 with $n = 2$. We will use it later for bounding the triple sum in Theorem 2.3.

Theorem 2.1. *Let $A, B \subseteq \mathbb{F}_p$, and χ_1, χ_2 be two multiplicative characters modulo p which are not simultaneously the trivial character χ_0. Let $c \neq c' \in \mathbb{F}_p$. Then*

$$S_1(\chi_1, \chi_2, A, B) := \Big| \sum_{(a,b)\in A\times B} \chi_1(a+b+c)\chi_2(a+b+c') \Big| \ll p^{1/4}|A|^{3/4}|B|^{1/2} + p^{1/8}|A|^{3/4}|B|.$$

We need to clarify the range of application of this bound in connection with known bounds. Interchanging if necessary the roles of A and B, we may rewrite our result in the following form.

Remarks. For $A, B \subset \mathbb{F}_p$ write $|A| = p^\alpha$ and $|B| = p^\beta$ and assume $\alpha \geq \beta$. Under the hypothesis and the notation in the theorem we have

$$S_1(\chi_1, \chi_2, A, B) \ll \begin{cases} p^{1/4}|A|^{1/2}|B|^{3/4} & \text{if } \beta \leq \alpha \leq 1/4, \\ p^{1/8}|A|^{3/4}|B| & \text{if } 1/4 \leq \beta \leq \alpha, \\ p^{1/4}|A|^{3/4}|B|^{1/2} & \text{if } 1/2 - \alpha \leq \beta \leq 1/4, \\ p^{1/8}|A||B|^{3/4} & \text{if } \beta \leq \min(1/2-\alpha, 1/4). \end{cases}$$

The first and the fourth alternatives are worse than the trivial bound $S_1 \leq |A||B|$. The second one needs $\alpha \geq 1/2$ to be nontrivial. In that case our bound is better than the bound $O(\sqrt{p|A||B|})$ given by Chung (cf. Theorem 2.3 of [2]) provided that $\alpha + 2\beta \leq 3/2$. Finally, the third one needs also $\alpha \geq 1 - 2\beta$ and then it is better than Chung's bound.

We may summarize the result as follows:

Corollary 2.2. *Let $A, B \subset \mathbb{F}_p$ with $|A| = p^\alpha$ and $|B| = p^\beta$ and assume $\alpha \geq \beta$. Let (χ_1, χ_2) be a pair of multiplicative characters that differs from (χ_0, χ_0), and let $c \neq c'$ be elements of $\mathbb{F}_p$. Then*

$$S_1(\chi_1, \chi_2, A, B) \ll \begin{cases} p^{1/8}|A|^{3/4}|B| & \text{if } 1/4 \leq \beta \leq 1/2 \leq \alpha \text{ and } \alpha + 2\beta \leq 3/2, \\ p^{1/4}|A|^{3/4}|B|^{1/2} & \text{if } 1 - \alpha \leq 2\beta \leq 1/2, \\ \sqrt{p|A||B|} & \text{if } \alpha + 2\beta \geq 3/2, \\ |A||B| & \text{otherwise.} \end{cases}$$

For $\beta \leq 1/4$, we might improve on the trivial bound using the Hölder inequality with large even parameters.

Turning to the question of triple sums associated with the vertices of triangles in a specific graph on ${\mathbb{F}_p}^2$, we get the following theorem:

Theorem 2.3. *Let $W \subset {\mathbb{F}_p}^3$ and $X, Y, Z \subseteq {\mathbb{F}_p}^2$ as in Theorem 1.5 and let $\chi \neq \chi_0$ be a non-principal multiplicative character modulo p. Then assuming $|X| \leq |Z| \leq |Y|$,*

we have

$$S_2(\chi, W) := \Big| \sum_{(a,b,c)\in W} \chi(a+b+c) \Big| \ll p^{1/2}|X|^{1/2}|Z|^{1/4}|Y|^{3/8} + p^{3/16}|X|^{1/2}|Z|^{1/2}|Y|^{3/8}. \quad (9)$$

One can notice that the upper bound above surpasses Chung's bound in Theorem 1.5 whenever $|Z|^2|Y| \ll p^{9/2}$.

In view of (4), the bound (9) in Theorem 2.3 will be nontrivial when

$$\max(p^{1/2}|X|^{1/2}|Z|^{1/4}|Y|^{3/8}, p^{3/16}|X|^{1/2}|Z|^{1/2}|Y|^{3/8}) \ll \min(p|X|, |X|^{1/2}|Y|^{1/2}|Z|^{1/2}),$$

since $S_2 \leq |W|$. This is equivalent to the conditions

$$\max(p^{3/2}, p^4|Z|^{-2}) \ll |Y| \ll \min(p^{4/3}|X|^{4/3}|Z|^{-2/3}, p^{13/6}|X|^{4/3}|Z|^{-4/3}).$$

Thus the bound (9) improves Chung's bound when $p^4 \ll |Y||Z|^2 \ll p^{9/2}$ holds. Let us write $|X| = p^x$, $|Y| = p^y$ and $|Z| = p^z$. By (5) we obtain that our Theorem 2.3 is meaningful when the parameters x, y, z satisfy

$$\begin{cases} y \geq z \geq x > 0, \quad 2 \geq y \geq 3/2, \quad y + 2z \geq 4, \quad 1 + x \geq y, \\ x + z \geq y, \quad 4x - 3y - 2z \geq -4, \quad 8x - 6y - 8z \geq -13. \end{cases}$$

Let us assume now that W is the Cartesian product $A \times B \times C$. Set the cardinalities of the sets in the series $|A| \leq |B| \leq |C|$, then we have $|A \times B| \leq |C \times A| \leq |B \times C|$. An easy computation shows that the bound in Theorem 2.3 is nontrivial whenever $|B||C| \gg p^{3/2}$ and $|A|^2|B||C|^3 \gg p^4$. These conditions are automatically satisfied when (8) holds. This gives a further argument that Theorem 2.3 is somewhat stronger than Theorem 1.5.

Remarks. We observe that the exponents of $|X|, |Y|, |Z|$ must not be unbalanced since these cardinalities cannot be strongly different. For instance, it is possible to prove the following bound in line with Theorem 1.5 but using Hölder's inequality instead of Cauchy's:

$$S_2(\chi, W) \ll p^{7/8}|X|^{3/4}|Y|^{1/8}|Z|^{1/8}. \quad (10)$$

This approach has the effect of unbalancing the final bound in terms of the cardinalities $|X|, |Y|, |Z|$. At first glance this bound seems better than Chung's if $p|X|^2 \ll |Y||Z|$ and better than our bound in Theorem 2.3 if $p^3|X|^2 \ll |Y|^2|Z|$ or $p^{11/2}|X|^2 \ll |Y|^2|Z|^3$. Moreover the requested condition $p^{7/8}|X|^{3/4}|Y|^{1/8}|Z|^{1/8} \ll p|X|$ for bound (10) to be effective leads to $p|X|^2 \gg |Y||Z|$. This condition is similar to that needed for Chung's bound in Theorem 1.4. It implies ultimately that bound (10) is either worse than Chung's or worse than the trivial $S_2 \leq p|X|$.

3. Proofs of the Results

Proof of Theorem 2.1. We simply write S_1 for $S_1(\chi_1, \chi_2, A, B)$ throughout the proof.

We start by using the triangle inequality and the Hölder inequality:

$$\begin{aligned} S_1 &= \Big| \sum_{(a,b)\in A\times B} \chi_1(a+b+c)\chi_2(a+b+c')\Big| \\ &\le |A|^{3/4}\bigg(\sum_{a\in A} \Big| \sum_{b\in B} \chi_1(a+b+c)\chi_2(a+b+c')\Big|^4\bigg)^{1/4}. \end{aligned}$$

We let

$$\begin{aligned} T_{c,c'}(\chi_1,\chi_2,b_1,b_2,b_1',b_2') := \sum_{a\in\mathbb{F}_p} &\chi_1(a+b_1+c)\chi_2(a+b_1+c')\chi_1(a+b_2+c)\chi_2(a+b_2+c') \\ &\times \overline{\chi_1(a+b_1'+c)\chi_2(a+b_1'+c')\chi_1(a+b_2'+c)\chi_2(a+b_2'+c')}, \end{aligned}$$

where $\chi_i(0) = 0$, $i = 1, 2$. Extending the summation to all $a \in \mathbb{F}_p$ and expanding the expression, we infer

$$S_1 \le |A|^{3/4}\bigg(\sum_{b_1,b_2,b_1',b_2'\in B} T_{c,c'}(\chi_1,\chi_2,b_1,b_2,b_1',b_2')\bigg)^{1/4}.$$

If there exist coincidences among the variables b_1, b_2, b_1', b_2', namely, they take altogether at most 2 distinct values, then the sum $T_{c,c'}(\chi_1, \chi_2, b_1, b_2, b_1', b_2')$ is trivially bounded by p. Otherwise we use the next lemma which gives a bound for some mixed character sums. The proof can be found in [4].

Lemma 3.1. *Let m be a positive integer, $\chi_1, \chi_2, \dots, \chi_m$ be any multiplicative characters modulo p and $b_1, b_2, \dots, b_m \in \mathbb{F}_p$ be distinct elements. Then*

$$\Big| \sum_{x\in\mathbb{F}_p} \chi_1(x+b_1)\chi_2(x+b_2)\dots\chi_m(x+b_m)\Big| \le (m-m_0+1)\sqrt{p} + m_0 + 1,$$

where m_0 is the number of χ_i's which coincide with the principal character χ_0.

If $\{b_1, b_2\} \ne \{b_1', b_2'\}$ and if $(b_1, b_1') \ne (b_2, b_2')$ the summand does not reduce to a constant, hence by Lemma 3.1 we have $\big|T_{c,c'}(\chi_1, \chi_2, b_1, b_2, b_1', b_2')\big| \le 8p^{1/2}$. Thus we obtain

$$S_1 \ll |A|^{3/4}\left(p|B|^2 + p^{1/2}|B|^4\right)^{1/4} \le p^{1/4}|A|^{3/4}|B|^{1/2} + p^{1/8}|A|^{3/4}|B|. \quad \square$$

Proof of Theorem 2.3. From Theorem 2.1 we deduce a new bound for the triple sum $S_2 = S_2(\chi, W)$ defined by (9). First through the triangle and Cauchy inequalities,

we have

$$\begin{aligned}
S_2 &\le \sum_{(a,b)\in X}\Big|\sum_{c\in Z^a\cap Y_b}\chi(a+b+c)\Big| \\
&\le |X|^{1/2}\Big(\sum_{(a,b)\in\mathbb{F}_p{}^2}\sum_{c,c'\in Z^a\cap Y_b}\chi(a+b+c)\overline{\chi(a+b+c')}\Big)^{1/2} \\
&= |X|^{1/2}\Big(\sum_{c\in\mathbb{F}_p}|Z_c\cap Z_{c'}||Y^c\cap Y^{c'}| + \sum_{c\ne c'\in\mathbb{F}_p}\sum_{\substack{a\in Z_c\cap Z_{c'}\\ b\in Y^c\cap Y^{c'}}}\chi(a+b+c)\overline{\chi(a+b+c')}\Big)^{1/2}
\end{aligned}$$

after interchanging the summation and separating the cases $c=c'$ and $c\ne c'$.

Now we apply Theorem 2.1 to treat the inner sum on a and b. Then

$$\begin{aligned}
S_2 &\ll |X|^{1/2}\Big(p|Z| + \sum_{c,c'\in\mathbb{F}_p}\Big(p^{1/4}|Z_{c'}|^{1/2}|Y^c|^{3/4} + p^{1/8}|Z_{c'}||Y^c|^{3/4}\Big)\Big)^{1/2} \\
&\ll p^{1/2}|X|^{1/2}|Z|^{1/2} + p^{1/8}|X|^{1/2}\Big(\sum_{c'\in\mathbb{F}_p}|Z_{c'}|^{1/2}\Big)^{1/2}\Big(\sum_{c\in\mathbb{F}_p}|Y^c|^{3/4}\Big)^{1/2} \\
&\qquad + p^{1/16}|X|^{1/2}|Z|^{1/2}\Big(\sum_{c\in\mathbb{F}_p}|Y^c|^{3/4}\Big)^{1/2} \\
&\ll p^{1/2}|X|^{1/2}|Z|^{1/2} + p^{1/2}|X|^{1/2}|Z|^{1/4}|Y|^{3/8} + p^{3/16}|X|^{1/2}|Z|^{1/2}|Y|^{3/8}.
\end{aligned}$$

By the Hölder and Cauchy inequalities we have

$$\sum_{c\in\mathbb{F}_p}|Y^c|^{3/4} \le p^{1/4}\Big(\sum_{c\in\mathbb{F}_p}|Y^c|\Big)^{3/4} = p^{1/4}|Y|^{3/4}$$

$$\sum_{c'\in\mathbb{F}_p}|Z_{c'}|^{1/2} \le p^{1/2}\Big(\sum_{c'\in\mathbb{F}_p}|Z_{c'}|\Big)^{1/2} = p^{1/2}|Z|^{1/2}.$$

Finally from the assumption $|Z|\le|Y|$ we get the desired result. □

4. Comments

We may apply Theorem 2.1 to estimate the number $N'(A,B)$ of pairs $(a,b)\in A\times B$ such that both $a+b$ and $a+b+1$ are squares in $\mathbb{F}_p^*$. We have

$$\begin{aligned}
N'(A,B) &= \frac{1}{4}\sum_{(a,b)\in A\times B}\Big(1+\Big(\frac{a+b}{p}\Big)\Big)\Big(1+\Big(\frac{a+b+1}{p}\Big)\Big) \\
&= \frac{|A||B|}{4} + R(A,B)
\end{aligned}$$

where $R(A, B)$ is an *error* term which can be bounded by one of the cases in Corollary 2.2.

Theorems 1.5 and 2.3 can be also used in a similar way in order to estimate character sums where the triples $(a, b, c) \in W$ satisfy some prescribed arithmetic properties, such as $a + b + c$ is a square in $\mathbb{F}_p^*$.

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