GRAPH COMPOSITIONS: DELETING EDGES FROM COMPLETE GRAPHS

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Abstract

Graph compositions are related to compositions of positive integers and partitions of finite sets, and have applications in electrical networks. This paper provides extensions of a previously known result which states that

$$C(K_N^-) = B(N) - B(N-2),$$

where $B(N)$ represents the N^{th} Bell number and $C(K_N^-)$ is the complete graph on N vertices with one edge deleted.

1. Introduction

Graph compositions were introduced by Knopfmacher and Mays in [3]. Their work defines graph compositions and develops formulae for the number of compositions of a few families of graphs (e.g., paths, cycles, trees, etc.). A study on the number of compositions of unions of graphs was conducted by Ridley and Mays [5]. Graph compositions were used in [2] to gain additional insight into series-parallel graphs (which are relevant to the study of electrical networks) and were connected to flats of matroid cycles by Mphako-Banda [4]. Graph compositions are also related to compositions of positive integers and partitions of finite sets. Specifically, the number of compositions of the path P_n is equal to the number of compositions of the positive integer n, and $C(K_n)$ is the number of set partitions of S, where $|S| = n$.

Let G be a graph and $E(G)$ and $V(G)$ represent the edge set and vertex set of G. A *composition* of G is defined as a partition of $V(G)$ into vertex sets of

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connected induced subgraphs of G. Hence, a composition of G yields a set of connected subgraphs of G, $\{G_1, G_2, ..., G_m\}$, with properties $\bigcup_{i=1}^{m} V(G_i) = V(G)$ and $V(G_i) \cap V(G_j) = \emptyset$ for $i \neq j$ [3].

Before discussing results, some concepts, notation, and previous results used throughout the paper are introduced. For all that follows, let G be a graph on n vertices, $\mathcal{C}$ be a composition of G, and H be a subgraph of G. Then $\mathcal{C} = \{V(G_1), V(G_2), ..., V(G_m)\}$ such that $\bigcup_{i=1}^{m} V(G_i) = V(G)$ and $V(G_i) \cap V(G_j) = \emptyset$ for $i \neq j$. Every $V(G_i) \in \mathcal{C}$ will be referred to as a *component* of $\mathcal{C}$ and will be denoted as G_i when there is no chance of confusion (given in [3]). Furthermore, G^{-H} will be used to denote the graph with vertex set $V(G)$ and edge set $E(G) \backslash E(H)$. We will refer to this graph as "the deletion of H from G" and refer to the process of obtaining the graph as "deleting H from G." Any $e \in E(G)$ is said to be *contained* in a composition $\mathcal{C}$ if there exists a component of G which contains both vertices of e. Additionally, if $G^{-e_1} \cong G^{-e_2}$ for every $e_1, e_2 \in E(G)$, then we denote G^{-e} as G^- for all $e \in E(G)$. The *composition number* of G is the number of distinct compositions of G and will be denoted as $C(G)$ (given in [3]). $C(K_N)$ is known to be $B(N)$ (given in [3]), where $B(N)$ is the N^{th} term in the *Bell number sequence* [1]. Finally, for $N \in \mathbb{Z}^+$, let G be a subgraph of K_N, $\mathcal{C}$ be a composition of K_N, and G_i be a component of $\mathcal{C}$. If $V(G_i) \subseteq V(G)$ and the complement of $G|_{G_i}$ is disconnected, then G_i will be referred to as a *bad* component of K_N^{-G}. Otherwise, G_i will be referred to as a *good* component of K_N^{-G}.

The theorems in this paper were motivated by this larger problem: for any graph G, is there an efficient method for locating an edge e_1 such that $C(G^{-e_1}) \leq C(G^{-e})$ for all $e \in E(G)$ (other than calculating $C(G^{-e})$ for every $e \in E(G)$)? Efforts to solve it have led to insights about graph compositions that perhaps would not have been obtained otherwise.

The focus of this paper is on the composition number of "dense graphs" (i.e., graphs whose complements have small edge sets); more specifically, we concern ourselves with the composition number of deletions of certain families of graphs (e.g., paths, cycles, stars, etc.) from complete graphs. The motivation for this study came from a result in [3] which states $C(K_N^-) = B(N) - B(N-2)$ for $N \geq 2$. The article goes on to state that adjacency of the edges must be taken into account when deleting more than a single edge from K_n, but does not

Theorem 1. *Let G be a subgraph of K_N for $N \in \mathbb{Z}^+$, $|V(G)| = n$, and $b_{j,k,n}$ represent the number of ways of choosing k disjoint bad components of K_N^{-G} such that the cardinality of the union of vertices of all components is j. Define $b_{j,n} = \sum_{k=0}^{n} (-1)^k \cdot b_{j,k,n}$. Then $C(K_N^{-G}) = \sum_{j=0}^{n} b_{j,n} B(N-j)$.*

Proof. Let $\mathcal{C}$ be a composition of K_N. Then $\mathcal{C}$ will *not* be a composition of K_N^{-G}

if and only if $\mathcal{C}$ contains a bad component of K_N^{-G}. Utilization of the sieve method shows

$$C(K_N^{-G}) = \sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^k b_{j,k,n} B(N-j).$$

We begin by denoting the set of all compositions of K_N by Γ. Then, for all $\gamma \in \Gamma$, denote the number of disjoint bad components contained in γ by $B(\gamma)$. We have

$$C(K_N^{-G}) = \sum_{\substack{\gamma\in\Gamma \\ B(\gamma)=0}} 1 = \sum_{\substack{\gamma\in\Gamma \\ B(\gamma)=0}} 1 + \sum_{\substack{\gamma\in\Gamma \\ B(\gamma)\neq 0}} 0 = \sum_{\gamma\in\Gamma}\sum_{k=0}^{B(\gamma)}(-1)^k \binom{B(\gamma)}{k}$$

since the alternating sum of the k^{th} row of Pascal's Triangle is 0 for $k > 0$ and 1 for $k = 0$. Note that $\binom{B(\gamma)}{k}$ will count the number of times γ is counted as a composition of K_N with at least k disjoint bad components for a fixed γ and k.

Next, let $B_{k,n}$ denote the set of all compositions of K_N which contain at least k disjoint bad components. We can write

$$|B_{k,n}| = \sum_{j=0}^{n} b_{j,k,n} B(N-j),$$

and

$$\sum_{k=0}^{n}|B_{k,n}| = \sum_{\gamma\in\Gamma}\sum_{k=0}^{B(\gamma)}\binom{B(\gamma)}{k},$$

and therefore

$$\begin{aligned} C(K_N^{-G}) &= \sum_{k=0}^{n}(-1)^k|B_{k,n}| = \sum_{k=0}^{n}\sum_{j=0}^{n}(-1)^k b_{j,k,n}B(N-j) \\ &= \sum_{j=0}^{n}\sum_{k=0}^{n}(-1)^k b_{j,k,n}B(N-j) = \sum_{j=0}^{n} b_{j,n}B(N-j). \end{aligned}$$

□

Remark 1. It should be noted that $b_{0,n} = 1$ (since we can choose zero vertices exactly one way) and $b_{1,n} = 0$ (since there is no way to choose one vertex to be a single component) for all $n \in \mathbb{Z}^+$. Also, $b_{j,n}$ is undefined for $j > n$ (since we cannot choose more than n vertices from a set of n vertices).

1.1. Deletions of Families of Graphs from Complete Graphs

1.1.1. Paths

Theorem 2. *Let P_n be a subgraph of K_N for $N \in \mathbb{Z}^+$. Define $p_{j,k,n}$ to be the number of ways of choosing k disjoint bad components of $K_N^{-P_n}$ such that the number* *of vertices of all bad components is* j *and* $p_{j,n} = \sum_{k=0}^{n} (-1)^k \cdot p_{j,k,n}$. *Then* $C(K_N^{-P_n}) = \sum_{j=0}^{n} p_{j,n} B(N-j)$ *and* $p_{j,n} = p_{j,n-1} - p_{j-2,n-2} - p_{j-3,n-3}$ *for* $j, n \geq 3$.

Proof. We have $C(K_N^{-P_n}) = \sum_{j=0}^{n} p_{j,n} B(N-j)$ by applying Theorem 1 to $K_N^{-P_n}$. Let $\mathcal{C}$ be a composition of K_N. Then $\mathcal{C}$ will *not* be a composition of $K_N^{-P_n}$ if and only if $\mathcal{C}$ contains a bad component of $K_N^{-P_n}$.

Note that if we delete a subpath of length $t \geq 3$ from P_n in K_N, then the complement is necessarily connected. The endpoints of P_{t+1} will be adjacent and every interior point of the path will be adjacent to both endpoints. Hence, the only bad components that exist when deleting P_n from K_N are subpaths of P_n of length 1 and 2.

Next, define $S_{j,n} = \sum_{k \text{ is even}} p_{j,k,n}$ and $L_{j,n} = \sum_{k \text{ is odd}} p_{j,k,n}$. Note that $S_{j,n}$ represents the number of ways of choosing an even number of disjoint bad components from $V(P_n)$ where the cardinality of the union of bad vertices is j, and $L_{j,n}$ analogously represents the number of ways of choosing an odd number of disjoint bad components from $V(P_n)$ where the cardinality of the union of bad vertices is j. We can write $p_{j,n} = S_{j,n} - L_{j,n}$.

Let u represent one of the terminal vertices of the deleted path and fix the component, C, which contains u. Then one of three cases occurs:

1. C is not a bad component. Then there are $S_{j,n-1}$ [$L_{j,n-1}$] ways of choosing an even [odd] number of disjoint bad components from $V(P_N)$ (with the cardinality of the union of vertices equal to j) which do not include C.

2. C is a 2-element bad component. Then there are $L_{j-2,n-2}$ [$S_{j-2,n-2}$] ways of choosing an even [odd] number of disjoint bad components from $V(P_n)$ (with the cardinality of the union of vertices equal to j) which include C.

3. C is a 3-element bad component. Then there are $L_{j-3,n-3}$ [$S_{j-3,n-3}$] ways of choosing an even [odd] number of disjoint bad components from $V(P_n)$ (with the cardinality of union of vertices equal to j) which include C.

The above "world encompassing" cases give us

$$\begin{aligned} S_{j,n} &= S_{j,n-1} + L_{j-2,n-2} + L_{j-3,n-3} \\ L_{j,n} &= L_{j,n-1} + S_{j-2,n-2} + S_{j-3,n-3} \end{aligned}$$

which yields $p_{j,n} = p_{j,n-1} - p_{j-2,n-2} - p_{j-3,n-3}$. □

A table of values for $p_{j,n}$ is given below.

n \ j	0	1	2	3	4	5	6	7	...	
0	1	-	-	-	-	-	-	-	...	
1	1	0	-	-	-	-	-	-	...	
2	1	0	-1	-	-	-	-	-	...	
3	1	0	-2	-1	-	-	-	-	...	
4	1	0	-3	-2	1	-	-	-	...	
5	1	0	-4	-3	3	2	-	-	...	
6	1	0	-5	-4	6	6	0	-	...	
7	1	0	-6	-5	10	12	-1	-3	...	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Remark 2. We have $p_{j,n} = -(n-1)$ for $j = 2$ since this implies the bad component being chosen from $V(P_n)$ is an edge and there are $n-1$ ways of choosing an edge from P_n.

Theorem 3. *Let* $F(x,y) = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} p_{j,n}x^n y^j$*. Then* $F(x,y) = \dfrac{1}{1+x^3y^3+x^2y^2-x}$.

Proof. Using our recurrence relation for $p_{j,n}$ we get

$$p_{j,n}x^n y^j = x\cdot p_{j,n-1}x^{n-1}y^j - x^2y^2 p_{j-2,n-2}x^{n-2}y^{j-2} - x^3y^3\cdot p_{j-3,n-3}x^{n-3}y^{j-3}$$

for $j,n \geq 3$. This in turn yields

$$\sum_{j=3}^{\infty}\sum_{n=3}^{\infty}p_{j,n}x^ny^j = x\sum_{j=3}^{\infty}\sum_{n=2}^{\infty}p_{j,n}x^ny^j - x^2y^2\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}p_{j,n}x^ny^j - x^3y^3\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}p_{j,n}x^ny^j.$$

If we consider $p_{j,n} = 0$ for $j > n$, we have

$$\sum_{j=3}^{\infty}\sum_{n=0}^{\infty}p_{j,n}x^ny^j = x\sum_{j=3}^{\infty}\sum_{n=0}^{\infty}p_{j,n}x^ny^j - x^2y^2\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}p_{j,n}x^ny^j - x^3y^3\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}p_{j,n}x^ny^j.$$

Rewriting the entire equation in terms of $F(x,y)$ then yields

$$\begin{aligned}
&F(x,y) - \sum_{n=0}^{\infty}p_{0,n}x^n - \sum_{n=0}^{\infty}p_{1,n}x^ny - \sum_{n=0}^{\infty}p_{2,n}x^ny^2\\
&= x(F(x,y) - \sum_{n=0}^{\infty}p_{0,n}x^n - \sum_{n=0}^{\infty}p_{1,n}x^ny - \sum_{n=0}^{\infty}p_{2,n}x^ny^2)\\
&- x^2y^2(F(x,y) - \sum_{n=0}^{\infty}p_{0,n}x^n) - x^3y^3F(x,y).
\end{aligned}$$

Application of Remark 1 and manipulation of the equation yields

$$F(x,y)(1 - x + x^2y^2 + x^3y^3) = 1,$$

which leads directly to the desired result. □

1.1.2. Cycles

After examination of $C(K_N^{-P_n})$, it is natural to consider $C(K_N^{-C_n})$ (since any path can be made into a cycle by connecting its terminal vertices). Hence, this section is concerned with the study of $C(K_N^{-C_n})$.

Theorem 4. *Let C_n be a subgraph of K_N where $N \in \mathbb{Z}^+$ and $p_{j,n}$ be the coefficients as defined in Theorem 2. Define $c_{j,k,n}$ to be the number of ways of choosing k disjoint bad components of $K_N^{-C_n}$ where the number of vertices of the bad components is j, and $c_{j,n} = \sum_{k=0}^{n} (-1)^k \cdot c_{j,k,n}$. Then $C(K_N^{-C_n}) = \sum_{j=0}^{n} c_{j,n} B(N-j)$ and $c_{j,n} = p_{j,n-1} - 2 \cdot p_{j-2,n-2} - 3 \cdot p_{j-3,n-3}$ for $j \geq 3$, $n \geq 3$, and $n \neq j$ for $n = 3, 4$.*

Proof. That $C(K_N^{-C_n}) = \sum_{j=0}^{n} c_{j,n} B(N-j)$ follows from the application of Theorem 1 to $K_N^{-C_n}$. Let $\mathcal{C}$ be a composition of K_N. Then, $\mathcal{C}$ will **not** be a composition of $K_N^{-C_n}$ if and only if $\mathcal{C}$ contains a bad component of $K_N^{-C_n}$.

Next, we describe the bad components of $K_N^{-C_n}$. Since there is always some $P_n \subseteq C_n$, $K_N^{-C_n}$ will inherit subpaths of length 1 and 2 of C_n as bad components.

Consider a composition of K_N for which all of C_n has been deleted. It is easily verified that if $n > 4$, then the composition is *good*. Given this, we also classify any component which contains a 3 or 4 element cyclic component as bad.

Define $S_{j,n} = \sum_{k \text{ is even}} c_{j,k,n}$ and $L_{j,n} = \sum_{k \text{ is odd}} c_{j,k,n}$. Then $S_{j,n}$ represents the number of ways of choosing an even number of disjoint bad components from $V(C_n)$ where the cardinality of the union of vertices is j and $L_{j,n}$ analogously represents the number of ways of choosing an odd number of disjoint bad components from $V(C_n)$ where the cardinality of the union of vertices is j. Note that $c_{j,n} = S_{j,n} - L_{j,n}$.

Let $w \in V(C_n)$ and fix the component $C_w \in \mathcal{C}$ which contains w. Then one of four cases occur:

1. C_w is not a bad component. Then there are $S_{j,n-1}$ [$L_{j,n-1}$] ways of choosing an even [odd] number of disjoint bad components from $V(C_N)$ (with cardinality of the union of vertices j) which do not include C_w.

2. C_w is a 2-element bad component. Then there are $2 \cdot L_{j-2,n-2}$ [$2 \cdot S_{j-2,n-2}$] ways of choosing an even [odd] number of disjoint bad components from $V(C_n)$ (with cardinality of the union of vertices j) which include C_w.

3. C_w is a 3-element bad component. If C_w is comprised of just a path ($n > 3$), then there are $3 \cdot L_{j-3,n-3}$ [$3 \cdot S_{j-3,n-3}$] ways of choosing an even [odd] number of disjoint bad components from $V(C_n)$ (with cardinality of the union of vertices j) which include C_w. If C_w is a cycle, then there is exactly one way of choosing an odd number of bad components and no way of choosing an even number of bad components.

4. C_w is a 4-element bad component. Then there is exactly one way of choosing

an odd number of bad components and no way of choosing an even number of bad components.

The above "world encompassing" cases give us

$$S_{j,n} = S_{j,n-1} + 2 \cdot L_{j-2,n-2} + 3 \cdot L_{j-3,n-3}$$
$$L_{j,n} = L_{j,n-1} + 2 \cdot S_{j-2,n-2} + 3 \cdot S_{j-3,n-3}$$

which yields $c_{j,n} = p_{j,n-1} - 2 \cdot p_{j-2,n-2} - 3 \cdot p_{j-3,n-3}$. □

The first few values of $c_{j,n}$ are shown in the table below.

n \ j	0	1	2	3	4	5	6	7	...	
0	1	-	-	-	-	-	-	-	...	
1	1	0	-	-	-	-	-	-	...	
2	1	0	-1	-	-	-	-	-	...	
3	1	0	-3	-1	-	-	-	-	...	
4	1	0	-4	-4	1	-	-	-	...	
5	1	0	-5	-5	5	5	-	-	...	
6	1	0	-6	-6	9	12	1	-	...	
7	1	0	-7	-7	14	21	0	-7	...	
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Remark 3. We see $c_{2,n} = -n$ for $n > 2$, since $c_{2,n}$ is the number of ways of choosing a single edge from C_n.

Theorem 5. *Let* $G(x, y) = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} c_{j,n}x^n y^j$. *Then*

$$G(x, y) = 1 + x^2y^2 + 2x^3y^3 - x^4y^4 + \frac{x - 2x^2y^2 - 3x^3y^3}{1 + x^3y^3 + x^2y^2 - x}.$$

Proof. Given that $c_{j,n} = 0$ for $j > n$ and $c_{1,n} = 0$, it is easily established that

$$\begin{aligned}\sum_{j=3}^{\infty}\sum_{n=3}^{\infty} c_{j,n}x^n y^j &= G(x, y) - \sum_{n=0}^{\infty} c_{0,n}x^n - y^2\sum_{n=2}^{\infty} c_{2,n}x^n \\ &= \sum_{j=3}^{\infty} c_{j,3}x^3y^j + \sum_{j=3}^{\infty}\sum_{n=4}^{\infty} c_{j,n}x^n y^j \\ &= c_{3,3}x^3y^3 + c_{4,4}x^4y^4 + y^3\sum_{n=4}^{\infty} c_{3,n}x^n + \sum_{j=4}^{\infty}\sum_{n=5}^{\infty} c_{j,n}x^n y^j.\end{aligned}$$

Substitution of known values and sums lets us write the right hand side as

$$-x^3y^3 + x^4y^4 + \frac{y^3x^4(3x - 4)}{(1 - x)^2} + \sum_{j=4}^{\infty}\sum_{n=5}^{\infty} c_{j,n}x^n y^j.$$

Now application of Theorem 4 lets us write this as

$$
\begin{aligned}
&- x^3y^3 + x^4y^4 + \frac{y^3x^4(3x-4)}{(1-x)^2} + x\sum_{j=4}^{\infty}\sum_{n=4}^{\infty} p_{j,n}x^ny^j - 2x^2y^2\sum_{j=2}^{\infty}\sum_{n=3}^{\infty} p_{j,n}x^ny^j \\
&- 3x^3y^3\sum_{j=2}^{\infty}\sum_{n=2}^{\infty} p_{j,n}x^ny^j \\
&= -x^3y^3 + x^4y^4 + \frac{y^3x^4(3x-4)}{(1-x)^2} + x[F(x,y) - \sum_{n=0}^{\infty} p_{0,n}x^n - y^2\sum_{n=2}^{\infty} p_{2,n}x^n \\
&- y^3\sum_{n=3}^{\infty} p_{3,n}x^n] \\
&= -2x^2y^2[y^2\sum_{n=3}^{\infty} p_{2,n}x^n + F(x,y) - \sum_{n=0}^{\infty} p_{0,n}x^n] - 3x^3y^3[F(x,y) - \sum_{n=0}^{\infty} p_{0,n}x^n].
\end{aligned}
$$

The above equation along with substitution and algebraic manipulation gives

$$
\begin{aligned}
G(x,y) &= 1 - x^2y^2 - x^3y^3 + x^4y^4 + \frac{(1-x^2)(2x^2y^2 + 3x^3y^3 - 2x^4y^4)}{(1-x)^2} \\
&\quad + F(x,y)[x - 2x^2y^2 - 3x^3y^3] \\
&= 1 + x^2y^2 + 2x^3y^3 - x^4y^4 + \frac{x - 2x^2y^2 - 3x^3y^3}{1 - x + x^2y^2 + x^3y^3.}
\end{aligned}
$$

□

1.1.3. Star Graphs

Recall that a star graph on $n+1$ vertices (denoted S_{n+1}) is a graph of $n+1$ vertices and n edges where exactly one vertex (which we call the central vertex) is incident to all other vertices.

Theorem 6. *Let $N, n \in \mathbb{Z}^+$ with $n+1 \leq N$. Then*

$$C(K_N^{-S_{n+1}}) = B(N) - \sum_{k=1}^{n} \binom{n}{k} \cdot B(N-(k+1)).$$

Proof. Let $\mathcal{C}$ be a composition of K_N. Then $\mathcal{C}$ is *not* a composition of $K_N^{-S_{n+1}}$ if and only if $\mathcal{C}$ contains a bad component of $K_N^{-S_{n+1}}$. Note that all bad components will be isomorphic to some S_{k+1}, $k \leq n$ and the vertex sets of all distinct bad components will have intersection that is exactly the central vertex of S_{n+1}. This implies that it is impossible to choose more than 1 unique disjoint bad component simultaneously. The number of compositions which contain exactly one bad component, G_i, where $|V(G_i)| = k+1$, is $\binom{n}{k} \cdot B(N-(k+1))$. Hence,

$$C(K_N^{-S_{n+1}}) = B(N) - \sum_{k=1}^{n} \binom{n}{k} \cdot B(N-(k+1)).$$

□

Remark 4. Note that $C(K_N^{-S_{n+1}})$ can be written as $\sum_{j=0}^{n} s_{j,n} B(N-j)$, where $s_{j,n}$ is undefined for $j > n$, $s_{0,n} = 1$ for all n, $s_{1,n} = 0$ for all $n > 0$, and $s_{j,n} = -\binom{n-1}{j-1}$ for $j, n \geq 2$ and $j \leq n$. Having an explicit formula for $s_{j,n}$ makes it trivial to establish that $s_{j,n} = s_{j-1,n-1} + s_{j,n-1}$ for $j, n \geq 3$. This allows us to recover a generating function as before.

Remark 5. Note also that $C(K_N^{-S_N}) = B(N-1)$ since $K_N^{-S_N} \cong K_{N-1}$. Setting $N = N+1$ yields $C(K_{N+1}^{-S_{N+1}}) = B(N)$ and applying Theorem 6 gives $C(K_{N+1}^{-S_{N+1}}) = B(N+1) - \sum_{k=1}^{N} \binom{N}{k} B(N-k)$. Setting the two equations equal yields $B(N+1) = B(N) + \sum_{k=1}^{N} \binom{N}{k} B(N-k) = \sum_{k=0}^{N} \binom{N}{k} B(N-k)$, a well-known recursion of the Bell numbers.

Theorem 7. *Let* $S(x,y) = \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} s_{j,n} x^n y^j$*. Then* $S(x,y) = \dfrac{1 - x - xy - x^2y^2}{1 - 2x + x^2 - xy + x^2y}$.

Proof. Using Remark 4, algebraic manipulation, and known values for $s_{j,n}$, we get

$$\begin{aligned}
S(x,y) - \sum_{n=0}^{\infty} s_{0,n} x^n - y^2 \sum_{n=0}^{\infty} s_{2,n} x^n &= \sum_{j=3}^{\infty} \sum_{n=3}^{\infty} s_{j,n} x^n y^j \\
&= xy \sum_{j=2}^{\infty} \sum_{n=2}^{\infty} s_{j,n} x^n y^j + x \sum_{j=3}^{\infty} \sum_{n=3}^{\infty} s_{j,n} x^n y^j \\
&= xy \cdot (S(x,y) - \sum_{n=0}^{\infty} s_{0,n} x^n) \\
&\quad + x \cdot (S(x,y) - \sum_{n=0}^{\infty} s_{0,n} x^n - y^2 \cdot \sum_{n=2}^{\infty} s_{2,n} x^n).
\end{aligned}$$

This yields

$$\begin{aligned} S(x,y)[1-x-xy] &= (1-x-xy)\sum_{n=0}^{\infty} s_{0,n}x^n + y^2(1-x)\sum_{n=2}^{\infty} s_{2,n}x^n \\ &= \frac{1-x-xy-x^2y^2}{1-x}. \end{aligned}$$

Hence, $S(x,y) = \dfrac{1-x-xy-x^2y^2}{(1-x)(1-x-xy)} = \dfrac{1-x-xy-x^2y^2}{1-2x+x^2-xy+x^2y}$. □

1.1.4. "Disjoint" Graphs

In this section, we refer to a graph of k disjoint edges on $2k$ vertices as a *disjoint graph* and denote it as D_k.

Theorem 8. *Let $N, n \in \mathbb{Z}^+$ such that $2n \leq N$. Then*

$$C(K_N^{-D_n}) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cdot B(N-2k).$$

Proof. Let $\mathcal{C}$ be a composition of K_N. Then C is *not* a composition of $K_N^{-D_n}$ if and only if $\mathcal{C}$ contains a bad component of $K_N^{-D_k}$.

Let Γ represent the set of all compositions of K_N and $B(\gamma)$ represent the number of bad components contained in $\gamma \in \Gamma$. Also, let $A_{k,n}$ represent the set of compositions of K_N with at least k bad components. Then, as in Theorem 2,

$$C(K_N^{-D_n}) = \sum_{\gamma\in\Gamma}\sum_{k=0}^{B(\gamma)} (-1)^k \tbinom{B(\gamma)}{k}.$$

Using $|A_{k,n}| = \binom{n}{k}B(N-2k)$, we get the result

$$C(K_N^{-D_n}) = \sum_{\gamma\in\Gamma}\sum_{k=0}^{B(\gamma)} (-1)^k \tbinom{B(\gamma)}{k} = \sum_{k=0}^{n} (-1)^k \cdot |A_{k,n}| = \sum_{k=0}^{n} (-1)^k \cdot \tbinom{n}{k} B(N-2k).$$

□

Remark 6. Setting $d_{j,n} = (-1)^j \binom{n}{j}$ yields $C(K_N^{-D_n}) = \sum_{j=0}^{n} d_{j,n} \cdot B(N-j)$, where $d_{j,n} \in \mathbb{Z}$ for all j, n. As with Remark 4, having the explicit formula for $d_{j,n}$ makes it easy to establish that $d_{j,n} = d_{j,n-1} - d_{j-1,n-1}$ for $j, n \geq 1$.

Theorem 9. *Let $D(x,y) = \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} d_{j,n}x^n y^j$. Then $D(x,y) = \dfrac{1}{1-x+xy}$.*

Proof. Using Remark 5, we obtain the result

$$\begin{aligned} D(x,y) - \sum_{n=0}^{\infty} d_{0,n}x^n \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} d_{j,n}x^n y^j \\ = x\sum_{j=1}^{\infty}\sum_{n=1}^{\infty} d_{j,n}x^n y^j - xy\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} d_{j,n}x^n y^j \\ = x(D(x,y) - \sum_{n=0}^{\infty} d_{0,n}x^n) - xy \cdot D(x,y). \end{aligned}$$

This implies that

$$D(x,y)[1-x+xy] = (1-x)\sum_{n=0}^{\infty} d_{0,n}x^n = \frac{1-x}{1-x} = 1.$$

Therefore, we get $D(x,y) = \dfrac{1}{1-x+xy}$. □

2. Compositions of Subgraphs

We can compare the composition number of a graph with the composition number of a subgraph.

Theorem 10. *Let G be a graph and H be a proper subgraph of G. Then $C(H) < C(G)$*

Proof. Since $H \subset G$, we know that $E(H) \subset E(G)$. This implies that every composition of H will also be a composition of G (i.e., $C(H) \leq C(G)$). Also, there exists an $e \in E(G) \backslash E(H)$. Let $\mathcal{C}$ be a composition of G which has e as a component. Then $\mathcal{C}$ cannot be a composition of H since $e \notin E(H)$. Hence, $C(H) < C(G)$. □

Theorem 11. *Let G be a graph and H be a proper subgraph of G such that $|E(H)| + k = |E(G)|$. Then $C(H) \geq \frac{1}{2^k} C(G)$.*

Proof. We prove this theorem via induction. Let G be a graph and H a proper subgraph of G such that $|E(H)|+1 = |E(G)|$. Then there exists an $e \in E(G) \backslash E(H)$. For every composition $\mathcal{C}$ of G that contains e, either $\mathcal{C}$ is or is not a composition of H. Assume that there are exactly s compositions which contain e that are compositions of H and t compositions which contain e that are not compositions of H. If $\mathcal{C}$ is not a composition of H which contains e, then deleting e will yield a composition of H which obviously does not contain e. This one-to-one correspondence yields $C(H) = s + t$ and $C(G) = 2t + s \leq 2t + 2s = 2C(H)$, which leads directly to the result of Theorem 11 for $k = 1$.

Assume that the theorem holds for $k \geq 1$. Consider a graph G and subgraph H of G such that $|E(H)|+(k+1) = |E(G)|$. Then, there exists $e \in E(G) \backslash E(H)$. $H \subset G^{-e}$ and $|E(G^{-e})| = |E(H)|+k$. By our induction hypothesis, $C(H) \geq \frac{1}{2^k} C(G^{-e})$. Additionally, $G^{-e} \subset G$ and $|E(G^{-e})|+1 = |E(G)|$ yields $C(G^{-e}) \geq \frac{1}{2} C(G)$. Hence, $C(H) \geq \frac{1}{2^k} C(G^{-e}) \geq \frac{1}{2^{k+1}} C(G)$ and Theorem 11 is true. □

3. Future Work

Bell numbers arise naturally in deleting graphs from the complete graph. What happens when the base graph is chosen from another family, perhaps complete bipartite graphs? One example would be to simply take the treatment of complete graphs from this paper and apply it to another family of graphs with the hopes of gaining more intuition about deleting families of graphs from any general graph. Another problem is the following: assume that G is any graph such that $|E(G)| = k$ and $k+1 \leq |V(G)| \leq 2k$. Is it possible to find some G_1 and G_2 such that $|E(G_1)| = |E(G_2)| = k$ and $C(K_N^{-G_1}) \leq C(K_N^{-G}) \leq C(K_N^{-G_2})$?

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