THE SUM OF DIGITS OF POLYNOMIAL VALUES

Shu-Yuan Mei
Department of Mathematics and Computer Science, Jiangsu Second Normal University, Nanjing, P. R. China
shuyuan_mei@sina.com

Received: 7/8/14, Revised: 3/5/15, Accepted: 6/9/15, Published: 7/17/15

Abstract

Let $s_q(n)$ denote the sum of the digits in the q-ary expansion of a nonnegative integer n, and let $p_1(x),\ p_2(x)$ be polynomials in $\mathbb{Z}[x]$ with distinct positive degrees. If $p_1(n) \geq 1$ and $p_2(n) \geq 1$ for all positive integers n, then for any $\varepsilon > 0$, we give lower bounds of the number of $n \leq N$ such that $s_q(p_1(n))/s_q(p_2(n)) < \varepsilon$.

1. Introduction

For any integer $q \geq 2$, let nonnegative integer

$$n = \sum_{i=0}^{k} \alpha_i(n) q^i, \quad \alpha_i(n) \in \{0, 1, \ldots, q-1\}.$$

Denote by $s_q(n) = \sum\limits_{i=0}^{k} \alpha_i(n)$ the sum of digits of n in base q. The study of the sum of digits mainly focuses on the sum of digits of some special sequences of integers, the average sum of the digits of integers, the asymptotic formula of the weighted sum-of-digits function, and the ratio of the sum of digits of polynomial values. For the study of the sum of digits of some special sequences of integers, several researchers investigated the properties of s_q of primes [14], polynomials [5, 7, 9, 11, 15, 16, 19], Fibonacci numbers [21] and Bernoulli numbers [2]. For the study of the average sum of the digits of integers, one may refer to [1, 4, 6, 17]. For the study of the asymptotic formula of the weighted sum-of-digits function, one may refer to [12, 18]. For the study of the ratio of the sum of digits of polynomial values, several researchers investigated the problems and a lot of academic achievements have been achieved.

This work was supported by the Project of Graduate Education Innovation of Jiangsu Province, CXZZ13_0385.

In 1978, Stolarsky [20] showed that

$$\liminf_{n\to\infty} \frac{s_2(n^2)}{s_2(n)} = 0$$

and conjectured that

$$\liminf_{n\to\infty} \frac{s_2(n^h)}{s_2(n)} = 0$$

for any integer $h \geq 2$.

In 2011, Hare, Laishram and Stoll [10] proved that for any integer $q \geq 2$ and for any polynomial $p(x) = \sum\limits_{i=0}^{t} c_i x^i \in \mathbb{Z}[x]$ with $t \geq 2$ and $c_t > 0$,

$$\liminf_{n\to\infty} \frac{s_q(p(n))}{s_q(n)} = 0.$$

In 2014, Madritsch and Stoll [13] proved that

$$\left(\frac{s_q(p_1(n))}{s_q(p_2(n))}\right)_{n\geq 1}$$

is dense in $\mathbb{R}^+$, where $p_1(x)$, $p_2(x)$ are polynomials in $\mathbb{Z}[x]$ of distinct positive degrees with $p_1(\mathbb{N})$, $p_2(\mathbb{N}) \subseteq \mathbb{N}$.

In this paper, we always assume that

$$p_1(x) = \sum_{i=0}^{h} a_i x^i \in \mathbb{Z}[x], \quad p_2(x) = \sum_{i=0}^{l} b_i x^i \in \mathbb{Z}[x]$$

with $h \geq 1$, $l \geq 1$, $a_h > 0$ and $b_l > 0$.

By employing the methods in [10] and [20], the following theorems are proved.

Theorem 1. *Let* $\deg p_1 > \deg p_2$. *If* $p_1(n) \geq 1$ *and* $p_2(n) \geq 1$ *for any positive integer* n, *then for any* $\varepsilon > 0$, *there exists a positive constant* C_1, *dependent only on* ε, q, $p_1(x)$ *and* $p_2(x)$, *such that*

$$\left|\left\{n \leq N : \frac{s_q(p_1(n))}{s_q(p_2(n))} < \varepsilon\right\}\right| \geq C_1 N^{\alpha}$$

for all sufficiently large integers N, *where* $\alpha = \varepsilon(2h(l+3)(h(l+3)+1)+\varepsilon)^{-1}$.

Theorem 2. *Let* $\deg p_1 < \deg p_2$. *If* $p_1(n) \geq 1$ *and* $p_2(n) \geq 1$ *for any positive integer* n, *then for any* $\varepsilon > 0$, *there exists a positive constant* C_2, *dependent only on* ε, q, $p_1(x)$ *and* $p_2(x)$, *such that*

$$\left|\left\{n \leq N : \frac{s_q(p_1(n))}{s_q(p_2(n))} < \varepsilon\right\}\right| \geq C_2 \log N$$

for all sufficiently large integers N.

2. Preliminary Lemmas

Let $[a, b]$ denote the interval of integers n such that $a \le n \le b$. For convenience, we write $f(s) \asymp g(s)$ $(s \in S)$ if $f(s)$ and $g(s)$ are positive for all $s \in S$ and $c_1 g(s) \le f(s) \le c_2 g(s)$ for all $s \in S$, where c_1 and c_2 are two positive constants.

Lemma 1. *(See [10, Proposition 2.1]) For any integers a, b, k with $1 \le b < q^k$ and a, $k \ge 1$, we have*

$$s_q(aq^k + b) = s_q(a) + s_q(b),$$
$$s_q(aq^k - b) = s_q(a-1) + (q-1)k - s_q(b-1).$$

Lemma 2. *Let n be a positive integer. Then for any integer $q \ge 2$, we have*

$$s_q(n) \le (q-1)(1 + \log_q n).$$

Proof. Let $n = \alpha_t q^t + \cdots + \alpha_1 q + \alpha_0$ with

$$\alpha_i \in \{0, 1, \ldots, q-1\}, \quad i = 0, 1, \ldots, t, \quad \alpha_t \ne 0.$$

Then

$$s_q(n) = \sum_{i=0}^{t} \alpha_i \le (q-1)(t+1) \le (q-1)(1 + \log_q n).$$

□

Lemma 3. *(See [8, Bose-Chowla Theorem] or [3]) Let $d \ge 2$ be an integer, and let M be a power of a prime. Then there exist integers $y_1, y_2, \ldots, y_{M+1}$ with $1 \le y_1 < y_2 < \cdots < y_{M+1} = M^d$ such that all sums*

$$y_{j_1} + y_{j_2} + \cdots + y_{j_d}, \quad 1 \le j_1 \le j_2 \le \cdots \le j_d \le M+1$$

are distinct.

Lemma 4. *Let l be a positive integer, and let*

$$t_m(x) = m + mx - x^2 - x^3 - \cdots - x^{l+1} + mx^{l+2} + mx^{l+3}$$

be a polynomial in $\mathbb{Z}[x]$. For any positive integer i, let

$$(t_m(x))^i = \sum_{j=0}^{(l+3)i} a_j^{(i,m)} x^j.$$

Then

(a) for all positive integers m and i, we have

$$|a_j^{(i,m)}| \le (4m+l)^i, \quad 0 \le j \le (l+3)i.$$

(b) there exists a positive constant c_0, dependent only on l, such that for any integer m with $m \geq c_0$ and any integer i with $1 \leq i \leq l$,

$$a_j^{(i,m)} \asymp m^i \text{ if and only if } j \in \bigcup_{0\leq k\leq i}([0,i]+(l+2)k),$$

and

$$-a_j^{(i,m)} \asymp m^{i-1} \text{ if and only if } j \in \bigcup_{0\leq k\leq i-1}([i+1,l+1]+(l+2)k).$$

(c) for any integer i with $i > l$, there exists a positive constant c_1, dependent only on i, such that for any integer m with $m \geq c_1$,

$$a_j^{(i,m)} \asymp m^i, \quad 0 \leq j \leq (l+3)i.$$

Proof. *(a)* Let

$$f_m(x) = m + mx + x^2 + x^3 + \cdots + x^{l+1} + mx^{l+2} + mx^{l+3}$$

and

$$(f_m(x))^i = \sum_{j=0}^{(l+3)i} b_j^{(i,m)} x^j.$$

Since

$$|a_j^{(i,m)}| \leq b_j^{(i,m)}, \quad \sum_{j=0}^{(l+3)i} b_j^{(i,m)} = (4m+l)^i,$$

it follows that

$$|a_j^{(i,m)}| \leq (4m+l)^i.$$

(b) We will complete the proof by induction on i. It is easy to see that Lemma 4 *(b)* is true for $i = 1, 2$. Suppose that Lemma 4 *(b)* is true for an integer i with $2 \leq i < l$. Let

$$(t_m(x))^i(m+mx) = \sum_{j=0}^{(l+3)i+1} c_j^{(i,m)} x^j,$$

$$(t_m(x))^i(mx^{l+2}+mx^{l+3}) = \sum_{j=l+2}^{(l+3)(i+1)} d_j^{(i,m)} x^j,$$

and

$$(t_m(x))^i(-x^2-x^3-\cdots-x^{l+1}) = \sum_{j=2}^{(l+3)i+l+1} e_j^{(i,m)} x^j.$$

By the induction hypothesis, for all sufficiently large integers m, it is easy to get that

$$c_j^{(i,m)} \asymp m^{i+1} \text{ if and only if } j \in \bigcup_{0\leq k\leq i}([0,i+1]+(l+2)k),$$

$$-c_j^{(i,m)} \asymp m^i \text{ if and only if } j \in \bigcup_{0\le k\le i-1}([i+2,l+1]+(l+2)k),$$

$$d_j^{(i,m)} \asymp m^{i+1} \text{ if and only if } j \in \bigcup_{1\le k\le i+1}([0,i+1]+(l+2)k),$$

$$-d_j^{(i,m)} \asymp m^i \text{ if and only if } j \in \bigcup_{1\le k\le i}([i+2,l+1]+(l+2)k),$$

and

$$-e_j^{(i,m)} \asymp m^i, \quad 2 \le j \le (l+3)i+l+1.$$

Therefore, for all sufficiently large integers m, we have

$$a_j^{(i+1,m)} \asymp m^{i+1} \text{ if and only if } j \in \bigcup_{0\le k\le i+1}([0,i+1]+(l+2)k),$$

and

$$-a_j^{(i+1,m)} \asymp m^i \text{ if and only if } j \in \bigcup_{0\le k\le i}([i+2,l+1]+(l+2)k).$$

(c) From the proof of Lemma 4 *(b)*, we see that Lemma 4 *(c)* is true for $i = l+1$. A proof is similar to the proof of Lemma 4 *(b)* by induction on $i \ge l+1$. This completes the proof of Lemma 4. □

3. Proof of Theorem 1

Proof. Let $t_m(x)$ and $a_j^{(i,m)}$ $(0 \le j \le (l+3)i)$ be as in Lemma 4, $a_0^{(0,m)} = 1$, $a_j^{(i,m)} = 0$ $(j > (l+3)i)$ and let

$$p_1(t_m(x)) = \sum_{0\le i\le h(l+3)} f_i^{(m)} x^i,$$

$$p_2(t_m(x)) = \sum_{0\le i\le l(l+3)} g_i^{(m)} x^i,$$

and

$$\lambda = \max\{|a_0|, |a_1|, \ldots, |a_h|, |b_0|, |b_1|, \ldots, |b_l|\}.$$

Then

$$f_j^{(m)} = \sum_{i=0}^{h} a_i a_j^{(i,m)}, \quad 0 \le j \le h(l+3) \tag{1}$$

and

$$g_j^{(m)} = \sum_{i=0}^{l} b_i a_j^{(i,m)}, \quad 0 \le j \le l(l+3). \tag{2}$$

Since $l^2+2l-1 > (l+3)(l-1)$, we have

$$a_{l^2+2l-1}^{(i,m)} = 0, \quad i \le l-1.$$

By (2) and Lemma 4 *(b)*, noting that $l^2+2l-1 = l+1+(l+2)(l-1)$, there exists a positive constant m_0, dependent only on $p_2(x)$, such that

$$g^{(m)}_{l^2+2l-1} = b_l a^{(l,m)}_{l^2+2l-1} < 0 \tag{3}$$

for all integers $m \geq m_0$. Since $h > l$, it follows from Lemma 4 *(b)* and Lemma 4 *(c)* that there exists a positive constant $m_1^{'}$, dependent only on h, such that

$$a_j^{(h,m)} \asymp m^h$$

and

$$a_j^{(i,m)} = O(m^{h-1}) \quad (i \leq h-1)$$

for all integers $m \geq m_1^{'}$. So there exists a positive constant $m_1^{''}$, dependent only on $p_1(x)$, such that

$$f_j^{(m)} > 0, \quad 0 \leq j \leq h(l+3)$$

for all integers $m \geq m_1^{''}$. Thus, by Lemma 4 *(a)* and (1), there exists a positive constant m_1, dependent only on $p_1(x)$ and $p_2(x)$ with $m_1 \geq l$, such that

$$0 < f_j^{(m)} \leq \lambda \sum_{0\leq i\leq h} (4m+l)^i \leq 2\lambda(5m)^h, \quad 0 \leq j \leq h(l+3) \tag{4}$$

and

$$|g_j^{(m)}| \leq \lambda \sum_{0\leq i\leq l} (4m+l)^i \leq 2\lambda(5m)^l, \quad 0 \leq j \leq l(l+3) \tag{5}$$

for all integers $m \geq m_1$. By Lemma 4 *(b)* and (2), there exists a positive constant m_2, dependent only on $p_2(x)$, such that $g_0^{(m)} > 0$ and $g_1^{(m)} > 0$ for all integers $m \geq m_2$. For all integers m with $m \geq m_0$, by (3), at least one coefficient of $p_2(t_m(x))$ is negative. For $m \geq \max\{m_0, m_2\}$, let j be the least positive integer with $g_j^{(m)} < 0$. Then $2 \leq j \leq l^2+2l-1$. If $m \geq \max\{m_0, m_1\}$ and $q^{k-2} > (2\lambda(5m)^l)^2$, then, by Lemma 1 and (5), we have

$$\begin{aligned}
& s_q(p_2(t_m(q^k))) && (6)\\
= \;& s_q(g_0^{(m)} + g_1^{(m)}q^k + g_2^{(m)}q^{2k} + \cdots + g^{(m)}_{l(l+3)}q^{kl(l+3)})\\
= \;& s_q(g_0^{(m)}) + s_q(g_1^{(m)} + g_2^{(m)}q^k + \cdots + g^{(m)}_{l(l+3)}q^{kl(l+3)-k})\\
\geq \;& s_q(g_1^{(m)} + g_2^{(m)}q^k + \cdots + g^{(m)}_{l(l+3)}q^{kl(l+3)-k})\\
\geq \;& \cdots\\
\geq \;& s_q(g_j^{(m)} + g_{j+1}^{(m)}q^k + \cdots + g^{(m)}_{l(l+3)}q^{kl(l+3)-jk})\\
\geq \;& (q-1)k - s_q(-g_j^{(m)} - 1)\\
\geq \;& (q-1)k - (q-1)(\log_q(-g_j^{(m)} - 1) + 1)\\
\geq \;& (q-1)k - (q-1)(\log_q(2\lambda(5m)^l) + 1)\\
> \;& \frac{1}{2}(q-1)k.
\end{aligned}$$

If $q^k > 2m$, then, by the definition of $t_m(x)$, we have

$$mq^{(l+3)k} < t_m(q^k) < 2mq^{(l+3)k} < q^{(l+4)k}. \tag{7}$$

If $q^k > 2\lambda(5m)^l$ and $m \geq m_1$, then, by (4) and Lemma 1, we have

$$\begin{aligned} & s_q(p_1(t_m(q^k))) \quad (8) \\ = \; & s_q(f_0^{(m)} + f_1^{(m)} q^k + f_2^{(m)} q^{2k} + \cdots + f_{h(l+3)}^{(m)} q^{kh(l+3)}) \\ = \; & s_q(f_0^{(m)}) + s_q(f_1^{(m)} + f_2^{(m)} q^k + \cdots + f_{h(l+3)}^{(m)} q^{kh(l+3)-k}) \\ = \; & \cdots \\ = \; & s_q(f_0^{(m)}) + s_q(f_1^{(m)}) + \cdots + s_q(f_{h(l+3)}^{(m)}) \\ \leq \; & (h(l+3)+1)(q-1)(1+\log_q(2\lambda(5m)^h)). \end{aligned}$$

Let $m_3 = \max\{m_0, m_1, m_2, l\}$. For any integers m and k with $m \geq m_3$ and $k \geq [2\log_q(2\lambda(5m)^l) + 2]$, by (6) and (8), we have

$$\frac{s_q(p_1(n))}{s_q(p_2(n))} \leq \frac{2(h(l+3)+1)(1+\log_q(2\lambda(5m)^h))}{k}, \tag{9}$$

where $n = t_m(q^k)$.

Without loss of generality, we can assume that $0 < \varepsilon \leq 1$. Let m be an integer with $m \geq m_3$,

$$k(m) = \left[\frac{2(h(l+3)+1)(1+\log_q(2\lambda(5m)^h)))}{\varepsilon}\right] + 1,$$

and $n(m) = t_m(q^{k(m)})$. Then $k(m) \geq [2\log_q(2\lambda(5m)^l) + 2]$. By (9), we have

$$\frac{s_q(p_1(n(m)))}{s_q(p_2(n(m)))} < \varepsilon. \tag{10}$$

Now we prove that all $n(m)$ ($m \geq m_3$) are distinct. Suppose that $m'' > m' \geq m_3$. Then $k(m'') \geq k(m')$. If $k(m'') = k(m')$, then

$$n(m'') = n(m') + (m'' - m')(1 + q^{k(m')} + q^{(l+2)k(m')} + q^{(l+3)k(m')}) > n(m').$$

If $k(m'') > k(m')$, then

$$\frac{n(m'')}{n(m')} \geq \frac{m'' q^{(l+3)k(m'')}}{2m' q^{(l+3)k(m')}} > \frac{q^{(l+3)(k(m'')-k(m'))}}{2} > 1.$$

By the definitions of $t_m(x)$ and $k(m)$, we have

$$\begin{aligned} n(m) &= t_m(q^{k(m)}) < 2mq^{(l+3)k(m)} \\ &\le 2mq^{(l+3)(2(h(l+3)+1)\varepsilon^{-1}+1)}q^{2(l+3)(h(l+3)+1)\varepsilon^{-1}\log_q(2\lambda(5m)^h)} \\ &= 2mq^{(l+3)(2(h(l+3)+1)\varepsilon^{-1}+1)}(2\lambda(5m)^h)^{2(l+3)(h(l+3)+1)\varepsilon^{-1}} \\ &< mq^{3(l+3)(h(l+3)+1)\varepsilon^{-1}}(10\lambda m)^{2h(l+3)(h(l+3)+1)\varepsilon^{-1}} \\ &< (10\lambda qm)^{2h(l+3)(h(l+3)+1)\varepsilon^{-1}+1}. \end{aligned}$$

Hence, if

$$m_3 \le m \le C_1^{'} N^{\alpha}, \tag{11}$$

then $n(m) \le N$, where $C_1^{'} = (10\lambda q)^{-1}$, and

$$\alpha = \varepsilon(2h(l+3)(h(l+3)+1)+\varepsilon)^{-1}.$$

Since m_3 is positive constant dependent only on $p_1(x)$ and $p_2(x)$, by (11), there exists a positive constant C_1, dependent only on ε, q, $p_1(x)$ and $p_2(x)$ such that

$$\begin{aligned} & \left|\left\{n \le N : \frac{s_q(p_1(n))}{s_q(p_2(n))} < \varepsilon\right\}\right| \\ \ge & \left|\left\{m : m_3 \le m \le C_1^{'} N^{\alpha}\right\}\right| \\ \ge & C_1^{'} N^{\alpha} - m_3 \\ > & C_1 N^{\alpha} \end{aligned}$$

for all sufficiently large integers N. This completes the proof of Theorem 1. □

4. Proof of Theorem 2

Proof. We follow the proof of Hare, Laishram and Stoll [10]. Let b be a positive integer, $p_1(x+b) = \sum_{i=0}^{h} u_i x^i$ and $p_2(x+b) = \sum_{i=0}^{l} v_i x^i$. Then all

$$u_i = \sum_{0 \le k \le h-i} a_{k+i} \binom{k+i}{i} b^k, \quad 0 \le i \le h$$

and

$$v_i = \sum_{0 \le k \le l-i} b_{k+i} \binom{k+i}{i} b^k, \quad 0 \le i \le l$$

are positive integers for all sufficiently large integers b. Without loss of generality, we may assume that all the coefficients of $p_1(x)$ and $p_2(x)$ are positive integers. Let

$\lambda = \max\{a_0, a_1, \dots, a_h, b_0, b_1, \dots, b_l\}$. Let M be a prime, $d = l$ and $y_1, y_2, \dots, y_{M+1}$ be as in Lemma 3. Let

$$T_M(x) = \sum_{i=1}^{M+1} x^{y_i},$$

and

$$p_2(T_M(x)) = \sum_{0\le j\le lM^l} q_j^{(M)} x^j.$$

Then

$$\begin{aligned} p_2(T_M(x)) &= \sum_{0\le i\le l} b_i (x^{y_1} + x^{y_2} + \cdots + x^{y_{M+1}})^i \qquad (12)\\ &= \sum_{0\le i\le l} b_i \sum_{h_1+h_2+\cdots+h_{M+1}=i} \frac{i!}{h_1!h_2!\cdots h_{M+1}!} x^{h_1y_1+h_2y_2+\cdots+h_{M+1}y_{M+1}}. \end{aligned}$$

By Lemma 3, for any fixed integer i with $0 \le i \le l$, we have all sums

$$h_1y_1 + h_2y_2 + \cdots + h_{M+1}y_{M+1}$$

with

$$h_1 + h_2 + \cdots + h_{M+1} = i, \quad h_j \ge 0, \quad 1 \le j \le M+1$$

are distinct. Then for any nonnegative integer $i \le lM^l$, we have

$$0 < q_i^{(M)} \le \sum_{0\le j\le l} b_j j! \le \lambda(l+1)!. \qquad (13)$$

By (12), we have

$$\begin{aligned} &\left|\left\{0 \le j \le lM^l : q_j^{(M)} > 0\right\}\right| \qquad (14)\\ \le &\sum_{0\le i\le l} \sum_{h_1+h_2+\cdots+h_{M+1}=i} 1\\ \le &\sum_{0\le i\le l} \binom{M+i}{M} = \binom{M+l+1}{M+1}. \end{aligned}$$

Let $n = T_M(q^k)$ and $k_0 = [\log_q(\lambda(l+1)!)] + 1$. Then for any integer $k \ge k_0$, we have $q^k > \lambda(l+1)!$. Since all coefficients of

$$x^{h_1y_1+h_2y_2+\cdots+h_{M+1}y_{M+1}}$$

with

$$h_1 + h_2 + \cdots + h_{M+1} = l, \quad h_j \ge 0, \quad 1 \le j \le M+1$$

are positive integers and all sums

$$h_1y_1 + h_2y_2 + \cdots + h_{M+1}y_{M+1}$$

with

$$h_1 + h_2 + \cdots + h_{M+1} = l, \quad h_j \geq 0, \quad 1 \leq j \leq M+1$$

are distinct, by Lemma 1, we have

$$s_q(p_2(n)) \geq \sum_{h_1+h_2+\cdots+h_{M+1}=l} 1 = \binom{M+l}{M}. \tag{15}$$

By (13) and Lemma 2, for any nonnegative integer $i \leq lM^l$, we have

$$s_q\left(q_i^{(M)}\right) \leq (q-1)(1+\log_q \lambda(l+1)!). \tag{16}$$

By (14), (16), and Lemma 1, noting that $q^k > \lambda(l+1)!$, we have

$$\begin{aligned} s_q(p_2(n)) &= \sum_{\substack{0\leq j\leq lM^l \\ q_j^{(M)}>0}} s_q\left(q_j^{(M)}\right) \\ &\leq \sum_{\substack{0\leq j\leq lM^l \\ q_j^{(M)}>0}} (q-1)\left(\log q_j^{(M)} + 1\right) \\ &\leq (q-1)(1+\log_q(\lambda(l+1)!))\binom{M+l+1}{M+1}. \end{aligned}$$

As a similar argument for $p_1(x)$, we have

$$s_q(p_1(n)) \leq (q-1)(1+\log_q(\lambda(h+1)!))\binom{M+h+1}{M+1}. \tag{17}$$

By (15) and (17), we have

$$\begin{aligned} \frac{s_q(p_1(n))}{s_q(p_2(n))} &\leq \frac{(q-1)(1+\log_q(\lambda(h+1)!))\binom{M+h+1}{M+1}}{\binom{M+l}{M}} \\ &\leq \frac{(q-1)l!(1+\log_q(\lambda(h+1)!))}{h!(M+1)^{l-h}}, \end{aligned} \tag{18}$$

where $n = T_M(q^k)$.

For any $\varepsilon > 0$ and for any integer $k \geq k_0$, by (18), there exists a prime M_0 such that

$$\frac{s_q(p_1(n))}{s_q(p_2(n))} < \varepsilon$$

for all integers $n = T_{M_0}(q^k)$. By $n = T_{M_0}(q^k) < q^{k(M_0^l+1)}$, we see that, if

$$k \leq \frac{\log N}{(M_0^l+1)\log q},$$

then $n \le N$. Since $T_{M_0}(q^k)$ $(k \ge k_0)$ are distinct, there exists a positive constant C_2, dependent only on ε, q, $p_1(x)$ and $p_2(x)$ such that

$$\begin{aligned} & \left|\left\{n \le N : \frac{s_q(p_1(n))}{s_q(p_2(n))} < \varepsilon\right\}\right| \\ \ge & \left|\left\{k : k_0 \le k \le \frac{\log N}{(M_0^l+1)\log q}\right\}\right| \\ \ge & C_2 \log N \end{aligned}$$

for all sufficiently large integers N. This completes the proof of Theorem 2. □

Acknowledgements The author would like to thank the anonymous referee for helpful comments and suggestions.

References

[1] R. Bellman, H.N. Shapiro, *On a problem in additive number theory.* Ann. of Math. **49** (1948), 333–340.

[2] A. Bérczes, F. Luca, *On the sum of digits of numerators of Bernoulli numbers.* Canad. Math. Bull. **56** (2013), 723–728.

[3] R.C. Bose, S. Chowla, *Theorems in the additive theory of numbers.* Comm. Math. Helv. **37** (1962/63), 141–147.

[4] L.E. Bush, *An asymptotic formula for the average sum of the digits of integers.* Amer. Math. Monthly **47** (1940), 154–156.

[5] C. Dartyge, G. Tenenbaum, *Congruences de sommes de chiffres de valeurs polynomiales.* Bull. London Math. Soc. **38** (2006), 61–69.

[6] H. Delange, *Sur la fonction sommatoire de la fonction "somme des chiffres".* Enseign. Math. **21** (1975), 31–47.

[7] M. Drmota, J. Rivat, *The sum-of-digits function of squares.* J. London Math. Soc. **72** (2005), 273–292.

[8] H. Halberstam, K.F. Roth, Sequences, second edition, Springer-Verlag, New York-berlin, 1983.

[9] B. Lindström, *On the binary digits of a power.* J. Number Theory **65** (1997), 321–324.

[10] K.G. Hare, S. Laishram and T. Stoll, *Stolarsky's conjecture and the sum of digits of polynomial values.* Proc. Amer. Math. Soc. **139** (2011), 39–49.

[11] K.G. Hare, S. Laishram and T. Stoll, *The sum of digits of n and n^2.* Int. J. Number Theory **7** (2011), 1737–1752.

[12] G. Larcher, H. Zellinger, *on irregularities of distribution of weighted sums-of-digits.* Discrete. Math. **311** (2011), 109–123.

[13] M. Madritsch and T. Stoll, *On simultaneous digital expansions of polynomial values.* Acta Math. Hungar. **143** (2014), 192–200.

[14] C. Mauduit, J. Rivat, *Sur un probléme de Gelfond: la somme des chiffres des nombres premiers.* Annals of Math. **171** (2010), 1591–1646.

[15] C. Mauduit, J. Rivat, *La somme des chiffres des carrés.* Acta Math. **203** (2009), 107–148.

[16] G. Melfi, *On simultaneous binary expansions of* n *and* n^2. J. Number Theory **111** (2005), 248–256.

[17] M. Peter, *The summatory function of the sum-of-digits function on polynomial sequences.* Acta Arith. **104** (2002), 85-96.

[18] F. Pillichshammer, *Uniform distribution of sequences connected with the weighted sum-of-digits function.* Unif. Distrib. Theory **2** (2007), 1–10.

[19] T. Rivoal, *On the bits counting function of real numbers.* J. Aust. Math. Soc. **85** (2008), 95–111.

[20] K.B. Stolarsky, *The binary digits of a power.* Proc. Amer. Math. Soc. **71** (1978), 1–5.

[21] D.C. Terr, *On the sum of digits of Fibonacci numbers.* Fibonacci Quart. **34** (1996), 349–355.