ROBIN'S INEQUALITY FOR 11-FREE INTEGERS

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Abstract

Let $\sigma(n)$ denote the sum of divisors function, and let γ be Euler's constant. We prove that if there is an $n \geq 5041$ for which $\sigma(n) \geq e^{\gamma} n \log\log n$, then n must be divisible by the eleventh power of some prime.

1. Introduction

Let $\sigma(n)$ denote the sum of divisors of n. Robin [8] proved that the Riemann hypothesis is equivalent to the inequality

$$\sigma(n) < e^{\gamma} n \log\log n, \quad (n \geq 5041), \tag{1}$$

which we refer to hereafter as *Robin's inequality*. We say that a number is t-free if it is not divisible by the tth power of any prime. Choie et al. [2] showed that (1) is true for all 5-free integers; Solé and Planat [10] showed that (1) is true for all 7-free integers. Therefore if there is some $n \geq 5041$ for which $\sigma(n) \geq e^{\gamma} n \log\log n$, then n must be divisible by the seventh power of some prime. The point of this note is to prove

Theorem 1. *If there is some $n \geq 5041$ for which $\sigma(n) \geq e^{\gamma} n \log\log n$, then n must be divisible by the eleventh power of some prime.*

It is easy to check that the only two positive integers $n \leq 5040$ that are divisible by an eleventh power of a prime, viz. 2^{11} and 2^{12}, both satisfy Robin's inequality. In other words

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Corollary 1. *The Riemann hypothesis is equivalent to Robin's inequality being satisfied by all integers greater than 2 that are divisible by the 11th power of at least one prime.*

Robin [8, Prop. 1, p. 192] showed that if Robin's inequality is true on consecutive *colossally abundant numbers* n_1 and n_2 then it is true for all $n \in [n_1, n_2]$. We say that n is colossally abundant if there exists a positive ϵ for which $\sigma(n)/n^{1+\epsilon} \geq \sigma(k)/k^{1+\epsilon}$ for all $k > 1$. Briggs [1] proved Robin's inequality for all colossally abundant numbers n with $5041 \leq n \leq 10^{10^{10}}$. This shows that Robin's inequality is true for all integers n with $5041 \leq n \leq 10^{10^{10}}$.

Solé and Planat prove their results using primorials. Let the nth primorial be defined as $N_n = \prod_{i=1}^{n} p_i$, where p_n denotes the nth prime with $p_1 = 2$. For an integer $t \geq 2$ define

$$\Psi_t(n) := n \prod_{p|n} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{t-1}}\right), \quad R_t(n) := \frac{\Psi_t(n)}{n \log\log n}.$$

Solé and Planat note that on a t-free integer n one has $\sigma(n) \leq \Psi_t(n)$. Using their method, for a fixed value of t we first find an integer $n_1(t)$ such that $R_t(N_{n_1(t)}) < e^{\gamma}$ and $N_{n_1(t)} < 10^{10^{10}}$. Solé and Planat showed [10, Cor. 9] that for all $N > N_{n_1(t)}$ we have $R_t(N) < e^{\gamma}$. It follows from the remark after Corollary 1 that, for our fixed value of t, Robin's inequality is true for all t-free integers $n \geq 5041$.

The main idea of this article is to use explicit estimates on sums over primes to bound the primorials. This makes for an easy computation, and one which we use to verify Theorem 1.

2. Bounds on Primorials

We proceed to estimate the function $R_t(n)$. For $t \geq 2$ we have

$$R_t(N_n) = \frac{1}{\log\log N_n} \prod_{k=1}^{n} \frac{1 - p_k^{-t}}{1 - p_k^{-1}} = \frac{\prod_{p > p_n}(1 - p^{-t})^{-1}}{\zeta(t) \log\log N_n} \prod_{p \leq p_n} (1 - p^{-1})^{-1}. \quad (2)$$

We estimate the first product on the right-side of (2) using Lemma 6 of Solé and Planat, namely,

$$\prod_{p \geq p_n} (1 - p^{-t})^{-1} \leq \exp(2/p_n), \quad (3)$$

for all $n \geq 2$. Although this could be improved, such an improvement has negligible influence on the final result.

To estimate the second product on the right-side of (2) we use the following result

$$\prod_{p \leq x} (1 - p^{-1})^{-1} \leq e^{\gamma} \log x \left(1 - \frac{1}{5 \log^2 x}\right)^{-1}, \quad (x \geq 2973), \quad (4)$$

given by Dusart [3, Thm 4]. This improves on a result given by Rosser and Schoenfeld [9, Thm 8 Cor. 1].

Since we wish to apply (3) and (4) with $x = p_n$ we need an explicit bound on the nth prime. It is possible to proceed without an explicit bound, though this increases greatly the computation time when n is large. In fact, if one considers the function

$$f(x) = \frac{\exp(2/x)\log x}{(1 - \frac{1}{5\log^2 x})}, \tag{5}$$

which is increasing for all $x \geq 5$, one sees that it is sufficient to consider only an explicit upper bound on p_n. We have

$$p_n \leq b_1(n) := n(\log n + \log\log n - \tfrac{1}{2}), \quad (n \geq 20), \tag{6}$$

and

$$p_n \leq b_2(n) := n(\log n + \log\log n - 0.9484), \quad (n \geq 39017), \tag{7}$$

due respectively to Rosser and Schoenfeld [9, (3.11)] and Dusart [4, §4].

We now bound the factor $\log\log N_n$ in (2). We make use of the function $\theta(x) = \sum_{p\leq x}\log p$. Since $\log N_n = \sum_{i=1}^{n}\log p_i = \theta(p_n)$ we can derive bounds on N_n using bounds on $\theta(p_n)$. We present these results in the following

Lemma 1. *For $k \geq 198$,*

$$k\left(\log k + \log\log k - 1 + \frac{\log\log k - 2.1454}{\log k}\right) \leq \log N_k$$
$$\leq k\left(\log k + \log\log k - 1 + \frac{\log\log k - 2}{\log k}\right). \tag{8}$$

Proof. The left inequality follows from Robin [7, Thm 7] and is valid for all $k \geq 3$; the right inequality follows from Massias and Robin [6, Thm B(v)] and is valid for all $k \geq 198$. □

We remark that the constant '2' in the right inequality in (8) cannot be improved. One could reduce the '2.1454' that appears in the left inequality at the expense of taking a much larger k. As shown in §3, this has very little influence on our problem.

We now use (3)-(7) and Lemma 1 to estimate the right side of (2). We have

$$R_t(N_n) \leq \begin{cases} e^{\gamma} g_1(t,n), & (430 \leq n \leq 39016), \\ e^{\gamma} g_2(t,n), & (n \geq 39017), \end{cases} \tag{9}$$

where

$$g_i(t,n) = \frac{f(b_i(n))}{\zeta(t)\log\left\{n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2.1454}{\log n}\right)\right\}},$$

for $i = 1, 2$. We require $n \geq 430$ in (9) to ensure that the conditions in (4) and Lemma 1 are met since $p_{429} = 2971$ and $p_{430} = 2999$. Following Solé and Planat, define $n_1(t)$ to be the least value of $n \geq 430$ for which $g_i(t, n) < 1$. Since $g_i(t, n)$ is decreasing in n we will have $g_i(t, n) < 1$ for all $n \geq n_1(t)$. For such an $n_1(t)$ we consider the size of the associated $N_{n_1(t)}$. We summarise the results in the following table. Since we have given upper bounds on $N_{n_1(t)}$ the decimals in Table 1 have been rounded up. We do not give an exact value of $N_{n_1}(t)$ for $t = 11, 12$ owing to the computational complexity of calculating the nth primorial exactly.

Table 1: Values of $n_1(t)$ and upper bounds on $N_{n_1(t)}$

t	$n_1(t)$	$N_{n_1(t)}$	Bound on $N_{n_1}(t)$ using Lemma 1
6	430	3.3×10^{1273}	1.4×10^{1276}
7	1847	3.3×10^{6836}	2.7×10^{6851}
8	39017	4.9×10^{202520}	2.3×10^{202725}
9	39017	4.9×10^{202520}	2.3×10^{202725}
10	234372	1.2×10^{1416098}	1.8×10^{1416984}
11	48304724	—	$2.8 \times 10^{411504586}$
12	162914433505	—	$> 10^{10^{12}}$

Given Briggs' result [1], to prove a statement such as 'Robin's inequality holds for all t-free integers' we need to show that $N_{n_1(t)} \leq 10^{10^{10}}$. The last entry in Table 1 shows that, at present, it is impossible to consider $t = 12$ without a new idea.

3. Conclusion

We discuss briefly the possibility of proving that Robin's inequality is satisfied by all 12-free integers. Dusart [5] (unpublished) has considered some finessed versions of (4) and (8), namely

$$\prod_{p \leq x} (1 - p^{-1})^{-1} \leq e^{\gamma} \log x \left(1 + \frac{1}{5 \log^2 x}\right), \quad (x \geq 2973),$$

and

$$k \left(\log k + \log\log k - 1 + \frac{\log\log k - 2.04}{\log k}\right) \leq \theta(p_k), \quad (p^k \geq 10^{15}).$$

These results appear respectively as Theorem 6.12 and Proposition 6.2 in [5]. Even with these improvements one still has $N_{n_1(12)} > 10^{10^{12}}$. Without injecting new ideas into the argument one would have to increase the range of Briggs' computations beyond $10^{10^{10}}$.

To extend the computation one need only check those numbers that are colossally abundant and are divisible by the 11th power of some prime. Presumably, this is a very thin set of numbers. Checking only these numbers may precipitate an extension of Briggs' computations and hence the possibility of extending the the results in this paper to $t = 12$.

References

[1] K. Briggs. Abundant numbers and the Riemann hypothesis. *Experiment. Math.*, 15(2):251–256, 2006.

[2] Y. Choie, N. Lichiardopol, P. Moree, and P. Solé. On Robin's criterion for the Riemann hypothesis. *J. Théor. Nombres Bordeaux*, 19(2):357–372, 2007.

[3] P. Dusart. Inégalités explicites pour $\psi(X)$, $\theta(X)$, $\pi(X)$ et les nombres premiers. *C. R. Math. Acad. Sci. Soc. R. Can.*, 21(2):53–59, 1999.

[4] P. Dusart. The kth prime is greater than $k(\ln k + \ln\ln k - 1)$ for $k \geq 2$. *Math. Comp.*, 68(225):411–415, 1999.

[5] P. Dusart. Estimates of some functions over primes without R.H. arXiv:1002.0442v1, 2010.

[6] J.-P. Massias and G. Robin. Bornes effectives pour certaines fonctions concernant les nombres premiers. *J. Théor. Nombres Bordeaux*, 8(1):215–242, 1996.

[7] G. Robin. Estimation de la fonction de Tchebychef θ sur le k-ième nombre premier et grandes valeurs de la fonction $\omega(n)$ nombre de diviseurs premiers de n. *Acta Arith.*, 42(4):367–389, 1983.

[8] G. Robin. Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. *J. Math. Pures Appl. (9)*, 63(2):187–213, 1984.

[9] J. B. Rosser and L. Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.*, 6:64–94, 1962.

[10] P. Solé and M. Planat. The Robin inequality for 7-free integers. *Integers*, 12(2):301–309, 2012.