SHORT EFFECTIVE INTERVALS CONTAINING PRIMES

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Abstract

We prove that if x is large enough, namely $x \geq x_0$, then there exists a prime between $x(1-\Delta^{-1})$ and x, where Δ is an effective constant computed in terms of x_0.

1. Introduction

In this article, we address the problem of finding short intervals containing primes. In 1845 Bertrand conjectured that for any integer $n > 3$, there always exists at least one prime number p with $n < p < 2n-2$. This was proven by Chebyshev in 1850, using elementary methods. Since then other intervals of the form $(kn, (k+1)n)$ have been investigated. We refer the reader to [1] for $k = 2$ and to [12] for $k = 3$. Assuming that x is arbitrarily large, the length of intervals containing primes can be drastically reduced. To date, the record is held by Baker, Harman, and Pintz [2] as they prove that there is at least one prime between x and $x + x^{0.525+\varepsilon}$. This is an impressive result since under the Riemann Hypothesis the exponent 0.525 can only be reduced to 0.5. On the other hand, maximal gaps for the first primes have been checked numerically up to $4 \cdot 10^{18}$ by Oliveira e Silva et al. [14]. In particular, they find that the largest prime gap before this limit is 1 476 and occurs at $1\,425\,172\,824\,437\,699\,411 = e^{41.8008\cdots}$. The purpose of this article is to obtain an effective result of the form: for all $x \geq x_0$, there exists $\Delta > 0$ such that the interval $(x(1-\Delta^{-1}), x)$ contains at least one prime. In 1976 Schoenfeld's Theorem 12 of [18] gave this for $x_0 = 2\,010\,881.1$ and $\Delta = 16\,598$. In 2003 Ramaré and Saouter improved on Schoenfeld's method by using a smoothing argument. They also extended the computations to many other values for x_0 ([16, Theorem 2 and

Table 1]). In [9], the first author generalized this theorem to primes in arithmetic progression and applied this to Waring's seven cubes problem. Here, our theorem improves [16] by making use of a new explicit zero-density for the zeros of the Riemann zeta function:

Theorem 1.1. *Let $x_0 \geq 4 \cdot 10^{18}$ be a fixed constant and let $x > x_0$. Then there exists at least one prime p such that $(1 - \Delta^{-1})x < p < x$, where Δ is a constant depending on x_0 and is given in Table 2.*

In Section 2, we prove a general theorem (Theorem 2.7) which provides conditions for intervals of the form $((1 - \Delta^{-1})x, x)$ to contain a prime. In Section 3, we apply this theorem to compute explicit values for Δ.

We present an example of the numerical improvement this theorem allows, for instance when $x_0 = e^{59}$, Ramaré and Saouter [16] found that the interval gap was given by $\Delta = 209\,257\,759$. In [5, page 74], Helfgott mentioned an improvement of Ramaré using Platt's latest verification of the Riemann Hypothesis [15]: $\Delta = 307\,779\,681$. Our Theorem 1.1 leads to $\Delta = 1\ 946\ 282\ 821$.

We now mention an application to the verification of the Ternary Goldbach conjecture. This conjecture was known to be true for sufficiently large integers (by Vinogradov), and Liu and Wang [11] prove it for all integers $n \geq e^{3100}$. On the other hand, the conjecture was verified for the first values of n. In [16, Corollary 1], Ramaré and Saouter verified it for $n \leq 1.132 \cdot 10^{22}$. Very recently, Oliveira e Silva et. al. [14, Theorem 2.1] extended this limit to $n \leq 8.370 \cdot 10^{26}$. In [5, Proposition A.1.], Helfgott applied the above result on short intervals containing primes ($\Delta = 307\,779\,681$) and found $n \leq 1.231 \cdot 10^{27}$. This allowed him to complete his proof [5, 6] of the Ternary Goldbach conjecture for the remaining integers. Here our main theorem gives:

Corollary 1.2. *Every odd number larger than* 5 *and smaller than*

$$1\ 966\ 196\ 911 \times 4 \cdot 10^{18} = 7.864 \ldots \cdot 10^{27}$$

is the sum of at most three primes.

As of today, Helfgott and Platt [7] have announced a verification up to $8.875 \cdot 10^{30}$.

2. Proof of Theorem 1.1

We recall the definition of the classical Chebyshev functions:

$$\theta(x) = \sum_{p \leq x} \log p, \quad \psi(x) = \sum_{n \leq x} \Lambda(n), \text{ with } \Lambda(n) = \begin{cases} 1 & \text{if } n = p^k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

For each x_0, we want to find the largest $\Delta > 0$ such that, for all $x > x_0$, there exists a prime between $x(1-\Delta^{-1})$ and x. This happens as soon as $\theta(x) - \theta(x(1-\Delta^{-1})) > 0$.

2.1. Introduction of Parameters

We list here the parameters we will be using throughout the proof.

$$
\begin{aligned}
&* \ m \text{ integer with } m \geq 2,\\
&* \ 0 \leq u \leq 0.0001, \ \ \delta = mu \text{ and } 0 \leq \delta \leq 0.0001,\\
&* \ 0 \leq a \leq 1/2,\\
&* \ \Delta = \left(1 - (1+\delta a)(1+\delta(1-a))^{-1} e^{-u}\right)^{-1},\\
&* \ X \geq X_0 \geq e^{38},\\
&* \ x = e^u X(1+\delta(1-a)) \geq x_0 = e^u X_0 (1+\delta(1-a)),\\
&* \ y = X(1+\delta a) = x\left(1-\Delta^{-1}\right).
\end{aligned}
\tag{1}
$$

2.2. Smoothing the Difference $\theta(x) - \theta(y)$

We follow here the smoothing argument of Ramaré and Saouter [16]. Let f be a positive function integrable on $(0,1)$. We denote

$$\|f\|_1 = \int_0^1 f(t)dt, \tag{2}$$

$$\nu(f,a) = \int_0^a f(t)dt + \int_{1-a}^1 f(t)dt, \tag{3}$$

$$\text{and } I_{\delta,u,X} = \frac{1}{\|f\|_1} \int_0^1 \left(\theta(e^u X(1+\delta t)) - \theta(X(1+\delta t))\right) f(t)dt. \tag{4}$$

Note that for all $a \leq t \leq 1-a$, $\theta(e^u X(1+\delta t)) - \theta(X(1+\delta t)) \leq \theta(x) - \theta(y)$. We integrate with the positive weight f and obtain:

$$\int_a^{1-a} \left(\theta(e^u X(1+\delta t)) - \theta(X(1+\delta t))\right) f(t)dt \leq (\theta(x) - \theta(y)) \int_a^{1-a} f(t)dt. \tag{5}$$

We extend the left integral to the interval $(0,1)$ and use a Brun-Titchmarsh inequality to control the primes on the extremities $(0,a)$ and $(1-a,1)$ of the interval (see [16, page 16, line -5] or [13, Theorem 2]):

$$\int_{t\in(0,a)\cup(1-a,1)} \left(\theta(e^u X(1+\delta t)) - \theta(X(1+\delta t))\right) f(t)dt$$
$$\leq 2(1+\delta)(e^u - 1)\frac{\log(e^u X)}{\log(X(e^u-1))}\nu(f,a)X. \tag{6}$$

Note that [16] uses the slightly larger bound

$$2.0004u\frac{\log X}{\log(uX)}\nu(f,a)X.$$

Combining (5) and (6) gives

$$I_{\delta,u,X} \le (\theta(x)-\theta(y))\,\frac{\int_a^{1-a} f(t)dt}{\|f\|_1} + 2(1+\delta)(e^u-1)\frac{\log(e^u X(1+\delta))}{\log(X(e^u-1))}\frac{\nu(f,a)}{\|f\|_1}X. \quad (7)$$

Thus $\theta(x)-\theta(y)>0$ when

$$I_{\delta,u,X} - 2(1+\delta)(e^u-1)\frac{\log(e^u X(1+\delta))}{\log(X(e^u-1))}\frac{\nu(f,a)}{\|f\|_1}X > 0. \quad (8)$$

It remains to establish a lower bound for $I_{\delta,u,X}$. To do so, we first approximate $\theta(x)$ with $\psi(x)$. This will allow us to translate our problem in terms of the zeros of the zeta function. We use approximations proven by Costa in [3, Theorem 5]:

Lemma 2.1. *Let $x \ge e^{38}$. Then*

$$0.999\sqrt{x} + \sqrt[3]{x} < \psi(x) - \theta(x) < 1.001\sqrt{x} + \sqrt[3]{x}. \quad (9)$$

We have that for all $0 < t < 1$,

$$\begin{aligned}&(\psi(e^u X(1+\delta t)) - \theta(e^u X(1+\delta t))) - (\psi(X(1+\delta t)) - \theta(X(1+\delta t)))\\ &< \sqrt{X}\sqrt{1+\delta}\left(1.001e^{u/2} - 0.999 + X^{-1/6}(1+\delta)^{-1/6}(e^{u/3}-1)\right) < \omega\sqrt{X}, \quad (10)\end{aligned}$$

where we can take, under our assumptions (1),

$$\omega = 2.05022 \cdot 10^{-3}. \quad (11)$$

We denote

$$J_{\delta,u,X} = \frac{1}{\|f\|_1}\int_0^1 (\psi(e^u X(1+\delta t)) - \psi(X(1+\delta t)))\,f(t)dt. \quad (12)$$

It follows from (10) that

$$I_{\delta,u,X} \ge J_{\delta,u,X} - \omega\sqrt{X}. \quad (13)$$

Note that [16] used older approximations from [18], which lead to $\omega = 0.0325$. To summarize, we want to find conditions on m, δ, u, a so that

$$J_{\delta,u,X} - \omega\sqrt{X} - 2(1+\delta)(e^u-1)\frac{\log(e^u X(1+\delta))}{\log(X(e^u-1))}\frac{\nu(f,a)}{\|f\|_1}X > 0. \quad (14)$$

We are now left with evaluating $J_{\delta,u,X}$, which we shall do by relating it to the zeros of the Riemann zeta function through an explicit formula.

2.3. An Explicit Inequality for $J_{\delta,u,X}$

Lemma 2.2. *[16, Lemma 4] Let $2 \le b \le c$, and let g be a continuously differentiable function on $[b,c]$. We have*

$$\int_b^c \psi(u)g(u)du = \int_b^c ug(u) - \sum_\varrho \int_b^c \frac{u^\varrho}{\varrho} g(u)du + \int_b^c \left(\log 2\pi - \frac{1}{2}\log(1-u^{-2})\right) g(u)du. \quad (15)$$

We apply this identity to $g(t) = f\left(\delta^{-1}\left(e^{-u}X^{-1}t - 1\right)\right)$, $b = e^u X$, $c = e^u X(1+\delta)$, and $g(t) = f\left(\delta^{-1}\left(X^{-1}t - 1\right)\right)$, $b = X$, $c = X(1+\delta)$, respectively. It follows that

$$J_{\delta,u,X} = \frac{(e^u - 1)X}{\|f\|_1}\int_0^1 (1+\delta t)f(t)dt - \frac{1}{\|f\|_1}\sum_\varrho \int_0^1 \frac{(e^{u\varrho}-1)X^\varrho (1+\delta t)^\varrho f(t)}{\varrho}dt - \frac{1}{2\|f\|_1}\int_0^1 \left(\log\left(1-(e^u X(1+\delta t))^{-2}\right) - \log\left(1-(X(1+\delta t))^{-2}\right)\right) f(t)dt.$$

Observe that the last term is at least $-\frac{u}{2X}$. We obtain

$$\frac{J_{\delta,u,X}}{(e^u-1)X} \ge \frac{\int_0^1 (1+\delta t)f(t)dt}{\|f\|_1} - \sum_\varrho \left| \frac{(e^{u\varrho}-1)}{(e^u-1)\varrho} \frac{\int_0^1 (1+\delta t)^\varrho f(t)dt}{\|f\|_1} \right| X^{\mathfrak{Re}\varrho - 1} - \frac{u}{2(e^u-1)X^2}. \quad (16)$$

We obtain some small savings by directly computing the first term, whereas [16, Equation (13)] uses the following bound in (16) instead:

$$\frac{\int_0^1 (1+\delta t)f(t)dt}{\|f\|_1} \ge \frac{u}{e^u - 1}.$$

Let s be a complex number. We let $G_{m,\delta,u}(s)$ be the summand

$$G_{m,\delta,u}(s) = \frac{(e^{us}-1)}{(e^u-1)s}\frac{\int_0^1 (1+\delta t)^s f(t)dt}{\|f\|_1}, \tag{17}$$

and we rewrite inequality (16) as

$$\frac{J_{\delta,u,X}}{(e^u-1)X} \geq G_{m,\delta,u}(1) - \sum_{\varrho} |G_{m,\delta,u}(\varrho)|\, X^{\mathfrak{Re}\varrho-1} - \frac{u}{2(e^u-1)}X^{-2}. \tag{18}$$

Since the right term increases with X, we can replace X with X_0 for $X \geq X_0$. Note that this is also the case for

$$\frac{\omega}{(e^u-1)\sqrt{X}} - 2(1+\delta)\frac{\log(e^u X(1+\delta))}{\log(X(e^u-1))}\frac{\nu(f,a)}{\|f\|_1}.$$

For simplicity we let

$$\Sigma = \Sigma_{m,\delta,u,X} = \sum_{\varrho} |G_{m,\delta,u}(\varrho)|\, X^{\mathfrak{Re}\varrho-1}. \tag{19}$$

The following Proposition gives a first inequality in terms of the zeros of the Riemann zeta function and conditions on m, u, δ, a (and thus Δ) so that $\theta(x) - \theta(x(1-\Delta^{-1})) > 0$:

Proposition 2.3. *Let $m, u, \delta, a, \Delta, X_0$ satisfy (1). If $X \geq X_0$ and*

$$G_{m,\delta,u}(1) - \Sigma_{m,\delta,u,X_0} - \frac{u}{2(e^u-1)}X_0^{-2} - \frac{\omega}{(e^u-1)}X_0^{-1/2} - \frac{2\nu(f,a)(1+\delta)}{\|f\|_1}\frac{\log(e^u X_0(1+\delta))}{\log(X_0(e^u-1))} > 0, \tag{20}$$

then there exists a prime number between $x(1-\Delta^{-1})$ and x.

We are now going to make this lemma more explicit by providing computable bounds for the sum over the zeros Σ_{m,δ,u,X_0}.

2.4. Evaluating $G_{m,\delta,u}$

Let f be an m-admissible function over $[0,1]$. We recall the properties it entitles according to the definition of [16]:

- f is an m-times differentiable function,
- $f^{(k)}(0) = f^{(k)}(1) = 0$ for $0 \leq k \leq m-1$,
- $f \geq 0$,
- f is not identically 0.

For $k = 0, \ldots, m$ and for $s = \sigma + i\tau$ a complex number with $\tau > 0$ and $0 \leq \sigma \leq 1$, we let

$$F_{k,m,\delta} = \frac{\int_0^1 (1+\delta t)^{1+k} |f^{(k)}(t)| dt}{\|f\|_1}. \tag{21}$$

We provide finer estimates than [16] for $G_{m,\delta,u}$. Observe that

$$\left|\frac{e^{us}-1}{s}\right| = \left|\int_1^u e^{xs} dx\right| \leq \int_1^u e^{x\sigma} dx = \frac{e^{u\sigma}-1}{\sigma}, \tag{22}$$

$$\left|\frac{e^{us}-1}{s}\right| \leq \frac{e^{u\sigma}+1}{\tau}, \tag{23}$$

$$\text{and } \left|\int_0^1 (1+\delta t)^s f(t) dt\right| \leq \frac{1}{\delta^k \tau^k} F_{k,m,\delta}. \tag{24}$$

We easily deduce bounds for $G_{m,\delta,u}(s)$ by combining (22) and (24) with respectively $k = 0$, $k = 1$, and $k = m$:

$$|G_{m,\delta,u}(s)| \leq F_{0,m,\delta} \frac{e^{u\sigma}-1}{(e^u-1)\sigma}, \tag{25}$$

$$|G_{m,\delta,u}(s)| \leq F_{1,m,\delta} \frac{e^{u\sigma}-1}{(e^u-1)\sigma\delta\tau}, \tag{26}$$

$$\text{and } |G_{m,\delta,u}(s)| \leq F_{m,m,\delta} \frac{e^{u\sigma}-1}{(e^u-1)\sigma\delta^m\tau^m}. \tag{27}$$

Lastly by combining (23) and (24) with $k = m$ we obtain

$$|G_{m,\delta,u}(s)| \leq F_{m,m,\delta} \frac{e^{u\sigma}+1}{(e^u-1)\delta^m\tau^{m+1}}. \tag{28}$$

2.5. Zeros of the Riemann Zeta Function

We denote each zero of the zeta funtion by $\varrho = \beta + i\gamma$, the number of zeros in the rectangle $0 < \beta < 1$, $0 < \gamma < T$, by $N(T)$, and the number of those zeros in the rectangle $\sigma_0 < \beta < 1$, $0 < \gamma < T$, by $N(\sigma_0, T)$. We assume that we have the following information.

Theorem 2.4.

1. *A numerical verification of the Riemann Hypothesis:*
   There exists $H > 2$ such that if $\zeta(\beta + i\gamma) = 0$ at $0 \leq \beta \leq 1$ and $0 \leq \gamma \leq H$, then $\beta = 1/2$.

2. *A direct computation of some finite sums over the first zeros:*

Let $0 < T_0 < H$ and $S_0 > 0$ satisfy

$$\sum_{\substack{0<\gamma\le T_0 \\ \beta=1/2}} 1 \le N_0 = N(T_0), \tag{29}$$

$$\text{and} \sum_{\substack{0<\gamma\le T_0 \\ \beta=1/2}} \frac{1}{\gamma} \le S_0. \tag{30}$$

3. *A zero-free region:*
   There exists $R_0 > 0$, a constant, such that $\zeta(\sigma + it)$ does not vanish in the region

$$\sigma \ge 1 - \frac{1}{R_0 \log |t|} \quad \text{and} \quad |t| \ge 2. \tag{31}$$

4. *An estimate for $N(T)$:*
   There exist positive constants a_1, a_2, a_3 such that, for all $T \ge 2$,

$$\begin{aligned} &|N(T) - P(T)| \le R(T), \text{ where} \\ &P(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8},\ R(T) = a_1 \log T + a_2 \log\log T + a_3. \end{aligned} \tag{32}$$

5. *An upper bound for $N(\sigma_0, T)$:*
   Let $3/5 < \sigma_0 < 1$. Then there exist constants c_1, c_2, c_3 such that, for all $T \ge H$,

$$N(\sigma_0, T) \le c_1 T + c_2 \log T + c_3. \tag{33}$$

Note that [16] did not use any information of the type (30), (31), or (33). Instead they used (29), the fact that all nontrivial zeros satisfied $\beta < 1$, and the classical bound (32) for $N(T)$ as given in [17, Theorem 19]. Our improvement will mainly come from using a new zero-density of the form of (33).

2.6. Evaluating the Sum Over the Zeros $\boldsymbol{\Sigma_{m,\delta,u,X_0}}$

We assume Theorem 2.4. We split the sum Σ_{m,δ,u,X_0} vertically at heights $\gamma = 0$ (so as to use the symmetry with respect to the x-axis) and consider

$$\tilde{G}_{m,\delta,u}(\beta + i\gamma) = |G_{m,\delta,u}(\beta + i\gamma)| + |G_{m,\delta,u}(\beta - i\gamma)|.$$

We then split at $\gamma = H$ (so as to take advantage of the fact that all zeros below this horizontal line satisfy $\beta = 1/2$), and again at $\gamma = T_0$ and $\gamma = T_1$ (where T_1 will be chosen between T_0 and H), and consider:

$$\Sigma_0 = \sum_{0<\gamma\le T_0} \tilde{G}_{m,\delta,u}(1/2+i\gamma)X_0^{-1/2}, \tag{34}$$

$$\Sigma_1 = \sum_{T_0<\gamma\le T_1} \tilde{G}_{m,\delta,u}(1/2+i\gamma)X_0^{-1/2}, \tag{35}$$

$$\text{and} \quad \Sigma_2 = \sum_{T_1<\gamma\le H} \tilde{G}_{m,\delta,u}(1/2+i\gamma)X_0^{-1/2}. \tag{36}$$

For the remaining zeros (those with $\gamma > H$), we make use of the symmetry with respect to the critical line, and we split at $\beta = \sigma_0$ for some fixed $\sigma_0 > 1/2$ (we will consider $9/10 \le \sigma_0 \le 99/100$ for our computations). We denote

$$\Sigma_3 = \sum_{\substack{\gamma>H\\ \beta=1/2}} \tilde{G}_{m,\delta,u}(1/2+i\gamma)X_0^{-1/2} + \sum_{\substack{\gamma>H\\ 1/2<\beta\le\sigma_0}} \left(\tilde{G}_{m,\delta,u}(\beta+i\gamma)X_0^{\beta-1} + \tilde{G}_{m,\delta,u}(1-\beta+i\gamma)X_0^{-\beta}\right), \tag{37}$$

$$\Sigma_4 = \sum_{\substack{\gamma>H\\ \sigma_0<\beta<1}} \left(\tilde{G}_{m,\delta,u}(\beta+i\gamma)X_0^{\beta-1} + \tilde{G}_{m,\delta,u}(1-\beta+i\gamma)X_0^{-\beta}\right). \tag{38}$$

As a conclusion, we have

$$\Sigma_{m,\delta,u,X_0} = \Sigma_0 + \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4. \tag{39}$$

We state here some preliminary results (see [4, Equations (2.18), (2.19), (2.20), (2.21), (2.26)]).

Lemma 2.5. *Let T_0, H, R_0, σ_0 be as in Theorem 2.4. Let $m \ge 2$, $X_0 > 10$, and T_1 between T_0 and H. We define*

$$S_1(T_1) = \left(\frac{1}{2\pi} + q(T_0)\right)\left(\log\frac{T_1}{T_0}\log\frac{\sqrt{T_1T_0}}{2\pi}\right)\frac{2R(T_0)}{T_0}, \tag{40}$$

$$S_2(m,T_1) = \left(\frac{1}{2\pi} + q(T_1)\right)\left(\frac{1+m\log\frac{T_1}{2\pi}}{m^2T_1^m} - \frac{1+m\log\frac{H}{2\pi}}{m^2H^m}\right) + \frac{2R(T_1)}{T_1^{m+1}}, \tag{41}$$

$$S_3(m) = \left(\frac{1}{2\pi} + q(H)\right)\left(\frac{1+m\log\frac{H}{2\pi})}{m^2H^m}\right) + \frac{2R(H)}{H^{m+1}}, \tag{42}$$

$$S_4(m,\sigma_0) = \left(c_1\left(1+\frac{1}{m}\right) + \frac{c_2\log H}{H} + \left(c_3 + \frac{c_2}{m+1}\right)\frac{1}{H}\right)\frac{1}{H^m}, \tag{43}$$

$$S_5(X_0,m,\sigma_0) = \left(c_1 + \frac{c_2\log H}{H} + \frac{c_3}{H} + \left(c_1 + \frac{c_2}{H}\right)\frac{R_0}{2\log X_0}\frac{(\log H)^2}{(\frac{mR_0}{\log X_0})(\log H)^2 - 1}\right)\frac{1}{H^m}. \tag{44}$$

We assume Theorem 2.4. Then

$$\sum_{T_0<\gamma\le T_1} \frac{1}{\gamma} \le S_1(T_1), \tag{45}$$

$$\sum_{T_1<\gamma\le H} \frac{1}{\gamma^{m+1}} \le S_2(m,T_1), \tag{46}$$

$$\sum_{\gamma>H} \frac{1}{\gamma^{m+1}} \le S_3(m), \tag{47}$$

$$\sum_{\substack{\gamma>H\\ \sigma_0<\beta<1}} \frac{1}{\gamma^{m+1}} \le S_4(m,\sigma_0). \tag{48}$$

Moreover, if $\log X_0 < R_0 m(\log H)^2$*, then*

$$\sum_{\substack{\gamma>H\\ \sigma_0<\beta<1}} \frac{X_0^{\frac{-1}{R_0\log\gamma}}}{\gamma^{m+1}} \le S_5(X_0,m,\sigma_0) X_0^{\frac{-1}{R_0\log H}}. \tag{49}$$

Lemma 2.6. *Let* m,δ,X_0 *satisfy* (1). *We assume Theorem 2.4. If* $\log X_0 < R_0 m(\log H)^2$*, then*

$$\begin{aligned}\Sigma_{m,\delta,u,X_0} \le\ & B_0(m,\delta)X_0^{-1/2} + B_1(m,\delta,T_1)X_0^{-1/2} + B_2(m,\delta,T_1)X_0^{-1/2}\\ &+ B_3(m,\delta)\left(X_0^{\sigma_0-1}+X_0^{-\sigma_0}\right) + B_{41}(X_0,m,\delta,\sigma_0)X_0^{-\frac{1}{R_0\log(H)}}\\ &+ B_{42}(m,\delta,\sigma_0)X_0^{-1+\frac{1}{R_0\log H}},\end{aligned} \tag{50}$$

where the B_i*'s are respectively defined in* (51), (54), (58), (60), (62), *and* (63).

Proof. We investigate two ways to evaluate Σ_0 and Σ_1. For Σ_0, we can either combine (26) with (30), which computes $\sum_{0<\gamma\le T_0}\gamma^{-1}$, or (25) with (29), which computes $\sum_{0<\gamma\le T_0} 1$. We denote

$$B_0(m,\delta) = \min(\Sigma_{01}(m,\delta), \Sigma_{02}(m,\delta)), \tag{51}$$

with

$$\Sigma_{01}(m,\delta) = \frac{4F_{1,m,\delta}}{(e^{u/2}+1)\delta} S_0 \quad\text{and}\quad \Sigma_{02}(m,\delta) = \frac{4F_{0,m,\delta}}{(e^{u/2}+1)} N_0. \tag{52}$$

We obtain

$$\Sigma_0 \le B_0(m,\delta)X_0^{-1/2}. \tag{53}$$

For Σ_1, we can either combine (26) with the bound (45) for $\sum_{T_0<\gamma\le T_1}\gamma^{-1}$, or (25) with the bound (32) for $N(T)$ from Theorem 2.4. We denote

$$B_1(m,\delta,T_1) = \min(\Sigma_{11}(m,\delta,T_1), \Sigma_{12}(m,\delta,T_1)), \tag{54}$$

with

$$\Sigma_{11}(m,\delta,T_1) = \frac{4F_{1,m,\delta}}{(e^{u/2}+1)\delta} S_1(T_1), \text{ and } \Sigma_{12}(m,\delta,T_1) = \frac{4F_{0,m,\delta}}{e^{u/2}+1}(N(T_1)-N_0). \tag{55}$$

We obtain

$$\Sigma_1 \leq B_1(m,\delta,T_1) X_0^{-1/2}. \tag{56}$$

It follows from (28) and (46) that

$$\Sigma_2 \leq B_2(m,\delta,T_1) X_0^{-1/2}, \tag{57}$$

with

$$B_2(m,\delta,T_1) = \frac{2F_{m,m,\delta}}{(e^{u/2}-1)\delta^m} S_2(m,T_1). \tag{58}$$

We use (28) to bound $\tilde{G}$ in Σ_3:

$$\Sigma_3 \leq \frac{2F_{m,m,\delta}}{(e^u-1)\delta^m} \sum_{\substack{\gamma>H \\ 1/2\leq\beta\leq\sigma_0}} \frac{(e^{u\beta}+1)X_0^{\beta-1} + (e^{u(1-\beta)}+1)X_0^{-\beta}}{\gamma^{m+1}}.$$

Note that since $\log X_0 > u$, then $(e^{u\beta}+1)X_0^{\beta-1} + (e^{u(1-\beta)}+1)X_0^{-\beta}$ increases with $\beta \geq 1/2$. Moreover, we use (47) to bound the sum $\sum_{\substack{\gamma>H \\ \beta\geq 1/2}} \gamma^{-(m+1)}$, and we obtain

$$\Sigma_3 \leq B_3(m,\delta,\sigma_0) X_0^{\sigma_0-1} + B_3(m,\delta,1-\sigma_0) X_0^{-\sigma_0}, \tag{59}$$

where

$$B_3(m,\delta,\sigma) = \frac{2F_{m,m,\delta}}{\delta^m} \frac{e^{u\sigma}+1}{e^u-1} S_3(m). \tag{60}$$

For Σ_4 we again use (28) to bound $\tilde{G}$ and the fact that $X_0^{\beta-1} + X_0^{-\beta}$ increases with β. Since $\beta \leq 1 - \frac{1}{R_0 \log\gamma}$ and $\gamma > H$, we obtain

$$\Sigma_4 \leq \frac{2(e^u+1)F_{m,m,\delta}}{(e^u-1)\delta^m} \Big(\sum_{\substack{\gamma>H \\ \sigma_0<\beta<1}} \frac{X_0^{-\frac{1}{R_0\log\gamma}}}{\gamma^{m+1}} + X_0^{-1+\frac{1}{R_0\log H}} \sum_{\substack{\gamma>H \\ \sigma_0<\beta<1}} \frac{1}{\gamma^{m+1}} \Big).$$

We apply (48) and (49) to bound the above sums over the zeros and obtain

$$\Sigma_4 \leq B_{41}(X_0,m,\delta,\sigma_0) X_0^{-\frac{1}{R_0\log(H)}} + B_{42}(m,\delta,\sigma_0) X_0^{-1+\frac{1}{R_0\log H}}, \tag{61}$$

with

$$B_{41}(X_0,m,\delta,\sigma_0) = \frac{2(e^u+1)F_{m,m,\delta}}{(e^u-1)\delta^m} S_5(X_0,m,\sigma_0), \tag{62}$$

$$B_{42}(m,\delta,\sigma_0) = \frac{2(e^u+1)F_{m,m,\delta}}{(e^u-1)\delta^m} S_4(m,\sigma_0). \tag{63}$$

□

Note that $G_{m,\delta,u}(1) = F_{0,m,\delta}$. Finally we apply Proposition 2.3 and Lemma 2.6.

2.7. Main Theorem

Theorem 2.7. *Let $m, u, \delta, a, \Delta, X_0$, and x satisfy* (1). *Let T_0, H, R_0, σ_0 be as in Theorem 2.4. We assume Theorem 2.4. If $X \geq X_0$ and*

$$\begin{aligned} F_{0,m,\delta} - B_0(m,\delta)X_0^{-1/2} - B_1(m,\delta,T_1)X_0^{-1/2} - B_2(m,\delta,T_1)X_0^{-1/2} \\ - B_3(m,\delta,\sigma_0)X_0^{\sigma_0-1} - B_3(m,\delta,1-\sigma_0)X_0^{-\sigma_0} - B_{41}(X_0,m,\delta,\sigma_0)X_0^{-\frac{1}{R_0 \log H}} \\ - B_{42}(m,\delta,\sigma_0)X_0^{-1+\frac{1}{R_0 \log H}} - \frac{u}{2(e^u-1)}X_0^{-2} - \frac{\omega}{(e^u-1)}X_0^{-1/2} \\ - \frac{2\nu(f,a)(1+\delta)}{\|f\|_1}\frac{\log(e^u X_0(1+\delta))}{\log(X_0(e^u-1))} > 0, \quad (64) \end{aligned}$$

then there exists a prime number between $x(1-\Delta^{-1})$ and x.

3. Computations

3.1. Introducing the Smooth Weight f

We choose the same weight as [16]. That is

$$f_m(t) = (4t(1-t))^m \text{ if } 0 \leq t \leq 1, \text{ and } 0 \text{ otherwise.}$$

Faber and Kadiri proved in [4] that a primitive form of f_m provided a near optimal weight to estimate $\psi(x)$. Thus we believe that the above weight should also be near optimal to evaluate $\psi(y) - \psi(x)$ when y is close to x. We recall [16, Lemma 6]:

$$\|f_m\|_1 = \frac{2^{2m}(m!)^2}{(2m+1)!}, \quad (65)$$

$$\|f_m^{(m)}\|_2 = \frac{2^{2m}m!}{\sqrt{2m+1}}. \quad (66)$$

We now provide estimates for $F_{k,m,\delta}$ as defined in (21).

Lemma 3.1. *Let $m \geq 2, \delta > 0$, and $0 < \sigma < 1$. We define*

$$\lambda_0(m,\delta) = \frac{(2m+1)!}{2^{2m-1}(m!)^2},$$

$$\lambda_1(m,\delta) = \frac{(1+\delta)^2(2m+1)!}{2^{2m-1}(m!)^2},$$

$$\lambda(m,\delta) = \sqrt{\frac{(1+\delta)^{2m+3}-1}{\delta(2m+3)}}\frac{(2m+1)!}{m!\sqrt{2m+1}}.$$

Then

$$1 \le F_{0,m,\delta} \le 1+\delta, \tag{67}$$

$$\lambda_0(m,\delta) \le F_{1,m,\delta}(\sigma) \le \lambda_1(m,\delta), \tag{68}$$

$$F_{m,m,\delta}(\sigma) \le \lambda(m,\delta). \tag{69}$$

Proof. Inequalities (67) follow trivially from the fact that $1 \le (1+\delta t) \le 1+\delta$.

To bound $F_{1,m,\delta}$, we note that

$$\frac{\|f_m'\|_1}{\|f_m\|_1} \le F_{1,m,\delta} \le \frac{(1+\delta)^2\|f_m'\|_1}{\|f_m\|_1}.$$

Since $f_m'(t)$ has the same sign as $1-2t$, we have

$$\|f_m'\|_1 = \int_1^{1/2} f_m'(t)dt - \int_{1/2}^1 f_m'(t)dt = 2f_m(1/2) - f_m(0) - f_m(1) = 2.$$

This together with (65) achieves (68).

Lastly, for $F_{m.m,\delta}$, we apply (66) together with the Cauchy-Schwarz inequality:

$$F_{m,m,\delta}(\sigma) \le \frac{\sqrt{\int_0^1 (1+\delta t)^{2(m+1)}dt}\sqrt{\int_0^1 |f_m^{(m)}(t)|^2 dt}}{\|f_m\|_1} = \sqrt{\frac{(1+\delta)^{2m+3}-1}{\delta(2m+3)}}\frac{\|f_m^{(m)}\|_2}{\|f_m\|_1}.$$

□

Note that, while $F_{0,m,\delta}$ and $F_{1,m,\delta}$ can be easily computed as integrals, this is not the case for $F_{m,m,\delta}$. The following observation helps us to compute $F_{m,m,\delta}$ directly. We recognize in the definition of $f_m^{(m)}$ the analogue of Rodrigues' formula for the shifted Legendre polynomials:

$$f_m^{(m)}(t) = 4^m m! P_m(1-2t),$$

where $P_m(x)$ is the m^{th} Legendre polynomial, and

$$P_m(1-2t) = (-1)^m \sum_{k=0}^m \binom{m}{k}\binom{m+k}{k}(-t)^k.$$

For each $P_m(1-2t)$, we denote $r_{j,m}$, with $j = 0, \ldots, m$, its $m+1$ roots. Since $P_m(1-2t)$ alternates sign between each root, we have

$$\begin{aligned} F_{m,m,\delta} = & \frac{\int_0^1 (1+\delta t)^{m+1}|P_m(1-2t)|dt}{\|f\|_1} \\ & = \frac{1}{\|f\|_1}\sum_{j=0}^{m-1}(-1)^j \int_{r_j}^{r_{j+1}} (1+\delta t)^{m+1} P_m(1-2t)dt, \end{aligned}$$

and GP-Pari is able to compute quickly this sum of polynomial integrals.

3.2. Explicit Results About the Zeros of the Riemann Zeta Function

We provide here the latest values for the constants appearing in Theorem 2.4:

Theorem 3.2.

1. *A numerical verification of the Riemann Hypothesis (Platt [15]):*

$$H = 3.061 \cdot 10^{10}.$$

2. *A direct computation of some finite sums over the first zeros (using A. Odlyzko's list of zeros):*

$$\text{For } T_0 = 1\,132\,491, \text{ then } N_0 = N(T_0) = 2\,001\,052, \text{ and } S_0 = 11.637732363.$$

3. *A zero-free region (Kadiri [8, Theorem 1.1]):*

$$R_0 = 5.69693.$$

4. *An estimate for $N(T)$ (Rosser [17, Theorem 19]):*

$$a_1 = 0.137,\ a_2 = 0.443,\ a_3 = 1.588.$$

5. *An upper bound for $N(\sigma_0, T)$ (Kadiri [10]): For all $T \geq H$,*

$$N(\sigma, T) \leq c_1 T + c_2 \log T + c_3,$$

where the c_i's are given in Table 1.

Table 1: $N(\sigma, T) \leq c_1 T + c_2 \log T + c_3$.

σ	c_1	c_2	c_3
0.90	5.8494	0.4659	$-1.7905 \cdot 10^{11}$
0.91	5.6991	0.4539	$-1.7444 \cdot 10^{11}$
0.92	5.5564	0.4426	$-1.7007 \cdot 10^{11}$
0.93	5.4206	0.4318	$-1.6592 \cdot 10^{11}$
0.94	5.2913	0.4215	$-1.6196 \cdot 10^{11}$
0.95	5.1680	0.4116	$-1.5819 \cdot 10^{11}$
0.96	5.0503	0.4023	$-1.5458 \cdot 10^{11}$
0.97	4.9379	0.3933	$-1.5114 \cdot 10^{11}$
0.98	4.8304	0.3848	$-1.4785 \cdot 10^{11}$
0.99	4.7274	0.3766	$-1.4470 \cdot 10^{11}$

Note that [17, Theorem 19] was recently improved by T. Trudgian in [19, Corollary 1] with $a_1 = 0.111,\ a_2 = 0.275,\ a_3 = 2.450$. Our results are valid with either Rosser's or Trudgian's bounds.

3.3. Understanding the Contribution of the Low Lying Zeros

We assume Theorem 3.2 and that

$$m \geq m_0 = 5, \delta < \delta_0 = 2 \cdot 10^{-8}, \text{ and } T_1 > t_1 = 10^9 \tag{70}$$

(this would be consistent with the values we choose in Table 2). We observe that

$$B_0(m,\delta) = \Sigma_{02}(m,\delta) \text{ and } B_1(m,\delta,T_1) = \Sigma_{12}(m,\delta).$$

where Σ_{02} and Σ_{12} are defined in (52) and (55), respectively. In other words, it turns out that we obtain a smaller bound for the sum over the small zeros ($0 < \gamma < T$) by using $N(T)$ directly instead of evaluating $\sum_{0<\gamma<T} \gamma^{-1}$. This essentially comes from the fact that our choice of parameters ensures $\delta \ll \frac{F_{1,m,\delta} S_0}{F_{0,m,\delta} N_0}$ and $\delta \ll \frac{F_{1,m,\delta} S_1(T_1)}{F_{0,m,\delta}(N(T_1)-N_0)}$. We first prove the inequality

$$\frac{S_1(t)}{N(t)} \geq c_0 \frac{\log t}{t}. \tag{71}$$

Proof. We denote

$$w_1 = \frac{1}{2}\left(\frac{1}{2\pi} + q(T_0)\right) = 0.0795\ldots,\ w_2 = -\log(2\pi)\left(\frac{1}{2\pi} + q(T_0)\right) = -0.2925\ldots,$$
$$w_3 = \left(\frac{1}{2\pi} + q(T_0)\right)\left(\frac{-\log^2(T_0)}{2} + \log(T_0)\log(2\pi)\right) + \frac{2R(T_0)}{T_0} = -11.3860\ldots,$$
$$v_1 = \frac{1}{2\pi} = 0.1591\ldots,\ v_2 = \frac{-\log(2\pi)}{2\pi} - 1 = -1.2925\ldots,\ v_3 = a_1 = 0.137,$$
$$v_4 = a_2 = 0.443,\ v_5 = a_3 + \frac{7}{8} = 2.463.$$

and

$$S_1(t) = w_1(\log t)^2 + w_2 \log t + w_3,\ P(t) + R(t) = v_1 t \log t + v_2 t + v_3 \log t + v_4 \log\log t + v_5.$$

From (40) and Theorem 3.2 (d), we have

$$\frac{S_1(t)}{N(t)} \geq \frac{S_1(t)}{P(t)+R(t)} = \frac{w_1(\log t)^2 + w_2 \log t + w_3}{v_1 t \log t + v_2 t + v_3 \log t + v_4 \log\log t + v_5}.$$

Since $t > t_1 = 10^9$, we deduce the bound

$$\frac{S_1(t)}{N(t)} \geq c_0 \frac{\log t}{t}, \tag{72}$$

where

$$c_0 = \frac{w_1 + \frac{w_2}{\log t_1} + \frac{w_3}{(\log t_1)^2}}{v_1 + \frac{v_3}{t_1} + \frac{v_4 \log\log t_1}{t_1 \log t_1} + \frac{v_5}{t_1 \log t_1}} \geq 0.7508. \tag{73}$$

$\square$

We now establish that

$$\max\left(\Sigma_{01}+\Sigma_{11},\Sigma_{01}+\Sigma_{12},\Sigma_{02}+\Sigma_{11}\right)\le\Sigma_{02}+\Sigma_{12}.$$

We make use of Lemma 3.1, to provide estimates for the $F_{k,m,\delta}$'s in (72), and of the assumptions (70) on m,δ,T_1.

Proof. We have

$$(\Sigma_{01}+\Sigma_{11})-(\Sigma_{02}+\Sigma_{12})=\frac{4}{e^{u/2}+1}\left(\frac{F_{1,m,\delta}}{\delta}\left(S_0+S_1(T_1)\right)-F_{0,m,\delta}N(T_1)\right)$$
$$>\frac{4(1+\delta)N(T_1)}{e^{u/2}+1}\left(\frac{(2m_0+1)!}{2^{2m_0-1}(m_0!)^2}\frac{1}{\delta_0(1+\delta_0)}\left(\frac{S_0}{P(t_1)+R(t_1)}+c_0\frac{\log t_1}{t_1}\right)-1\right)$$
$$>\frac{4(1+\delta)N(T_1)}{e^{u/2}+1}\left(2.4796-1\right)>0.$$

We have

$$(\Sigma_{01}+\Sigma_{12})-(\Sigma_{02}+\Sigma_{12})=\left(\frac{S_0}{\delta}F_{1,m,\delta}-N_0F_{0,m,\delta}\right)\frac{4}{e^{u/2}+1}$$
$$>\frac{4(1+\delta)N_0}{e^{u/2}+1}\left(\frac{(2m_0+1)!}{2^{2m_0-1}(m_0!)^2}\frac{1}{\delta_0(1+\delta_0)}\frac{S_0}{N_0}-1\right)>\frac{4(1+\delta)N_0}{e^{u/2}+1}\left(1574-1\right)>0.$$

Finally,

$$(\Sigma_{02}+\Sigma_{11})-(\Sigma_{02}+\Sigma_{12})=\frac{4}{e^{u/2}+1}\left(\frac{F_{1,m,\delta}}{\delta}S_1(T_1)-F_{0,m,\delta}(N(T_1)-N_0)\right)$$
$$>\frac{4(1+\delta)(N(T_1)-N_0)}{e^{u/2}+1}\left(\frac{(2m_0+1)!}{2^{2m_0-1}(m_0!)^2}\frac{1}{\delta_0(1+\delta_0)}\frac{S_1(t_1)}{(\frac{S(t_1)t_1}{c_0\log t_1}-N_0)}-1\right)$$
$$>\frac{4(1+\delta)(N(T_1)-N_0)}{e^{u/2}+1}\left(1.3737-1\right)>0.$$

□

3.4. Table of Computations

The values for T_1 and a given in the next table are rounded down to the last digit. We start the computations at $x_0\ge 4\cdot 10^{18}$, that is $\log x_0\ge 42.8328\ldots$.

Table 2: For all $x \geq x_0$, there exists a prime between $x(1-\Delta^{-1})$ and x.

$\log x_0$	m	δ	T_1	σ_0	a	Δ
$\log(4\cdot 10^{18})$	5	$3.580\cdot 10^{-8}$	272 519 712	0.92	0.2129	36 082 898
43	5	$3.349\cdot 10^{-8}$	291 316 980	0.92	0.2147	38 753 947
44	6	$2.330\cdot 10^{-8}$	488 509 984	0.92	0.2324	61 162 616
45	7	$1.628\cdot 10^{-8}$	797 398 875	0.92	0.2494	95 381 241
46	8	$1.134\cdot 10^{-8}$	1 284 120 197	0.92	0.2651	148 306 019
47	9	$8.080\cdot 10^{-9}$	1 996 029 891	0.92	0.2836	227 619 375
48	11	$6.000\cdot 10^{-9}$	3 204 848 430	0.93	0.3050	346 582 570
49	15	$4.682\cdot 10^{-9}$	5 415 123 831	0.93	0.3275	518 958 776
50	20	$3.889\cdot 10^{-9}$	8 466 793 105	0.93	0.3543	753 575 355
51	28	$3.625\cdot 10^{-9}$	12 399 463 961	0.93	0.3849	1 037 917 449
52	39	$3.803\cdot 10^{-9}$	16 139 006 408	0.93	0.4127	1 313 524 036
53	48	$4.088\cdot 10^{-9}$	18 290 358 817	0.93	0.4301	1 524 171 138
54	54	$4.311\cdot 10^{-9}$	19 412 056 863	0.93	0.4398	1 670 398 039
55	56	$4.386\cdot 10^{-9}$	19 757 119 193	0.93	0.4445	1 770 251 249
56	59	$4.508\cdot 10^{-9}$	20 210 075 547	0.93	0.4481	1 838 818 070
57	59	$4.506\cdot 10^{-9}$	20 219 045 843	0.93	0.4496	1 886 389 443
58	61	$4.590\cdot 10^{-9}$	20 495 459 359	0.93	0.4514	1 920 768 795
59	61	$4.589\cdot 10^{-9}$	20 499 925 573	0.93	0.4522	1 946 282 821
60	61	$4.588\cdot 10^{-9}$	20 504 393 735	0.93	0.4527	1 966 196 911
150	64	$4.685\cdot 10^{-9}$	21 029 543 983	0.96	0.4641	2 442 159 714

3.5. Verification of the Ternary Goldbach Conjecture

Proof of Corollary 1.2. Let $N = 4\cdot 10^{18}$. We follow Oliveira e Silva, Herzog and Pardi's argument in [14] where the authors computed all the prime gaps up to $4\cdot 10^{18}$. From Table 2, we have that for $x = e^{60}$ and $\Delta = 1\ 966\ 090\ 061$, there exists at least one prime in the interval $(x - x/\Delta, x]$. This one has length $5.8082\cdot 10^{16}$. Then $N\Delta = 7.8647\cdot 10^{27}$ and we may infer that the gap between consecutive primes up to $N\Delta$ can be no larger than N (since $N\Delta/\Delta = N$). The corollary follows by using all the odd primes up to $N\Delta$ to extend the minimal Goldbach partitions of $4, 6, \ldots, N$ up to $N\Delta$ (the method of computation is explained in [14, Section 1]). We also note that $N+2 = 211 + (N-209)$ and $N+4 = 313 + (N-309)$, where $211, 313, N-209$, and $N-309$ are all prime. Thus, there is at least one way to write each odd number greater than 5 and smaller than $N\Delta$ as the sum of at most 3 primes. $\square$

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