HANDLING A LARGE BOUND FOR A PROBLEM ON THE GENERALIZED PILLAI EQUATION $\pm ra^x \pm sb^y = c$.

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Abstract

We consider N, the number of solutions (x, y, u, v) to the equation $(-1)^u ra^x + (-1)^v sb^y = c$ in nonnegative integers x, y and integers $u, v \in \{0, 1\}$, for given integers $a > 1$, $b > 1$, $c > 0$, $r > 0$ and $s > 0$. Previous work showed that there are nine essentially distinct (a, b, c, r, s) for which $N \geq 4$, except possibly for cases in which the solutions have r, a, x, s, b, and y each bounded by $8 \cdot 10^{14}$ or $2 \cdot 10^{15}$. In this paper we show that there are no further cases with $N \geq 4$ within these bounds. We note that $N = 3$ for an infinite number of (a, b, c, r, s), even if we eliminate from consideration cases which are directly derived from other cases in one of several completely designated ways. Our work differs from previous work in that we allow x and y to be zero and also allow choices of (u, v) other than $(0, 1)$.

1. Introduction

The problem of finding N, the number of solutions (x, y, u, v) to the equation

$$(-1)^u ra^x + (-1)^v sb^y = c \qquad (1)$$

in nonnegative integers x, y and integers $u, v \in \{0, 1\}$, for given integers $a > 1$, $b > 1$, $c > 0$, $r > 0$ and $s > 0$, has been considered by many authors with various restrictions on the variables ([1], [5], [6], [12], [14], [16]). See [1], [2], [20], [14], [15] for histories of the problem.

In [14] we showed that $N > 3$ implies $\max(a, b, r, s, x, y) < 8 \cdot 10^{14}$ (or, in some cases, $2 \cdot 10^{15}$). The purpose of this paper is to show that there are exactly nine essentially different cases with $N > 3$ within those bounds.

To state our main result we need to summarize some definitions from [14].

We will refer to a *set of solutions* to (1) which we write as

$$(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$$

and by which we mean the (unordered) set of ordered pairs (x_1, y_1), (x_2, y_2), $\ldots$, (x_N, y_N) giving solutions to (1), with $N > 2$, for given integers a, b, c, r, and s. We say that two sets of solutions $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ and $(A, B, C, R, S; X_1, Y_1, X_2, Y_2, \ldots, X_N, Y_N)$ belong to the same *family* if a and A are both powers of the same integer, b and B are both powers of the same integer, there exists a positive rational number k such that $kc = C$, and for every i there exists a j such that $kra^{x_i} = RA^{X_j}$ and $ksb^{y_i} = SB^{Y_j}$, $1 \leq i, j \leq N$. One can show [14] that each family contains a unique member $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ with the following properties: $\gcd(r, sb) = \gcd(s, ra) = 1$; $\min(x_1, x_2, \ldots, x_N) = \min(y_1, y_2, \ldots, y_N) = 0$; and neither a nor b is a perfect power. We say that a set of solutions with these properties is in *basic form*.

The *associate* of a set of solutions $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ is the set of solutions $(b, a, c, s, r; y_1, x_1, y_2, x_2, \ldots, y_N, x_N)$.

A *subset* of a set of solutions $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ is a set of solutions with the same (a, b, c, r, s) and all its pairs (x, y) among the pairs (x_i, y_i), $1 \leq i \leq N$. Note that this subset may be (and, in our usage, usually is) the set of solutions $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ itself.

We are now ready to state the result of this paper:

Theorem 1. *Any set of solutions $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ to (1) with $N > 3$ must be in the same family as a subset (or an associate of a subset) of one of the following:*

$$(3, 2, 1, 1, 2; 0, 0, 1, 0, 1, 1, 2, 2)$$
$$(3, 2, 5, 1, 2; 0, 1, 1, 0, 1, 2, 2, 1, 3, 4)$$
$$(3, 2, 7, 1, 2; 0, 2, 2, 0, 1, 1, 2, 3)$$
$$(5, 2, 3, 1, 2; 0, 0, 0, 1, 1, 0, 1, 2, 3, 6)$$
$$(5, 3, 2, 1, 1; 0, 0, 0, 1, 1, 1, 2, 3)$$
$$(7, 2, 5, 3, 2; 0, 0, 0, 2, 1, 3, 3, 9)$$
$$(6, 2, 8, 1, 7; 0, 0, 1, 1, 2, 2, 3, 5)$$
$$(2, 2, 3, 1, 1; 0, 1, 0, 2, 1, 0, 2, 0)$$
$$(2, 2, 4, 3, 1; 0, 0, 1, 1, 2, 3, 2, 4).$$

Note that there are an infinite number of cases with $N = 3$, even if we consider only sets of solutions in basic form (see (11) through (18) in Section 2).

2. Preliminary Results

If, for a given choice of (a, b, c, r, s), (1) has two solutions (x_i, y_i) and (x_j, y_j), $1 \le i, j \le N$, we have

$$ra^{\min(x_i, x_j)} \left(a^{|x_j - x_i|} + (-1)^\gamma \right) = sb^{\min(y_i, y_j)} \left(b^{|y_j - y_i|} + (-1)^\delta \right) \qquad (2)$$

where $\gamma, \delta \in \{0, 1\}$.

In the following lemma, we summarize some results which follow from Theorem 1.1 of [14] and from the proof of Theorem 1.2 of [14].

Lemma 1. *Any set of solutions violating Theorem 1 above must be in the same family as a basic form (or the associate of a basic form) which satisfies one of the following:*

$$(ra, sb) = 1, \max(a, b, r, s, x, y) < 8 \cdot 10^{14}, 0 = x_1 < x_2 < x_3 < \cdots < x_N,$$
$$0 = y_1 < y_2 < y_3 < \cdots < y_N, \quad (3)$$

$$(ra, sb) = 1, \max(a, b, r, s, x, y) < 8 \cdot 10^{14}, 0 = x_1 < x_2 < x_3 < \cdots < x_N,$$
$$0 = y_2 < y_1 < y_3 < \cdots < y_N, \quad (4)$$

$$(ra, sb) = 1, \max(a, r, s, x, y) < 8 \cdot 10^{14}, 0 = x_1 < x_2 < x_3 < \cdots < x_N,$$
$$0 = y_1 = y_2 < y_3 < \cdots < y_N. \quad (5)$$

Lemma 1 is derived in part from work of Matveev [7] on lower bounds on linear forms in three logarithms as quoted in [8]. We will also need the following two lemmas from [14].

Lemma 2. *(Lemma 4.1 of [14]) Suppose $(ra, sb) = 1$ and suppose (1) has four solutions (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) with $x_1 < x_2 < x_3 < x_4$. Let $Z = \max(x_4, y_1, y_2, y_3, y_4)$. Then*

$$a^{x_3 - x_2} \le Z, s \le Z + 1.$$

Similarly, if (1) has four solutions (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) with $y_1 < y_2 < y_3 < y_4$, $\max(y_4, x_1, x_2, x_3, x_4) = Z$, and $(ra, sb) = 1$, then

$$b^{y_3 - y_2} \le Z, r \le Z + 1.$$

Lemma 3. *(Lemma 4.5 of [14]) Let $a > 1$ and $b > 1$ be relatively prime integers. For $1 \le i \le m$, let p_i be one of the m distinct prime divisors of a. Let $p_i^{g_i} || b^{n_i} \pm 1$,*

where n_i is the least positive integer for which there exists a positive integer k such that $|b^{n_i} - kp_i| = 1$, and $\pm$ is read as the sign that maximizes g_i. Write

$$\sigma = \sum_i g_i \log(p_i)/\log(a).$$

Then, if

$$a^x \mid b^y \pm 1,$$

where the $\pm$ sign is independent of the above, we must have

$$a^x \mid a^\sigma y.$$

Define $\sigma_a(b)$ to be the σ of this lemma, and let $\sigma_b(a)$ be the σ of this lemma with the roles of a and b reversed.

In the course of this paper, we will often need to show that a given set of three solutions does not have a fourth solution. We eliminate the possibility of a fourth solution by one of three methods: using the method known as 'bootstrapping' (see [4] or [18]), using bounds derived from LLL basis reduction (see [17]), or using x_4 to calculate y_4 and seeing if y_4 is an integer (or using y_4 to calculate x_4).

The technique known as 'bootstrapping' assumes one knows the values of a, b, r, s, x_3, and y_3. For simplicity of exposition assume $\gamma = \delta = 1$ (the other cases are only slightly more complicated) and consider (2) with $(i,j) = (3,4)$ where $x_3 < x_4$ and $y_3 < y_4$:

$$ra^{x_3}(a^{x_4-x_3} - 1) = sb^{y_3}(b^{y_4-y_3} - 1).$$

Let $\mathrm{ord}(n,p)$ be the least positive integer such that $p \mid n^{\mathrm{ord}(n,p)} - 1$. When $(ra, sb) = 1$, for each prime $p \mid sb^{y_3}$, we have $\mathrm{ord}(a,p) \mid x_4 - x_3$. Let $x_0 = \mathrm{lcm}\{\mathrm{ord}(a,p) : p \mid sb^{y_3}\}$ so $x_0 \mid x_4 - x_3$. Similarly, define $y_0 = \mathrm{lcm}\{\mathrm{ord}(b,p) : p \mid ra^{x_3}\}$ so $y_0 \mid y_4 - y_3$. Now we begin the bootstrapping steps. For each prime $p \mid a^{x_0} - 1$ such that $p \nmid sb$, $\mathrm{ord}(b,p)$ must divide $y_4 - y_3$; setting $Y_0 = \mathrm{lcm}(y_0, \{\mathrm{ord}(b,p) : p \mid a^{x_0} - 1\})$, we have Y_0 in the role of a new y_0 with $Y_0 \mid y_4 - y_3$. Now write y_0 for Y_0. For each prime $p \mid b^{y_0} - 1$ such that $p \nmid ra$, $\mathrm{ord}(a,p) \mid x_4 - x_3$; setting $X_0 = \mathrm{lcm}(x_0, \{\mathrm{ord}(a,p) : p \mid b^{y_0} - 1\})$, we have X_0 in the role of a new x_0 with $X_0 \mid x_4 - x_3$. Now write x_0 for X_0. We alternately use x_0 to find a larger y_0 if possible, the new y_0 to find a larger x_0 if possible, etc., continuing to bootstrap back and forth until x_0 or y_0 exceeds $8 \cdot 10^{14}$, in which case $x_4 - x_3$ or $y_4 - y_3$ must exceed this bound, contradicting Lemma 1, so no fourth solution exists.

For the LLL basis reduction algorithm, we follow the exposition in [17]. We have $\pm c = ra^{x_4} - sb^{y_4}$ so

$$\frac{ra^{x_4}}{sb^{y_4}} = 1 \pm \frac{c}{sb^{y_4}}$$

and thus

$$\log\left(\frac{r}{s}\right) + x_4 \log(a) - y_4 \log(b) = \log\left(1 \pm \frac{c}{sb^{y_4}}\right). \tag{6}$$

Since $|\log(1 \pm x)| < 2x$ for $0 < x < 0.7968$, one derives from (6) that, if $c/(sb^{y_4}) < 0.7968$,

$$\left| \log\left(\frac{r}{s}\right) + x_4 \log(a) - y_4 \log(b) \right| < 2\frac{c}{sb^{y_4}}$$

so

$$\left| \log\left(\frac{r}{s}\right) + x_4 \log(a) - y_4 \log(b) \right| < \frac{2c}{s}e^{-\log(b)y_4}.$$

Choosing $C = 10^{36}$, we set

$$A = \begin{pmatrix} 1 & [C\log(a)] \\ 0 & [-C\log(b)] \end{pmatrix}, Y = \begin{pmatrix} 0 & [-C\log\left(\frac{r}{s}\right)] \end{pmatrix},$$

where we set $[X]$ to be the closest integer to X. Note that rows in the computer algebra program Maple correspond to columns in [17]. Let B be the LLL basis reduction of the rows of A, and let b_1 and b_2 be the rows of B. Set $b_2^* = b_2 - \frac{b_2 \cdot b_1}{b_1 \cdot b_1}b_1$. Further, define the vector $\sigma = YB^{-1}$, define the number $\sigma_2 = \sigma[2]$, and let $\{\sigma_2\} = \sigma_2 - [\sigma_2]$ be the distance from σ_2 to the nearest integer. In our context, Lemma VI.1 in [17] becomes

Lemma 4. *Let* $S = \left(8 \cdot 10^{14}\right)^2$, $T = 8 \cdot 10^{14} + 0.5$, $c_1 = \max(1, ||b_1||^2/||b_2^*||^2)$, $c_2 = \frac{2c}{s}$, $c_3 = \log(b)$, *and* $c_4 = c_1^{-1}\{\sigma_2\}||b_1||^2$. *Assume* $c/(sb^{y_4}) < 0.7968$. *If* $c_4^2 > S + T^2$ *then*

$$y_4 \leq \frac{1}{c_3}\left(\log(Cc_2) - \log\left(\sqrt{c_4^2 - S} - T\right)\right). \tag{7}$$

Given a, b, c, r, s, and verifying that $c/(sb^{y_4}) < 0.7968$, we can often use this lemma to find that $\max(x_4, y_4) < \min(a, s - 1)$, contradicting Lemma 2, and so we can conclude that no fourth solution exists.

Finally, the third useful method to eliminate the possibility of a fourth solution assumes we are given a, b, r, s, and either a potential x_4 or a potential y_4. Suppose we have a bound $c < 10^{1000}$ and suppose $y_4 \geq b \geq 1000$ so $b^{y_4} > 10^{3000}$. Using $|\log(1 \pm x)| < 2x$ for $0 < x < 0.7968$, we see that (6) implies

$$\left| \log\left(\frac{r}{s}\right) + x_4 \log(a) - y_4 \log(b) \right| < 2 \cdot 10^{-2000}.$$

Since $\log(b) > 2$, we have, to at least 2000 places of accuracy,

$$y_4 = x_4 \frac{\log(a)}{\log(b)} + \frac{\log(r/s)}{\log(b)} \tag{8}$$

and

$$x_4 = y_4 \frac{\log(b)}{\log(a)} + \frac{\log(s/r)}{\log(a)}. \tag{9}$$

If we know a, b, r, s, and x_4, we can calculate y_4 from (8) and if this y_4 is not an integer to 2000 places of accuracy, then no fourth solution can exist. Similarly,

given y_4 we can calculate x_4 from (9) and show that this x_4 is not an integer, thus showing no fourth solution can exist.

We will also need the following.

Lemma 5. *Suppose (1) has three solutions* (x_1, y_1, u_1, v_1), (x_2, y_2, u_2, v_2), *and* (x_3, y_3, u_3, v_3) *and further assume that the following four conditions hold:*

1.) $x_1 < x_2 < x_3$ *and* $y_1 < y_2 < y_3$,

2.) $u_1 \neq v_1$,

3.)   *any solution* (x, y) *to (1) such that* $x > x_1$ *and* $y > y_1$ *must also satisfy* $x \geq x_2$ *and* $y \geq y_2$,

4.) $R = \frac{ra^{x_1}}{\gcd(ra^{x_1}, sb^{y_1})} > 2$ *and* $S = \frac{sb^{y_1}}{\gcd(ra^{x_1}, sb^{y_1})} > 2$.

Then $x_2 - x_1 \mid x_3 - x_1$ *and* $y_2 - y_1 \mid y_3 - y_1$.

Proof. Suppose we have three solutions to (1) satisfying all four conditions of the lemma. Considering (2) with $(i, j) = (1, 2)$ and $(1, 3)$ we have

$$R(a^{x_2 - x_1} + (-1)^{\gamma_2}) = S(b^{y_2 - y_1} + (-1)^{\delta_2})$$

and

$$R(a^{x_3 - x_1} + (-1)^{\gamma_3}) = S(b^{y_3 - y_1} + (-1)^{\delta_3}).$$

Since $u_1 \neq v_1$, we must have $\gamma_2 = \delta_2$ and $\gamma_3 = \delta_3$. Let $\alpha = 1$ if $\gamma_2 = \gamma_3 = 1$, otherwise let $\alpha = 0$. Let

$$t = \frac{a^{x_2 - x_1} + (-1)^{\gamma_2}}{S} = \frac{b^{y_2 - y_1} + (-1)^{\gamma_2}}{R}$$

and

$$T = \frac{a^{x_3 - x_1} + (-1)^{\gamma_3}}{S} = \frac{b^{y_3 - y_1} + (-1)^{\gamma_3}}{R}.$$

Note that t and T are both integers.

Let $g_1 = \gcd(x_2 - x_1, x_3 - x_1)$ and $g_2 = \gcd(y_2 - y_1, y_3 - y_1)$. Let k be the least integer such that $b^k + (-1)^{\alpha}$ is divisible by R. Then k must divide both $y_2 - y_1$ and $y_3 - y_1$, so that k divides g_2, and

$$b^{g_2} + (-1)^{\alpha} = Rl_2$$

for some integer l_2. (Note that, when $\alpha = 0$, $2^n || k$ implies $2^n || y_2 - y_1$ when $\gamma_2 = 0$ and $2^{n+1} \mid y_2 - y_1$ when $\gamma_2 = 1$, similarly for $y_3 - y_1$, so that, since $\min(\gamma_2, \gamma_3) = 0$, we have $2^n || g_2$.) Similarly,

$$a^{g_1} + (-1)^{\alpha} = Sl_1$$

for some integer l_1. Since g_1 divides both $x_2 - x_1$ and $x_3 - x_1$, l_1 divides t and T. There must be an integer j which is the least positive integer such that $b^j + (-1)^{\alpha}$ is divisible by Rl_1, and j must divide both $y_2 - y_1$ and $y_3 - y_1$, so that j divides g_2. Therefore $l_1 | l_2$.

A similar argument with the roles of a and b reversed shows that $l_2|l_1$, so that $l_1 = l_2$, and we have

$$ra^{x_1}(a^{g_1} + (-1)^\alpha) = sb^{y_1}(b^{g_2} + (-1)^\alpha). \tag{10}$$

(10) shows that $(x_1 + g_1, y_1 + g_2)$ is a solution to (1). If $x_1 + g_1 \neq x_2$, then, using Condition 3 in the formulation of the lemma, we see that we must have $x_1 + g_1 > x_2$, which is impossible by the definition of g_1. So $x_1 + g_1 = x_2$ and, similarly, $y_1 + g_2 = y_2$. □

When $N = 3$, we find many sets of solutions. Here we list several types of sets of solutions, each one of which generates an infinite number of basic forms (and therefore an infinite number of families) giving three solutions to (1). We list these sets of solutions in the form $(a, b, c, r, s; x_1, y_1, x_2, y_2, x_3, y_3)$:

$$\left(a, \frac{a^{kd} + (-1)^{u+v}}{a^d + (-1)^u}, \frac{a^d b + (-1)^{u+v+1}}{h}, \frac{b + (-1)^v}{h}, \frac{a^d + (-1)^u}{h}; 0, 1, d, 0, kd, 2\right) \tag{11}$$

where a and $b = \frac{a^{kd} + (-1)^{u+v}}{a^d + (-1)^u}$ are integers greater than 1, d and k are positive integers, $h = \gcd(a^d + (-1)^u, b + (-1)^v)$, and u and v are in the set $\{0, 1\}$. When $u = 0$, we take $k - v$ odd; when $(u, v) = (1, 1)$, we take $a^d \leq 3$. When $a = d = 2$ and $(u, v) = (1, 1)$, we can take k to be a half integer. When $k = 2$ and $u - v$ is odd, the same choice of (a, b, r, s) as in (11) gives the additional set of solutions

$$\left(a, a^d + (-1)^v, \frac{2a^d + (-1)^v}{h}, \frac{a^d + (-1)^v 2}{h}, \frac{a^d + (-1)^{v+1}}{h}; 0, 0, d, 1, 3d, 3\right). \tag{12}$$

Other sets of solutions can be constructed with specified values of a. When $a = 3$ we have

$$\left(3, \frac{3^g + (-1)^v}{2}, \frac{3^{g+1} + (-1)^v}{2^{2+v-\alpha}}, \frac{3(3^{g-1} + (-1)^v)}{2^{2+v-\alpha}}, 2^{1-v+\alpha}; 0, 1, 1, 0, 2g, 3\right) \tag{13}$$

where $v \in \{0, 1\}$, g is a positive integer, $\alpha = 0$ when $2 \mid g - v$, $\alpha = 1$ when g is odd and $v = 0$, and $\alpha = 2$ when g is even and $v = 1$.

When $a = 2$ we have

$$(2, 2^g + (-1)^v, 2^g + (-1)^{v+1}, 2, 1; 0, 1, g - 1, 0, g, 1) \tag{14}$$

where $v \in \{0, 1\}$ and g is a positive integer.

Also, it is easy to construct sets of solutions for which $x_1 = y_1 = y_2 = 0$. For example, we have, for a even and $x > 0$,

$$(a, 2a^x \pm 1, a^x \pm 1, 2, a^x \mp 1; 0, 0, x, 0, 2x, 1). \tag{15}$$

More generally,

$$\left(a, b, \frac{a^{x_2} + (-1)^t}{2^m}, 2^{1-m}, \frac{a^{x_2} + (-1)^{t+1}}{2^m}; 0, 0, x_2, 0, x_3, y_3\right), \tag{16}$$

with $b^{y_3} = \frac{2a^{x_3} + (-1)^{t+w+1} a^{x_2} + (-1)^{w+1}}{a^{x_2} + (-1)^{t+1}}$, where $x_2 > 0$, $x_3 > 0$, $x_2 \mid x_3$, and $a^{x_3} \equiv (-1)^w \bmod \frac{a^{x_2} + (-1)^{t+1}}{2^m}$, for $t \in \{0, 1\}$, $w \in \{0, 1\}$, and $m = 1$ or 0 according as a is odd or even.

We also find an infinite family for which $\gcd(a, b) > 1$:

$$(a, b, c, r, s; x_1, y_1, x_2, y_2, x_3, y_3) =$$
$$\left(a, ta, \frac{a(t + (-1)^{u+v+1})}{h}, \frac{ta + (-1)^v}{h}, \frac{a + (-1)^u}{h}; 0, 0, 1, 1, m+1, 2\right) \tag{17}$$

where $m \geq 0$ is an integer, $t = \frac{a^m + (-1)^v}{a + (-1)^u}$ is an integer, $h = \gcd(ta + (-1)^v, a + (-1)^u)$, and u and v are in the set $\{0, 1\}$. Closely related to (17) is the following:

$$\left(2, 4t, \frac{4t+4}{h_1}, \frac{4t+1}{h_1}, \frac{3}{h_1}; 0, 0, 2, 1, m_1 + 2, 2\right) \tag{18}$$

where $m_1 \geq -1$ is an odd integer, $t = \frac{2^{m_1}+1}{3}$, $h_1 = 3$ or 1 according as $m_1 \equiv 5 \bmod 6$ or not, and $u, v \in \{0, 1\}$.

The Maple$^{\text{TM}}$ worksheets with the calculations for the following sections can be found at [19].

3. Case (3) $0 = x_1 < x_2 < x_3 < x_4$, $0 = y_1 < y_2 < y_3 < y_4$

Lemma 6. *Suppose* $\min(a, b) \leq 170000$. *Then any set of solutions to (1) satisfying conditions (3) has $N < 4$.*

Proof. Suppose (3) holds with $N > 3$. By symmetry, we may assume $a > b$. Further assume a is not a perfect power. Taking (2) with $(i, j) = (2, 3)$ we have $ra^{x_2}(a^{x_3 - x_2} + (-1)^\gamma) = sb^{y_2}(b^{y_3 - y_2} + (-1)^\delta)$. Lemma 2 shows $b^{y_3 - y_2} \leq Z$ so Lemma 1 shows $y_3 - y_2 < \log(8 \cdot 10^{14})/\log(b)$. For each choice of $b \leq 170000$, $\delta \in \{0, 1\}$, and $y_3 - y_2 < \log(8 \cdot 10^{14})/\log(b)$, we see that a^{x_2} is a factor of $b^{y_3 - y_2} + (-1)^\delta < 8 \cdot 10^{14}$; this is small enough to factor easily, so we can list all factors a^{x_2}, hence we know all possible a and x_2. Lemma 2 bounds $a^{x_3 - x_2} \leq Z$. For each choice of $x_3 - x_2 < \log(8 \cdot 10^{14})/\log(a)$ and of $\gamma \in \{0, 1\}$, we calculate the power of b dividing $a^{x_3 - x_2} + (-1)^\gamma$ which gives us the maximal possible value for y_2, call it $y_{2,max}$. If $y_{2,max} > 0$, for each y_2 with $1 \leq y_2 \leq y_{2,max}$ and $h = \gcd(a^{x_3 - x_2} + (-1)^\gamma, b^{y_3 - y_2} + (-1)^\delta)$, we can solve for $r = (b^{y_3 - y_2} + (-1)^\delta)/(a^{x_2}h)$ and $s = (a^{x_3 - x_2} + (-1)^\gamma)/(b^{y_2}h)$. We have $y_3 = (y_3 - y_2) + y_2$ and $x_3 = (x_3 - x_2) + x_2$.

We see that $c = |ra^{x_2} - sb^{y_2}|$. We now check if $c = |r \pm s|$. If so, we have values a, b, c, r, and s for which (1) has at least three solutions.

For each set of at least three solutions found for $b \leq 170000$, we use the LLL method or bootstrapping to show that there is no fourth solution. $\qquad\square$

Lemma 7. *If (a, b, c, r, s) are chosen as in (12) with $d = 1$ and $\min(a, b) > 4$, then (1) has only the three solutions given in (12).*

Proof. Assume some instance of (12) with $d = 1$ and $\min(a, b) > 4$ has a fourth solution. Note that no subset or associate of a subset of any of the sets of solutions listed in Theorem 1 is in the same family as a set of solutions which is an instance of (12) having $d = 1$, $\min(a, b) > 4$, and a fourth solution for the same (a, b, c, r, s). So by Lemma 1 we must have (3) (note that we cannot have (5) since (12) with $d = 1$ and $\min(a, b) > 4$ requires that a fourth solution must have $\min(x, y) > 0$). By Lemma 6, we need only consider $a > 170000$.

Now if the fourth solution in question is (x, y) with $y = 2$, then

$$hra^x = \pm hc \pm hsb^2 = \pm(2a + (-1)^v) \pm (a + (-1)^{v+1})(a + (-1)^v)^2 \not\equiv 0 \bmod a^2,$$

contradicting Lemma 1, which (considering (12)) requires $x = 2$. So if $d = 1$, any fourth solution (x_4, y_4) to (12) must satisfy $x_4 > 3$ and $y_4 > 3$.

Lemma 2 shows that $\max(x_4, y_4) \geq a > 170000$ while Lemma 1 shows that $c \leq r + s < 16 \cdot 10^{14}$ so certainly $x_4 > 100$.

When $d = 1$ and $v = 0$, we have $b = a + 1$, $rh = a + 2$, $sh = a - 1$, and $ch = 2a + 1$, where $h = \gcd(a + 2, a - 1)$. Considering the solution $(x_3, y_3) = (3, 3)$, we get

$$(2a + 1) + (a - 1)b^3 = (a + 2)a^3. \tag{19}$$

Considering the solution (x_4, y_4) we get

$$(2a + 1) + (a - 1)b^{y_4} \equiv 0 \bmod (a + 2)a^{100}. \tag{20}$$

Combining (19) and (20) we find

$$(a - 1)(b^{y_4 - 3} - 1) \equiv 0 \bmod (a + 2)a^3, \tag{21}$$

which requires $a^2 \mid y_4 - 3$ except possibly when $a \equiv 2 \bmod 4$ (see Lemma 1 of [12]). If $a \equiv 2 \bmod 4$, let $2^g \mid\mid a + 2 = b + 1$ and let $a/2 = a_0$; then instead of (21) we can use

$$(b^{y_4 - 3} - 1) \equiv 0 \bmod 2^{g+3} a_0^3$$

which again requires $a^2 \mid y_4 - 3$. So we can write $y_4 = 3 + ja^2$ for some integer

$j > 0$. For some integers M_1 and M_2 we have

$$(2a + 1) + (a - 1)\left(1 + (3 + ja^2)a + \frac{(3 + ja^2)(3 + ja^2 - 1)}{2}a^2 \right.$$
$$+ \frac{(3 + ja^2)(3 + ja^2 - 1)(3 + ja^2 - 2)}{6}a^3$$
$$\left. + \frac{(3 + ja^2)(3 + ja^2 - 1)(3 + ja^2 - 2)(3 + ja^2 - 3)}{24}a^4 + M_1 a^5 \right) = M_2 a^{100}$$

from which we derive, for some integer M_3,

$$(2 - j)a^3 + \left(1 - \frac{3}{2}j \right) a^4 = \frac{M_3}{6}a^5 \tag{22}$$

so that $j = (wa/6) + 2$ for some integer $w \geq 0$. If $w = 0$ then (22) becomes impossible. If $w > 0$ then $y_4 \geq 3 + \left(\frac{a}{6} + 2\right) a^2 > 8 \cdot 10^{14}$ when $a > 170000$, so we have a contradiction to Lemma 1, showing the impossibility of a fourth solution.

When $d = 1$, $v = 1$, we have $b = a - 1$, which is equivalent to the previous case after reversing the roles of a and b. Thus, in each case, no instance of (12) with $d = 1$ and $\min(a, b) > 4$ has a fourth solution. $\square$

Lemma 8. *Any set of solutions to (1) satisfying (3) has $N < 4$.*

Proof. Using Lemma 6, assume $a > b > 170000$. Suppose (1) satisfying (3) has four solutions with $0 = x_1 < x_2 < x_3 < x_4$ and $0 = y_1 < y_2 < y_3 < y_4$. To fix notation, let

$$c = -(-1)^\alpha r + (-1)^\beta s = (-1)^\gamma (ra^{x_2} - sb^{y_2}) = (-1)^\delta (ra^{x_3} - sb^{y_3})$$
$$= (-1)^\epsilon (ra^{x_4} - sb^{y_4})$$

for some $\alpha, \beta, \gamma, \delta, \epsilon \in \{0, 1\}$. Considering (2) with $(i, j) = (1, 2)$ and $(i, j) = (2, 3)$, we have

$$r(a^{x_2} + (-1)^{\alpha + \gamma}) = s(b^{y_2} + (-1)^{\beta + \gamma}) \tag{23}$$

and

$$ra^{x_2}(a^{x_3 - x_2} - (-1)^{\gamma + \delta}) = sb^{y_2}(b^{y_3 - y_2} - (-1)^{\gamma + \delta}). \tag{24}$$

Suppose $x_3 - x_2 = y_3 - y_2 = 1$. Since $(ra, sb) = 1$, from (24) we have $a^{x_2} \mid b - (-1)^{\gamma + \delta}$; therefore, since $a > b$, we have $x_2 = 1$, $(-1)^{\gamma + \delta} = -1$, and $a = b + 1$. Then $b^{y_2} \mid a - (-1)^{\gamma + \delta} = a + 1 = b + 2$ which is impossible for $b > 2$. Thus we have $\max(x_3 - x_2, y_3 - y_2) \geq 2$. By Lemma 2, $\max(x_3 - x_2, y_3 - y_2) \leq 2$ since $a > b > 170000$. Thus,

$$\max(x_3 - x_2, y_3 - y_2) = 2. \tag{25}$$

Taking the ratio of (23) and (24),

$$\frac{a^{x_2}(a^{x_3-x_2} - (-1)^{\gamma+\delta})}{a^{x_2} + (-1)^{\alpha+\gamma}} = \frac{b^{y_2}(b^{y_3-y_2} - (-1)^{\gamma+\delta})}{b^{y_2} + (-1)^{\beta+\gamma}}. \tag{26}$$

We rewrite this as

$$(a^{x_3-x_2} - (-1)^{\gamma+\delta})\left(1 - \frac{(-1)^{\alpha+\gamma}}{a^{x_2} + (-1)^{\alpha+\gamma}}\right) =$$
$$(b^{y_3-y_2} - (-1)^{\gamma+\delta})\left(1 - \frac{(-1)^{\beta+\gamma}}{b^{y_2} + (-1)^{\beta+\gamma}}\right).$$

From this we obtain

$$a^{x_3-x_2} - b^{y_3-y_2} =$$
$$a^{x_3-x_2}\frac{(-1)^{\alpha+\gamma}}{a^{x_2} + (-1)^{\alpha+\gamma}} - b^{y_3-y_2}\frac{(-1)^{\beta+\gamma}}{b^{y_2} + (-1)^{\beta+\gamma}} - \frac{(-1)^{\alpha+\delta}}{a^{x_2} + (-1)^{\alpha+\gamma}} + \frac{(-1)^{\beta+\delta}}{b^{y_2} + (-1)^{\beta+\gamma}}$$

so

$$|a^{x_3-x_2} - b^{y_3-y_2}| \le$$
$$a^{x_3-x_2}\frac{1}{a^{x_2} + (-1)^{\alpha+\gamma}} + b^{y_3-y_2}\frac{1}{b^{y_2} + (-1)^{\beta+\gamma}} + \frac{1}{a^{x_2} + (-1)^{\alpha+\gamma}} + \frac{1}{b^{y_2} + (-1)^{\beta+\gamma}}.$$

Since $a^{x_2} + (-1)^{\alpha+\gamma} \ge b$ and $b^{y_2} + (-1)^{\beta+\gamma} \ge b - 1$,

$$|a^{x_3-x_2} - b^{y_3-y_2}| \le \frac{a^{x_3-x_2}}{b} + \frac{b^{y_3-y_2}}{b-1} + \frac{1}{b} + \frac{1}{b-1}$$

and since (25) gives $\max(a^{x_3-x_2}, b^{y_3-y_2}) \ge b^2$,

$$|a^{x_3-x_2} - b^{y_3-y_2}| < \max(a^{x_3-x_2}, b^{y_3-y_2})\left(\frac{1}{b} + \frac{1}{b-1} + \frac{1}{b^3} + \frac{1}{(b-1)b^2}\right).$$

Therefore,

$$|a^{x_3-x_2} - b^{y_3-y_2}| < \frac{2}{b-1}\max\left(a^{x_3-x_2}, b^{y_3-y_2}\right)$$

so that, since $b > 3$,

$$\left(1 - \frac{2}{b-1}\right)b^{y_3-y_2} < a^{x_3-x_2} < \left(1 - \frac{2}{b-1}\right)^{-1}b^{y_3-y_2} \tag{27}$$

and similarly

$$\left(1 - \frac{2}{b-1}\right)a^{x_3-x_2} < b^{y_3-y_2} < \left(1 - \frac{2}{b-1}\right)^{-1}a^{x_3-x_2}. \tag{28}$$

We now show $x_2 \leq x_3 - x_2$. From (24) and (28) we have

$$a^{x_2} \mid b^{y_3-y_2} - (-1)^{\gamma+\delta} < \left(1 - \frac{2}{b-1}\right)^{-1} a^{x_3-x_2} + 1 < a^{x_3-x_2+1},$$

so $x_2 \leq x_3 - x_2$. Similarly, one can show that $y_2 \leq y_3 - y_2$.

We can further show $x_2 < x_3 - x_2$. Suppose $z = x_2 = x_3 - x_2$. Then (24) gives $a^z \mid b^{y_3-y_2} - (-1)^{\gamma+\delta}$ so (28) shows

$$a^z \mid b^{y_3-y_2} - (-1)^{\gamma+\delta} < \left(1 - \frac{2}{b-1}\right)^{-1} a^z + 1 < 2a^z.$$

Thus, $a^z = b^{y_3-y_2} - (-1)^{\gamma+\delta}$ and (26) becomes

$$(b^{y_3-y_2} - (-1)^{\gamma+\delta}2)(b^{y_2} + (-1)^{\beta+\gamma}) = b^{y_2}(b^{y_3-y_2} + (-1)^{\alpha+\gamma} - (-1)^{\gamma+\delta})$$

which is impossible modulo b. Thus, $1 \leq x_2 < x_3 - x_2$; similar arguments show $1 \leq y_2 < y_3 - y_2$.

Recalling (25) we find $x_2 = y_2 = 1$ and $x_3 = y_3 = 3$. If $(-1)^{\gamma+\delta} = 1$ then (26) can be rewritten as $a(a - (-1)^{\alpha+\gamma}) = b(b - (-1)^{\beta+\gamma})$. This implies $a = b - (-1)^{\beta+\gamma}$ and so $b = a - (-1)^{\alpha+\gamma}$, $\beta \neq \gamma$ and $\alpha = \gamma$, so $c = -(-1)^{\alpha}r + (-1)^{\beta}s > 0$ shows $\alpha = \gamma = 1$ and $\beta = 0$. Then $r = (a-2)/h$, $s = (a+1)/h$, and $c = (2a-1)/h$ where $h = \gcd(a-2, a+1) \leq 3$. We see that the case under consideration in this paragraph satisfies (12) with $d = v = 1$. By Lemma 7, this cannot lead to a fourth solution.

So $(-1)^{\gamma+\delta} = -1$, and (26) becomes

$$a(a^2 + 1)(b + (-1)^{\beta+\gamma}) = b(b^2 + 1)(a + (-1)^{\alpha+\gamma}).$$

Since $\gcd(a^2 + 1, a \pm 1) \leq 2$ and $\gcd(b^2 + 1, b \pm 1) \leq 2$, we must have

$$a + (-1)^{\alpha+\gamma} \mid 2(b + (-1)^{\beta+\gamma}), b + (-1)^{\beta+\gamma} \mid 2(a + (-1)^{\alpha+\gamma}). \qquad (29)$$

Note that (27) gives

$$a < \frac{b}{\sqrt{1 - \frac{2}{b-1}}} < b + 2$$

for $b \geq 6$ so $a = b + 1$. But then (29) is impossible. □

4. Case (4) $0 = x_1 < x_2 < x_3 < x_4$, $0 = y_2 < y_1 < y_3 < y_4$

We begin with some preliminaries. To fix notation, let

$$c = (-1)^{\alpha}r + sb^{y_1} = ra^{x_2} + (-1)^{\beta}s = (-1)^{\gamma}(ra^{x_3} - sb^{y_3}) = (-1)^{\delta}(ra^{x_4} - sb^{y_4}) \quad (30)$$

for some α, β, γ, and $\delta \in \{0, 1\}$. Applying (2) with $(i, j) = (2, 3)$, $(1, 2)$, and $(1, 3)$, we have

$$r(a^{x_3} - (-1)^\gamma a^{x_2}) = s(b^{y_3} + (-1)^{\beta+\gamma}), \tag{31}$$

$$r(a^{x_2} - (-1)^\alpha) = s(b^{y_1} - (-1)^\beta), \tag{32}$$

and

$$r(a^{x_3} - (-1)^{\alpha+\gamma}) = s(b^{y_3} + (-1)^\gamma b^{y_1}). \tag{33}$$

Taking the ratio of (31) and (32), we obtain

$$\frac{a^{x_3} - (-1)^\gamma a^{x_2}}{a^{x_2} - (-1)^\alpha} = \frac{b^{y_3} + (-1)^{\beta+\gamma}}{b^{y_1} - (-1)^\beta}. \tag{34}$$

Similarly, the ratio of (33) with (32) gives

$$\frac{a^{x_3} - (-1)^{\alpha+\gamma}}{a^{x_2} - (-1)^\alpha} = \frac{b^{y_3} + (-1)^\gamma b^{y_1}}{b^{y_1} - (-1)^\beta}, \tag{35}$$

and considering (31) with (33), one gets

$$\frac{a^{x_3} - (-1)^\gamma a^{x_2}}{a^{x_3} - (-1)^{\alpha+\gamma}} = \frac{b^{y_3} + (-1)^{\beta+\gamma}}{b^{y_3} + (-1)^\gamma b^{y_1}}. \tag{36}$$

Lemma 9. *For $b \leq 10^3$ and $a < 8 \cdot 10^{14}$ we must have $b^{\sigma_b(a)} < 10^{22}$, where $\sigma_b(a)$ is defined immediately following Lemma 3.*

Proof. Let $b = p$ be an odd prime less than 1000. Let $k = \lceil 22 \log(10)/\log(b) \rceil$, so $p^k \geq 10^{22}$. If $p^{\sigma_p(a)} \geq 10^{22}$ then $\sigma_p(a) \geq k$. Choose $\alpha \in \{0, 1\}$ to minimize the positive integer n such that $p \mid a^n + (-1)^\alpha$. Note that $n \mid (p-1)/2$. By definition, $p^{\sigma_p(a)} \mid a^n + (-1)^\alpha$. Let a_0 be any solution to $x^n + (-1)^\alpha \equiv 0 \bmod p$. Using Hensel's lifting lemma, we find a unique solution a_1 to the congruence $a_1^n + (-1)^\alpha \equiv 0 \bmod p^k$ with $a_1 \equiv a_0 \bmod p$. For each prime $p < 10^3$, for each $n \mid (p-1)/2$, and for each solution $a_0 \bmod p$, calculations show that the associated $a_1 \bmod p^k$ exceeds $8 \cdot 10^{14}$. In other words, for every prime $p < 10^3$, if $p^{\sigma_p(a)} \geq 10^{22}$, then $a > 8 \cdot 10^{14}$.

Now suppose $b = p_1^{\beta_1} p_2^{\beta_2}$. By definition of σ, $b^{\sigma_b(a)} = p_1^{g_1} p_2^{g_2}$ for some positive integers g_1 and g_2. For any given $1 \leq k_2 \leq 22 \log(10)/\log(p_2)$, let

$$k_1 = \left\lceil \frac{22 \log(10) - k_2 \log(p_2)}{\log(p_1)} \right\rceil.$$

Note that if $b^{\sigma_b(a)} = p_1^{g_1} p_2^{g_2} \geq 10^{22}$ for some a, then there exists $1 \leq k_2 \leq 22 \log(10)/\log(p_2)$ such that $k_2 \leq g_2$ and $k_1 \leq g_1$.

Suppose $b^{\sigma_b(a)} = p_1^{g_1} p_2^{g_2}$ with $k_1 \leq g_1$ and $k_2 \leq g_2$. By definition of σ, there exists $n_1 \mid (p_1 - 1)/2$, $n_2 \mid (p_2 - 1)/2$, and $\alpha_1, \alpha_2 \in \{0, 1\}$ such that

$$p_1^{g_1} \mid a^{n_1} + (-1)^{\alpha_1}, p_2^{g_2} \mid a^{n_2} + (-1)^{\alpha_2},$$

so
$$a^{n_1} + (-1)^{\alpha_1} \equiv 0 \bmod p_1^{k_1}, a^{n_2} + (-1)^{\alpha_2} \equiv 0 \bmod p_2^{k_2}.$$

For each choice of $n_1 \mid (p_1 - 1)/2$, we can list all values of $a \bmod p_1$ with $a^{n_1} + (-1)^{\alpha_1} \equiv 0 \bmod p_1$. Using Hensel's lifting lemma, we can obtain a complete list of all possible values of $a \bmod p_1^{k_1}$ satisfying $a^{n_1} + (-1)^{\alpha_1} \equiv 0 \bmod p_1^{k_1}$. Similarly, for each choice of $n_2 \mid (p_2 - 1)/2$, we can obtain a complete list of all possible values of $a \bmod p_2^{k_2}$ satisfying $a^{n_2} + (-1)^{\alpha_2} \equiv 0 \bmod p_2^{k_2}$. Thus, for each choice of p_1, p_2, k_2, $n_1 \mid (p_1 - 1)/2$, $n_2 \mid (p_2 - 1)/2$, and $\alpha_1, \alpha_2 \in \{0, 1\}$, we can obtain every possible $a \bmod p_1^{k_1} p_2^{k_2}$ satisfying

$$p_1^{k_1} \mid a^{n_1} + (-1)^{\alpha_1}, p_2^{k_2} \mid a^{n_2} + (-1)^{\alpha_2}.$$

Calculations show that each potential a exceeds $8 \cdot 10^{14}$. In other words, if $b^{\sigma_b(a)} \geq 10^{22}$ then $a > 8 \cdot 10^{14}$.

Suppose $b = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3}$ with $p_1 < p_2 < p_3$. By definition of σ, $b^{\sigma_b(a)} = p_1^{g_1} p_2^{g_2} p_3^{g_3}$ for some positive integers g_1, g_2, and g_3. For any given $k_3 \leq 22 \log(10) / \log(p_3)$, and any $k_2 \leq (22 \log(10) - k_3 \log(p_3)) / \log(p_2)$, let

$$k_1 = \left\lceil \frac{22 \log(10) - k_3 \log(p_3) - k_2 \log(p_2)}{\log(p_1)} \right\rceil.$$

If $b^{\sigma_b(a)} = p_1^{g_1} p_2^{g_2} p_3^{g_3} \geq 10^{22}$ for some a, there exist k_3 and k_2 such that $k_3 \leq g_3$, $k_2 \leq g_2$, and $k_1 \leq g_1$. For each choice of p_1, p_2, p_3, k_3, k_2, $n_1 \mid (p_1 - 1)/2$, $n_2 \mid (p_2 - 1)/2$, $n_3 \mid (p_3 - 1)/2$, $\alpha_1, \alpha_2, \alpha_3 \in \{0, 1\}$, we proceed as in the previous paragraph to obtain every possible $a \bmod p_1^{k_1} p_2^{k_2} p_3^{k_3}$ satisfying

$$p_1^{k_1} \mid a^{n_1} + (-1)^{\alpha_1}, p_2^{k_2} \mid a^{n_2} + (-1)^{\alpha_2}, p_3^{k_3} \mid a^{n_3} + (-1)^{\alpha_3}.$$

The calculations verify that each possible a exceeds $8 \cdot 10^{14}$.

Finally, suppose $b < 10^3$ is the product of four primes. The same procedure works. Since each $b \leq 10^3$ has four or fewer distinct prime factors, we conclude that if $b^{\sigma_b(a)} \geq 10^{22}$ then $a > 8 \cdot 10^{14}$. $\qquad\square$

The following two lemmas apply to the following set of solutions to (1):

$$(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N), N \geq 3, 0 = x_1 < x_2 < x_3 < \cdots < x_N,$$
$$0 = y_2 < y_1 < y_3 < \cdots < y_N. \quad (37)$$

Lemma 10. *If (37) holds, then either $x_2 \leq x_3 - x_2$ or $y_1 \leq y_3 - y_1$.*

Proof. Assume the set of solutions $(a, b, c, r, s; x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ satisfies (37), and further assume $x_2 > x_3 - x_2$ and $y_1 > y_3 - y_1$.

From (34) we get

$$a^{x_3-x_2}-(-1)^\gamma+\frac{(-1)^\alpha a^{x_3-x_2}-(-1)^{\alpha+\gamma}}{a^{x_2}-(-1)^\alpha}=b^{y_3-y_1}+\frac{(-1)^\beta b^{y_3-y_1}+(-1)^{\beta+\gamma}}{b^{y_1}-(-1)^\beta}. \quad (38)$$

If $\min(a,b)>2$, we see that $\left|\frac{(-1)^\alpha a^{x_3-x_2}-(-1)^{\alpha+\gamma}}{a^{x_2}-(-1)^\alpha}\right|\le\frac{3+1}{9-1}=\frac{1}{2}$. Similarly, we see that $\left|\frac{(-1)^\beta b^{y_3-y_1}+(-1)^{\beta+\gamma}}{b^{y_1}-(-1)^\beta}\right|\le\frac{1}{2}$. In both cases the value $1/2$ is possible only when a (respectively, b), equals 3, x_2 (respectively, y_1) equals 2, and x_3-x_2 (respectively, y_3-y_1) equals 1. By Lemma 2.2 of [14], we cannot have $(a,b)>1$ when (37) holds, so we must have

$$|a^{x_3-x_2}-b^{y_3-y_1}-(-1)^\gamma|<1, \quad (39)$$

so the left side of (39) must be zero.

But now from (36) we have

$$(-1)^\gamma a^{x_3-x_2}-(-1)^\gamma b^{y_3-y_1}-1=\frac{(-1)^{\beta+\gamma}a^{x_3}-(-1)^{\alpha+\gamma}b^{y_3}-(-1)^{\alpha+\beta}}{a^{x_2}b^{y_1}}. \quad (40)$$

But, from (39), the left side of (40) must be zero, which is impossible since the numerator on the right side of (40) cannot be zero by Mihailescu's theorem [9] since x_3 and y_3 both are greater than 2.

So we can assume $\min(a,b)=2$. We see that $\left|\frac{(-1)^\alpha a^{x_3-x_2}-(-1)^{\alpha+\gamma}}{a^{x_2}-(-1)^\alpha}\right|\le\frac{2+1}{4-1}=1$. Similarly, we have $\left|\frac{(-1)^\beta b^{y_3-y_1}+(-1)^{\beta+\gamma}}{b^{y_1}-(-1)^\beta}\right|\le1$. In both cases the value 1 is possible only when a (respectively, b), equals 2, x_2 (respectively, y_1) equals 2, and x_3-x_2 (respectively, y_3-y_1) equals 1. $(a,b)=1$, so we must have

$$\left|a^{x_3-x_2}-b^{y_3-y_1}-(-1)^\gamma\right|<2, \quad (41)$$

so the left side of (41) must be zero or one. If the left side of (41) is zero, then again we can use (36) to obtain a contradiction as above. If the left side of (41) equals one, then recall $(a,b)=1$ and note that $a^{x_3-x_2}-b^{y_3-y_1}=\pm2$ is impossible when $\min(a,b)=2$. $\qquad\square$

Lemma 11. *If $\min(a,b)>6$ in (37), then $x_2=x_3-x_2$ implies $y_1\le y_3-y_1$, and also $y_1=y_3-y_1$ implies $x_2\le x_3-x_2$.*

Proof. By symmetry, it suffices to prove that $x_2=x_3-x_2$ implies $y_1\le y_3-y_1$. Assume $y_1>y_3-y_1$ and $x_2=x_3-x_2$. Since $\frac{b^{y_3-y_1}+(-1)^\gamma}{b^{y_1}-(-1)^\beta}<1$, we cannot have $\alpha=\gamma$ in (38), so that $\frac{a^{x_3-x_2}-(-1)^\gamma}{a^{x_2}-(-1)^\alpha}=1\pm\frac{2}{A\mp1}$ where $A=a^{x_2}=a^{x_3-x_2}$. Since $\frac{2}{A\mp1}+\frac{b^{y_3-y_1}+(-1)^\gamma}{b^{y_1}-(-1)^\beta}<1$, we must have $\frac{b^{y_3-y_1}+(-1)^\gamma}{b^{y_1}-(-1)^\beta}=\frac{2}{A\mp1}$, so that $b^{y_3-y_1}+(-1)^\gamma\mid2(b^{y_1}-(-1)^\beta)$. Then, using the elementary divisibility properties of $b^y\pm1$ (for

general integer y), we see that, since $b > 3$, we must have $y_3 - y_1 \mid y_1$. Let $B = b^{y_3 - y_1}$. Let $j = \frac{2y_1 - y_3}{y_3 - y_1}$, noting that j is a positive integer. Then

$$\frac{1 + \frac{(-1)^\gamma}{B}}{B^j \left(1 - \frac{(-1)^\beta}{B^{j+1}}\right)} = \frac{2}{A(1 \mp \frac{1}{A})},$$

so that, letting $k = \min(a, b) > 6$,

$$B \leq \frac{A(1 + \frac{1}{k})^2}{2(1 - \frac{1}{k^2})} = \frac{A(1 + \frac{1}{k})}{2(1 - \frac{1}{k})} \leq \frac{2}{3}A. \tag{42}$$

But from (38) we get

$$B \geq A - 2 = A(1 - \frac{2}{A}) \geq A(1 - \frac{2}{k}) \geq \frac{5}{7}A,$$

contradicting (42). $\qquad\square$

Lemma 12. *Suppose* $\min(a, b) \leq 1000$. *Then any set of solutions to (1) satisfying conditions (4) has $N < 4$.*

Proof. Assume (4) holds with $N > 3$. By symmetry, we may assume $a > b$. Let $b \leq 10^3$ and assume a is not a perfect power. Considering (2) with $(i, j) = (3, 4)$ and applying Lemma 3 and Lemma 9, we have

$$b^{y_3} \mid b^{\sigma_b(a)} (x_4 - x_3) < 10^{22} Z < 8 \cdot 10^{36}, \tag{43}$$

so $y_3 < \log(8 \cdot 10^{36}) / \log(b)$.

Choose $b \leq 1000$ and $\nu \in \{0, 1\}$ where $\nu = \beta + \gamma$ as in (31). Considering (31) and noting that $(ra, sb) = 1$, a^{x_2} must be a divisor of $b^{y_3} + (-1)^\nu$. It is easy to factor $b^{y_3} + (-1)^\nu < 8 \cdot 10^{36}$, so for each b, y_3, and ν we obtain every possible a^{x_2} hence a complete list of possible values for a and its associated x_2 exponent. Lemma 2 gives a bound $x_3 - x_2 < \log(8 \cdot 10^{14}) / \log(a)$. For each $x_3 - x_2$ within this bound and each $\mu \in \{0, 1\}$, we can solve $r = (b^{y_3} + (-1)^\nu) / (a^{x_2} h)$ and $s = (a^{x_3 - x_2} + (-1)^\mu) / h$ where $h = \gcd(a^{x_3 - x_2} + (-1)^\mu, b^{y_3} + (-1)^\nu)$.

Considering (2) with $(i, j) = (1, 3)$, we should have $r(a^{x_3} + (-1)^\eta) = sb^{y_1}(b^{y_3 - y_1} + (-1)^\theta)$ for some $\eta, \theta \in \{0, 1\}$. We now determine if there is a value $\eta \in \{0, 1\}$ for which $b \mid a^{x_3} + (-1)^\eta$, in which case we determine y_1 such that $b^{y_1} \| a^{x_3} + (-1)^\eta$. Now we see if there is a value $\theta \in \{0, 1\}$ for which $r(a^{x_3} + (-1)^\eta) = sb^{y_1}(b^{y_3 - y_1} + (-1)^\theta)$. If so, we have three solutions to (1) with $c = |ra^{x_3} - sb^{y_3}|$.

Now apply bootstrapping as outlined above; our calculations show that in each case x_4 or y_4 exceeds $8 \cdot 10^{14}$, hence there is no fourth solution. $\qquad\square$

We can reformulate (11) as

$$\left(\frac{b^{kd} + (-1)^{u+v}}{b^d + (-1)^u}, b, \frac{ab^d - (-1)^{u+v}}{h}, \frac{b^d + (-1)^u}{h}, \frac{a + (-1)^v}{h}; 0, d, 1, 0, 2, kd \right), \quad (44)$$

where $b > 1$, $a = \frac{b^{kd} + (-1)^{u+v}}{b^d + (-1)^u}$, d and k are positive integers, $h = \gcd(a + (-1)^v, b^d + (-1)^u)$, u and v are in the set $\{0, 1\}$, and if $u = 0$ then $k - v$ must be odd, or if $u = 1$ then $v = 0$. (Note that this reformulation excludes the case $u = v = 1$ with $b^d \le 3$ and also excludes the case in which k is a half integer.)

Lemma 13. *If (a, b, c, r, s) are chosen as in (44), then (1) has only the three solutions given in (44).*

Proof. Whenever (44) holds, we cannot have $r + sb^d = ra + s = c$, since then $u = v = 1$, contradicting the conditions of (44). So we must have either

$$sb^d - r = c \qquad (45)$$

or

$$ra - s = c. \qquad (46)$$

Assume there is an instance of (44) having a fourth solution (x_4, y_4). Note that no subset or associate of a subset of any of the sets of solutions listed in Theorem 1 is in the same family as a set of solutions which is an instance of (44) having a fourth solution. So by Lemma 1, we can assume (4) holds, so that

$$x_4 > 2, y_4 > kd, k > 1. \qquad (47)$$

By Lemma 12, we may assume $b > 1000$. Now $a \ge b^{(k-1)d} - b^{(k-2)d} > 8 \cdot 10^{14}$ when $b > 1000$ and $(k-1)d \ge 5$, so, by Lemma 1, we must have $k < 6$. From (44) we derive $sh \equiv \pm(k+1) \bmod rh$, so that $h \mid k + 1$, giving $h < 7$, so that we certainly have

$$\min(r, s) > 2. \qquad (48)$$

We will now use Lemma 5 to show that $d \mid y_4$. When (45) holds, we apply Lemma 5 to the solutions $(0, d)$, $(2, kd)$, and (x_4, y_4), noting that (47) gives Conditions (1.) and (3.) of Lemma 5, (45) gives Condition (2.), and (48) gives Condition (4.) (recall (4) requires $(ra, sb) = 1$). Now we can use Lemma 5 to get $(k-1)d \mid y_4 - d$. Similarly, when (46) holds, we can apply Lemma 5 to the solutions $(1, 0)$, $(2, kd)$, and (x_4, y_4) to get $kd \mid y_4$. In either case, we have $d \mid y_4$. So, in considering the set of solutions $(a, b, c, r, s; 0, d, 1, 0, 2, kd, x_4, y_4)$, we can assume $d = 1$ without loss of generality, noting that we have reformulated the meanings of b and y_4.

Thus we have $d = 1$ and $2 \le k \le 5$. We can apply the LLL basis reduction method to show that there are no solutions for $b \le 6 \cdot 10^5$. There are a handful of b values for which the LLL method fails, and we handle these by bootstrapping.

Using $a \geq b-1$ and $\max(x_4, y_4) \geq a$ we see that $y_4 = \max(x_4, y_4)$ implies $y_4 \geq a \geq b-1$, while $x_4 = \max(x_4, y_4)$ implies $a^a - (a(a+1)+1) \leq ra^{x_4} - c \leq sb^{y_4} \leq (a+1)^{y_4+1}$. In either case, we certainly have $y_4 > 100$.

Suppose $d = 1$ and $k = 4$ or 5. Then $a \geq b^3 - b^2 + b - 1 > 8 \cdot 10^{14}$ for $b > 6 \cdot 10^5$ so these cases cannot lead to a fourth solution.

Suppose $d = 1$ and $k = 3$, so $a = b^2 \mp b + 1$, $rh = b \pm 1$, $sh = b^2 \mp b + 2$, and $ch = b^3 \mp b^2 + b \mp 1$, where we take the upper sign when $u = v = 0$ and the lower sign when $u = 1$ and $v = 0$. From (44) we have

$$ra^{x_4} + c \equiv ra^2 + c \equiv 0 \bmod sb^3. \tag{49}$$

When b is even, h is odd and s is even, so

$$ra^{x_4} + c \equiv ra^2 + c \equiv 0 \bmod 2b^3. \tag{50}$$

When $b \equiv 2 \bmod 4$, then, letting $2^g || a + 1 = sh$, we have, noting $2 \nmid h$,

$$ra^{x_4} + c \equiv ra^2 + c \equiv 0 \bmod 2^{g+3}(b/2)^3. \tag{51}$$

Let n be the least number such that $a^n \equiv 1 \bmod G$ where $G = b^3$ when b is odd, $G = 2b^3$ when $b \equiv 0 \bmod 4$, and $G = 2^{g+3}(b/2)^3$ when $b \equiv 2 \bmod 4$. Then, using the elementary divisibility properties of $a^x - 1$ (for general integer x), we find that when b is odd, $n = b^2$, and when b is even, $n = 2b^2$. Now from (49), (50), and (51) we see that we have $x_4 = 2 + jb^2$ where $j \geq 0$ is an integer. (We note that when b is even, j is even, although for this case we will not need this.) We have, for some integer M,

$$a^{x_4} = (b^2 \mp b + 1)^{x_4} = 1 + x_4(b^2 \mp b) + \frac{x_4(x_4 - 1)}{2}(b^2 \mp b)^2 +$$
$$\frac{x_4(x_4 - 1)(x_4 - 2)}{6}(b^2 \mp b)^3 + M(b^2 \mp b)^4$$

and thus

$$rha^{x_4} + ch = (b \pm 1)\left(1 + (2 + jb^2)(b^2 \mp b) + \frac{(2 + jb^2)(1 + jb^2)}{2}(b^2 \mp b)^2 +\right.$$
$$\left.\frac{(2 + jb^2)(1 + jb^2)jb^2}{6}(b^2 \mp b)^3 + M(b^2 \mp b)^4\right) + (b^3 \mp b^2 + b \mp 1).$$

Collecting like powers we find (recalling $y_4 > 100$)

$$rha^{x_4} + ch = (2 - j)b^3 + \frac{M_1}{6}b^4 = M_2 b^{100}$$

where M_1 and M_2 are integers. Thus, $j = wb/6 + 2$ for some integer $w \geq 0$, so that, when $w > 0$, $x_4 \geq 2 + 2b^2 + b^3/6 > 8 \cdot 10^{14}$ since $b > 6 \cdot 10^5$, contradicting Lemma 1.

So $w = 0$ and $x_4 = 2 + 2b^2 < 8 \cdot 10^{14}$, so that $b < 2 \cdot 10^7$ and $c < 10^{22}$. We apply (8) for each b with $6 \cdot 10^5 < b \le 2 \cdot 10^7$. The calculations show that for every b in this range, (8) never gives an integral value for y_4 within 25 places of accuracy, hence $x_4 = 2 + 2b^2$ cannot lead to a fourth solution. Thus, when $d = 1$ and $k = 3$, (44) cannot have a fourth solution.

Suppose $d = 1$, $k = 2$, $u = 1$, and $v = 0$, so $a = b+1$, $rh = b-1$, $sh = b+2$, and $ch = b^2 + b + 1$. From (44) we have $ra^{x_4} + c \equiv ra^2 + c \equiv 0 \bmod sb^2$. Now proceeding as in the case $k = 3$, we derive $x_4 = 2 + jb$ where j is even when b is even. Write $x_4 = 2 + e_1 b + e_2 b^2$ with $0 \le e_1 < b$, noting $2 | e_1$ when $2 | b$. For some integer M we have

$$rha^{x_4} + ch = (b-1)\left(1 + (2 + e_1 b + e_2 b^2)b + \frac{(2 + e_1 b + e_2 b^2)(1 + e_1 b + e_2 b^2)}{2}b^2\right.$$
$$\left. + Mb^3\right) + (b^2 + b + 1).$$

Noting $\big(x_4(x_4-1)/2\big) - 1$ is an integer divisible by b, and noting $y_4 > 100$, we have, for some integers M_1 and M_2,

$$rha^{x_4} + ch = (2 - e_1)b^2 + M_1 b^3 = M_2 b^{100}$$

so that $e_1 \equiv 2 \bmod b$, so $e_1 = 2$.

Write $x_4 = 2 + 2b + e_2 b^2$ with $0 \le e_2$. For some integer M we have

$$rha^{x_4} + ch = (b-1)\left(1 + (2 + 2b + e_2 b^2)b + \frac{(2 + 2b + e_2 b^2)(1 + 2b + e_2 b^2)}{2}b^2\right.$$
$$\left. + \frac{(2 + 2b + e_2 b^2)(1 + 2b + e_2 b^2)(2b + e_2 b^2)}{6}b^3 + Mb^4\right) + (b^2 + b + 1).$$

From this we obtain, for some integers M_1 and M_2,

$$rha^{x_4} + ch = -e_2 b^3 + \frac{M_1}{6}b^4 = M_2 b^{100}.$$

Thus $e_2 = wb/6$ for some integer $w \ge 0$. As before, we show $w = 0$, so it remains only to deal with $x_4 = 2 + 2b$. We have $c \le b^2 + b + 1 < 6.4 \cdot 10^{29} + 8 \cdot 10^{14} + 1$, so that we can use (8), which implies, for some ϵ such that $|\epsilon| < 10^{-2000}$,

$$y_4 = \frac{1}{\log(b)}\left(x_4 \log(b+1) - \log(b+2) + \log(b-1)\right) + \epsilon$$
$$= \frac{1}{\log(b)}\left(x_4 \log(b) + x_4 \log\left(1 + \frac{1}{b}\right) - \log\left(1 + \frac{2}{b}\right) + \log\left(1 - \frac{1}{b}\right)\right) + \epsilon. \quad (52)$$

From this one gets

$$y_4 - x_4 = \frac{1}{\log(b)} \left(x_4 \log \left(1 + \frac{1}{b} \right) - \log \left(1 + \frac{2}{b} \right) + \log \left(1 - \frac{1}{b} \right) \right) + \epsilon.$$

Plugging in $x_4 = 2 + 2b$ into $x_4 \log \left(1 + \frac{1}{b} \right) - \log \left(1 + \frac{2}{b} \right) + \log \left(1 - \frac{1}{b} \right)$ and taking Taylor series in terms of $z = 1/b < 1/1000$, one can show that the Taylor series $2 - 2z + 7z^2/6 - 17z^3/6 + \ldots$ is alternating and has terms decreasing in absolute value, hence, since by Lemma 1 we have $\log(b) < 35$,

$$0 < \frac{1}{\log(b)} \left(2 - 2\frac{1}{b} \right) + \epsilon < y_4 - x_4 < \frac{2}{\log(b)} + \epsilon < 1$$

which contradicts y_4 being an integer. So $x_4 = 2 + 2b$ is not possible in this case, completing the proof that no fourth solution is possible when $d = 1$, $k = 2$, $u = 1$, $v = 0$.

Suppose $d = 1$, $k = 2$, $u = 0$, and $v = 1$. Then $a = b - 1$ (recall we have reformulated b to represent b^d), and reversing the roles of a and b, this becomes identical to the case just completed.

Thus (44) does not have a fourth solution. $\square$

Lemma 14. *Any set of solutions to (1) satisfying conditions (4) has $N < 4$.*

Proof. By Lemma 12 we can assume $a > b > 1000$. Suppose (1) has four or more solutions and (4) holds.

From (34) we get

$$a^{x_3 - x_2} + \frac{(-1)^\alpha a^{x_3 - x_2} - (-1)^\gamma a^{x_2}}{a^{x_2} - (-1)^\alpha} = b^{y_3 - y_1} + \frac{(-1)^\beta b^{y_3 - y_1} + (-1)^{\beta + \gamma}}{b^{y_1} - (-1)^\beta}$$

and so

$$|a^{x_3 - x_2} - b^{y_3 - y_1}| \leq \frac{a^{x_3 - x_2}}{a^{x_2} - 1} + 1 + \frac{1}{a^{x_2} - 1} + \frac{b^{y_3 - y_1}}{b^{y_1} - 1} + \frac{1}{b^{y_1} - 1}. \qquad (53)$$

Then the right side of (53) can be bounded as follows:

$$|a^{x_3 - x_2} - b^{y_3 - y_1}| \leq \frac{a^{x_3 - x_2}}{b} + 1 + \frac{1}{b} + \frac{b^{y_3 - y_1}}{b - 1} + \frac{1}{b - 1}, \qquad (54)$$

and noting that $\max(a^{x_3 - x_2}, b^{y_3 - y_1}) \geq b$,

$$|a^{x_3 - x_2} - b^{y_3 - y_1}| \leq \max(a^{x_3 - x_2}, b^{y_3 - y_1}) \left(\frac{1}{b} + \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b - 1} + \frac{1}{(b - 1)b} \right)$$

so

$$|a^{x_3 - x_2} - b^{y_3 - y_1}| < \max(a^{x_3 - x_2}, b^{y_3 - y_1}) \left(\frac{3}{b - 1} \right).$$

Since $b > 4$,

$$\left(\left(1 - \frac{3}{b-1}\right) b^{y_3 - y_1}\right)^{\frac{1}{x_3 - x_2}} < a < \left(\left(1 - \frac{3}{b-1}\right)^{-1} b^{y_3 - y_1}\right)^{\frac{1}{x_3 - x_2}}. \tag{55}$$

Since $a > b$, this implies $x_3 - x_2 \le y_3 - y_1$.

Since $1000 < b$ and $b^{y_3 - y_1} < 8 \cdot 10^{14}$ we must have $y_3 - y_1 \le 4$. We divide this proof into a number of cases:

$$2 \le x_3 - x_2 < y_3 - y_1 \le 4,$$

$$3 \le x_3 - x_2 = y_3 - y_1 \le 4$$

$$x_3 - x_2 = y_3 - y_1 = 2$$

$$x_3 - x_2 = 1, y_1 \ge y_3 - y_1$$

$$x_3 - x_2 = 1, y_1 < y_3 - y_1 = 3, 4$$

$$x_3 - x_2 = 1, 1 = y_1 < y_3 - y_1 = 2.$$

In each case, a key idea is to use (53).

If $2 \le x_3 - x_2 \le y_3 - y_1 \le 4$, noting that $\max(a^{x_3 - x_2}, b^{y_3 - y_1}) \ge b^2$, (54) implies

$$|a^{x_3 - x_2} - b^{y_3 - y_1}| \le \max(a^{x_3 - x_2}, b^{y_3 - y_1}) \left(\frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \frac{1}{b-1} + \frac{1}{(b-1)b^2}\right)$$

so

$$|a^{x_3 - x_2} - b^{y_3 - y_1}| < \max(a^{x_3 - x_2}, b^{y_3 - y_1}) \left(\frac{2}{b-2}\right).$$

Since $b > 4$,

$$\left(\left(1 - \frac{2}{b-2}\right) b^{y_3 - y_1}\right)^{\frac{1}{x_3 - x_2}} < a < \left(\left(1 - \frac{2}{b-2}\right)^{-1} b^{y_3 - y_1}\right)^{\frac{1}{x_3 - x_2}}. \tag{56}$$

Suppose $2 \le x_3 - x_2 < y_3 - y_1 \le 4$. Given $y_3 - y_1$ and $x_3 - x_2$ with these bounds, consider each b with $1000 < b < (8 \cdot 10^{14})^{\frac{1}{y_3 - y_1}}$. Then for each a satisfying (56) with $\gcd(a, b) = 1$, we consider (2) with $(i, j) = (3, 4)$ and apply Lemma 3 to get $b^{y_3} | b^{\sigma_b(a)}(x_4 - x_3) < b^{\sigma_b(a)} 8 \cdot 10^{14}$. Thus, $y_3 < \sigma_b(a) + \log(8 \cdot 10^{14})/\log(b)$. From (31) $a^{x_2} | b^{y_3} + (-1)^{\beta + \gamma}$. For each $y_3 < \sigma_b(a) + \log(8 \cdot 10^{14})/\log(b)$ and each β, $\gamma \in \{0, 1\}$, let $x_{2,max}$ be the largest power of a dividing $b^{y_3} + (-1)^{\beta + \gamma}$. If $x_{2,max} > 0$ then for each $1 \le x_2 \le x_{2,max}$ and each $\alpha \in \{0, 1\}$, we set $y_1 = y_3 - (y_3 - y_1)$, $r = (b^{y_1} - (-1)^\beta)/h$, and $s = (a^{x_2} - (-1)^\alpha)/h$ where $h = \gcd(a^{x_2} - (-1)^\alpha, b^{y_1} - (-1)^\beta)$. Let $c = (-1)^\alpha r + s b^{y_1}$. If $c = r a^{x_2} + (-1)^\beta s = (-1)^\gamma (r a^{x_3} - s b^{y_3})$ then we have

three solutions to (1). Our calculations show that for $(x_3 - x_2, y_3 - y_1) = (2, 3)$, $(2, 4)$, or $(3, 4)$, only two closely related cases of three solutions occur:

$$(a, b, c, r, s; x_1, y_1, x_2, y_2, x_3, y_3) = (56744, 1477, 83810889, 1478, 56743; 0, 1, 1, 0, 3, 4),$$
$$(56745, 1477, 41906182, 739, 28373; 0, 1, 1, 0, 3, 4).$$

We apply bootstrapping to these; our calculations show that either there is no fourth solution or else $y_4 > 8 \cdot 10^{14}$, impossible, so these two sets of three solutions do not extend to a fourth solution.

Suppose $x_3 - x_2 = y_3 - y_1 = z$ where $z = 3$ or 4. From (56) we obtain the impossibility

$$b < a < \left(1 - \frac{2}{b-2}\right)^{\frac{-1}{z}} b \le \left(1 + \frac{2}{b-4}\right)^{1/3} b < b + 1. \tag{57}$$

Suppose $x_3 - x_2 = y_3 - y_1 = 2$. We consider several subcases.

Suppose $x_2 \ge x_3 - x_2 = 2$ and $y_1 \ge y_3 - y_1 = 2$. Then, since $b > 4$, (53) implies the impossibility

$$a^2 - b^2 < 1 + \frac{1}{b} + 1 + \frac{1}{b} + 1 + \frac{1}{b-1} + \frac{1}{b-1} < 4.$$

Suppose $x_2 \ge x_3 - x_2 = 2$, $1 = y_1 < y_3 - y_1 = 2$. Then, since $b > 4$, (53) implies the impossibility

$$2b + 1 \le a^2 - b^2 < 1 + \frac{1}{b} + 1 + \frac{1}{b} + \frac{b^2}{b-1} + \frac{1}{b-1} < b + 4.$$

Suppose $1 = x_2 < x_3 - x_2 = 2$, $y_1 \ge y_3 - y_1 = 2$. Then, since $b > 4$, (53) implies the impossibility

$$2a - 1 \le a^2 - b^2 < \frac{a^2}{a-1} + 1 + \frac{1}{a-1} + 1 + \frac{1}{b-1} + \frac{1}{b-1} < a + 4.$$

Suppose $1 = x_2 < x_3 - x_2 = 2$, $1 = y_1 < y_3 - y_1 = 2$. Then (56) shows that

$$b < a < \frac{b}{\sqrt{1 - \frac{2}{b-2}}} < b\left(1 + \frac{1}{b-4}\right) < b + 2$$

for $b > 8$ so $a = b + 1$. Now (34) implies

$$(b+1)((b+1)^2 - (-1)^\gamma)(b - (-1)^\beta) - (b + 1 - (-1)^\alpha)(b^3 + (-1)^{\beta+\gamma}) = 0. \tag{58}$$

Expanding in powers of b, one gets a polynomial $q_3 b^3 + q_2 b^2 + q_1 b + q_0$ where each coefficient satisfies $|q_i| \le 7$ and $|q_2| > 0$. No such polynomial can have a zero for $b > 1000$, hence (58) is impossible, so this case cannot lead to a fourth solution.

So we may now assume $x_3 - x_2 = 1$.

Consider first the case $x_3 - x_2 = 1$ and $y_1 \geq y_3 - y_1$. By Lemmas 10 and 11, we can assume $y_1 = y_3 - y_1$ and $x_2 = x_3 - x_2 = 1$, in which case (38) is possible only when $\alpha = \gamma \neq \beta$, so that we have (44) with $k = 2$, $u = \alpha$, and $v = \beta$, which has no fourth solution by Lemma 13.

Consider next the case $x_3 - x_2 = 1$ and $y_1 < y_3 - y_1$. Since $x_2 \geq x_3 - x_2 = 1$, (53) can be bounded as

$$|a - b^{y_3 - y_1}| \leq 1 + \frac{1}{b} + 1 + \frac{1}{b} + \frac{b^{y_3 - y_1}}{b - 1} + \frac{1}{b - 1}.$$

Since $y_1 < y_3 - y_1$, we have

$$|a - b^{y_3 - y_1}| \leq b^{y_3 - y_1} \left(\frac{1}{b^2} + \frac{1}{b^3} + \frac{1}{b^2} + \frac{1}{b^3} + \frac{1}{b - 1} + \frac{1}{(b - 1)b^2} \right),$$

and therefore

$$|a - b^{y_3 - y_1}| \leq b^{y_3 - y_1} \left(\frac{1}{b - 3} \right). \tag{59}$$

One can now derive

$$\left(1 - \frac{1}{b - 3} \right) b^{y_3 - y_1} < a < \left(1 + \frac{1}{b - 3} \right) b^{y_3 - y_1}. \tag{60}$$

If $y_3 - y_1 = 3$ or 4, then for each b with $1000 < b < (8 \cdot 10^{14})^{\frac{1}{y_3 - y_1}}$, each y_1 with $y_1 \leq y_3 - y_1$, and each $\beta, \gamma \in \{0, 1\}$, we can calculate $y_3 = y_1 + (y_3 - y_1)$ and then factor $b^{y_3} + (-1)^{\beta + \gamma}$. Since $a^{x_2} \mid b^{y_3} + (-1)^{\beta + \gamma}$, we now consider each factor $a \mid b^{y_3} + (-1)^{\beta + \gamma}$ that satisfies (60), and calculate the maximal value of x_2 such that $a^{x_2} \mid b^{y_3} + (-1)^{\beta + \gamma}$; call this $x_{2, max}$. For each x_2 up to $x_{2, max}$, let $x_3 = x_2 + 1$. For each $\alpha \in \{0, 1\}$ and for the chosen β, set $r = (b^{y_1} - (-1)^{\beta})/h$, $s = (a^{x_2} - (-1)^{\alpha})/h$, where $h = \gcd(a^{x_2} - (-1)^{\alpha}, b^{y_1} - (-1)^{\beta})$. We must have $(-1)^{\alpha} r + s b^{y_1} = r a^{x_2} + (-1)^{\beta} s = (-1)^{\gamma} (r a^{x_3} - s b^{y_3})$. Our calculations now show that the only possible sets of three solutions belong to the infinite class (44); by Lemma 13, none of these extends to four solutions.

Consider now the case $x_3 - x_2 = 1$ and $1 = y_1 < y_3 - y_1 = 2$. We still have (60) so $\left(1 - \frac{1}{b - 3} \right) b^2 < a < \left(1 + \frac{1}{b - 3} \right) b^2$. Now $a^{x_2} \mid b^3 + (-1)^{\beta + \gamma}$ but $a^2 > \left(1 - \frac{1}{b - 3} \right)^2 b^4 > b^3 + 1$, so $x_2 = 1$.

If $\alpha \neq \gamma$ then, using (34) and clearing denominators, we get

$$a(a + (-1)^{\alpha})(b - (-1)^{\beta}) = (b^3 + (-1)^{\beta + \gamma})(a - (-1)^{\alpha}).$$

Since $\gcd(a - (-1)^{\alpha}, a + (-1)^{\alpha}) \leq 2$, we must have $a - (-1)^{\alpha} \mid 2(b - (-1)^{\beta})$ so $a \leq 2b + 3$. This contradicts $\left(1 - \frac{1}{b - 3} \right) b^2 < a$.

If $\alpha = \gamma = 0$ then, using (34) and dividing out $a - 1$,

$$a(b - (-1)^\beta) = b^3 + (-1)^\beta. \tag{61}$$

Since $\gcd(b - (-1)^\beta, b^3 + (-1)^\beta) \leq 2$ and $b - (-1)^\beta > 2$, this is impossible.

Therefore $\alpha = \gamma = 1$. In this case, using (34) as before and dividing out $a + 1$, we get (61), so

$$a = b^2 + (-1)^\beta b + 1.$$

So we have a member of the infinite class (44) with $d = 1$, $k = 3$, $u = \beta + 1$, $v = 0$; by Lemma 13 this has no fourth solution. $\qquad\square$

5. Case (5) $0 = x_1 < x_2 < x_3 < x_4$, $0 = y_1 = y_2 < y_3 < y_4$

Suppose (5) holds. To fix notation, let $c = s - (-1)^\alpha r = ra^{x_2} - s = (-1)^\beta(ra^{x_3} - sb^{y_3}) = (-1)^\gamma(ra^{x_4} - sb^{y_4})$ for some α, β, $\gamma \in \{0, 1\}$. Considering (2) with $(i, j) = (1, 2)$, we see that $r(a^{x_2} + (-1)^\alpha) = 2s$, so $r = 1$ or 2. Applying (2) with $(i, j) = (1, 3)$ and $(2, 3)$, we find that $r(a^{x_3} + (-1)^{\alpha+\beta}) = s(b^{y_3} + (-1)^\beta)$ and $ra^{x_2}(a^{x_3-x_2} + (-1)^{\beta+1}) = s(b^{y_3} + (-1)^{\beta+1})$. We find that, for any instance of case (5), we have

$$r \in \{1, 2\}, 2 \mid a - r, s = \frac{r(a^{x_2} + (-1)^\alpha)}{2}, c = s + (-1)^{\alpha+1}r, b^{y_3} =$$
$$\frac{2(a^{x_3} + (-1)^{\alpha+\beta})}{a^{x_2} + (-1)^\alpha} - (-1)^\beta. \tag{62}$$

Since b^{y_3} is an integer, we can use the elementary divisibility properties of $a^x \pm 1$ to see that $x_2 \mid x_3$ except possibly when $x_2 = 2$ and $a \leq 3$. And, since (62) holds when x_3 is replaced by x_i where $3 < i \leq N$, we have, whenever either $x_2 > 2$ or $a > 3$,

$$x_2 \mid x_i, 3 \leq i \leq N. \tag{63}$$

Note that (62) corresponds to the infinite family (16).

Lemma 15. *Suppose $a \leq 134000$. Then any set of solutions to (1) satisfying conditions (5) has $N < 4$, except for $(a, b, c, r, s; x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = (2, 7, 5, 2, 3; 0, 0, 2, 0, 3, 1, 9, 3)$ or $(2, 5, 3, 2, 1; 0, 0, 1, 0, 2, 1, 6, 3)$.*

Proof. Suppose (5) holds and $a \leq 134000$.

We first get bounds on x_2 and $x_3 - x_2$. Lemma 2 gives $a^{x_3-x_2} \leq Z$ so $x_3 - x_2 < \log(8 \cdot 10^{14})/\log(a)$. If $x_2 > 2$, then, by (63), $x_2 \mid x_3 - x_2 < \log(8 \cdot 10^{14})/\log(a)$.

Given a, x_2, $x_3 - x_2$, and choosing α, $\beta \in \{0, 1\}$, and $r \in \{1, 2\}$ with $2 \mid a - r$, we have (62). Suppose for some choice of a, α, β, x_2, and $x_3 - x_2$, we find that $\frac{2(a^{x_3}+(-1)^{\alpha+\beta})}{a^{x_2}+(-1)^\alpha} - (-1)^\beta$ is an integer; it is b^{y_3}, so we know b and its

associated y_3 (without loss of generality we may assume b is not a perfect power). We now have three solutions to (1) with this a, b, c, r, s and choice of signs. We use LLL (then bootstrapping if LLL fails) to show no fourth solutions exist except when $(a, b, c, r, s; x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = (2, 7, 5, 2, 3; 0, 0, 2, 0, 3, 1, 9, 3)$ or $(2, 5, 3, 2, 1; 0, 0, 1, 0, 2, 1, 6, 3)$. A few cases with $a = 2$ and $b = (2^x + 1)/3$ require a further elementary argument modulo 8 to eliminate a fourth solution. □

Lemma 16. *Any set of solutions to (1) satisfying conditions (5) has $N < 4$, except for $(a, b, c, r, s; x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = (2, 7, 5, 2, 3; 0, 0, 2, 0, 3, 1, 9, 3)$ or $(2, 5, 3, 2, 1; 0, 0, 1, 0, 2, 1, 6, 3)$.*

Proof. Suppose (5) holds with $N > 3$ and further assume we do not have $(a, b, c, r, s; x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = (2, 7, 5, 2, 3; 0, 0, 2, 0, 3, 1, 9, 3)$ or $(2, 5, 3, 2, 1; 0, 0, 1, 0, 2, 1, 6, 3)$. By Lemma 15 we may assume $a > 134000$ so that (63) holds. From (62) we have $r = 1$ if a is odd, $r = 2$ if a is even. If $s \leq 2$, then, considering the solution (x_2, y_2), we must have $a \leq 5$, contradicting Lemma 15. So we can assume $s > 2$. We can also assume $ra^{x_2} > 2$. And we certainly can assume that the solutions $(x_1, y_1), \ldots, (x_N, y_N)$ include all solutions to (1) for (a, b, c, r, s), because any instance of (5) with $N > 3$ and $a > 134000$ must be a subset of an instance of (5) (perhaps itself) since it cannot be in the same family as a subset (or an associate of a subset) of one of the sets of solutions listed in Theorem 1. Now we can apply Lemma 5 to the solutions (x_2, y_2), (x_3, y_3), (x_4, y_4) to get

$$y_3 \mid y_4. \tag{64}$$

Since we have $a > 134000$, we see that $a^{x_3 - x_2} > 8 \cdot 10^{14}$ when $x_3 - x_2 \geq 3$, hence, by Lemma 2, we have $x_3 - x_2 \leq 2$. By (63) we see that, in considering the set of solutions $(a, b, c, r, s; 0, 0, x_2, 0, x_3, y_3, x_4, y_4)$, we can assume $x_2 = 1$ without loss of generality, noting that we are reformulating a, x_3 and x_4. Now one can see that the only possibilities satisfying (62) have $x_2 = 1$ with

$x_3 - x_2$	α	β
2	0	0
2	1	0
1	0	1
1	1	0

Note that $c \leq r + s < 8 \cdot 10^{14} + 2$ by Lemma 1, and $\pm c = ra^{x_4} - sb^{y_4}$. By Lemma 2, $134000 < a < \max(x_4, y_4)$, so certainly $100 < x_4$, and we have $sb^{y_4} \pm c \equiv 0 \bmod a^{100}$. We will expand y_4 in powers of a; we have $y_4 < 8 \cdot 10^{14} < 134000^3 < a^3$, so we do not need powers of a higher than two. By (64), we can assume $y_3 = 1$ without loss of generality, noting that in doing so we have reformulated b and y_4. Thus we are considering the following set of solutions: $(a, b, c, r, s; 0, 0, 1, 0, x_3, 1, x_4, y_4)$, noting the reformulations of variables as indicated above.

Case 1: Suppose $x_2 = 1$ and $x_3 - x_2 = 2$. Then $\beta = 0$. Using (62) and taking the upper sign when $\alpha = 0$ and the lower sign when $\alpha = 1$, we have $b = 2a^2 \mp 2a + 1$ and, for either choice of the parity of a,

$$(a \mp 1) + (a \pm 1)b^{y_4} \equiv (a \mp 1) + (a \pm 1)b \equiv 0 \bmod 2a^3$$

so that $(a \pm 1)(b^{y_4-1} - 1) \equiv 0 \bmod 2a^3$, so that, for either choice of the parity of a, $b^{y_4-1} \equiv 1 \bmod 2a^3$ which requires $a^2 \mid y_4 - 1$, by Lemma 1 of [13]. So, letting $y_4 = 1 + ja^2$ for some positive integer j, and letting M_i be an integer for $1 \le i \le 3$,

$$\frac{a \mp 1}{2} + \frac{a \pm 1}{2}(2a^2 \mp 2a + 1)^{y_4} = \frac{a \mp 1}{2} +$$

$$\frac{a \pm 1}{2}\left(1 + y_4(2a^2 \mp 2a) + \frac{y_4(y_4 - 1)}{2}(2a^2 \mp 2a)^2 + \frac{y_4(y_4 - 1)(y_4 - 2)}{6}(2a^2 \mp 2a)^3\right.$$

$$\left. + \frac{y_4(y_4 - 1)(y_4 - 2)(y_4 - 3)}{24}(2a^2 \mp 2a)^4 + 2M_1 a^5\right) = M_2 a^{100} \quad (65)$$

since $x_4 > 100$. This simplifies to

$$(1 - j)a^3 \pm ja^4 + \frac{M_3}{3}a^5 = M_2 a^{100}. \quad (66)$$

From this we see that $j = 1 + wa/3$ for some integer $w \ge 0$. If $w = 0$ then (66) becomes impossible. And if $w > 0$, then $y_4 > \frac{a}{3}a^2 > 8 \cdot 10^{14}$ when $a > 134000$, giving a contradiction to Lemma 1.

Case 2: Suppose $x_2 = x_3 - x_2 = 1$, so that $\alpha \ne \beta$ and $b = 2a - (-1)^\alpha$. Considering the solution $(x_3, y_3) = (2, 1)$, we have, for either choice of the parity of a,

$$(a - (-1)^\alpha) - (-1)^\alpha(a + (-1)^\alpha)b \equiv 0 \bmod 2a^2. \quad (67)$$

Let $t = 1$ if y_4 is odd and $\alpha = 0$, otherwise let $t = 0$. Then considering the solution (x_4, y_4) we have

$$(a - (-1)^\alpha) + (-1)^t(a + (-1)^\alpha)b^{y_4} \equiv 0 \bmod 2a^{100} \quad (68)$$

since $x_4 > 100$. Combining (67) and (68) we have $(a + (-1)^\alpha)(b^{y_4-1} + (-1)^{t+\alpha}) \equiv 0 \bmod 2a^2$, so that, for either choice of the parity of a, $b^{y_4-1} + (-1)^{t+\alpha} \equiv 0 \bmod 2a^2$, which requires $a \mid y_4 - 1$ by Lemma 1 of [13]. So we can write $y_4 = 1 + e_1 a + e_2 a^2$ where $0 \le e_1 < a$.

Assume first $\alpha = 0$. Let M_1 and M_2 be integers. Then using (68) we have

$$\frac{a - 1}{2} + \frac{a + 1}{2}\left(1 - y_4(2a) + \frac{y_4(y_4 - 1)}{2}(2a)^2 -\right.$$

$$\left. \frac{y_4(y_4 - 1)(y_4 - 2)}{6}(2a)^3 + 2M_1 a^4\right) = M_2 a^{100}. \quad (69)$$

This simplifies to

$$(-1 - e_1)a^2 = M_3 a^3$$

for some integer M_3. Thus $e_1 = a - 1$. So $y_4 = 1 + (a-1)a + e_2 a^2$. Now from (69) we obtain

$$(-1 - e_2)a^3 = \frac{M_4}{3}a^4$$

for some integer M_4, so that $e_2 = (wa/3) - 1$ for some integer $w \geq 1$. But then $y_4 \geq 1 + (a-1)a + \left(\frac{a}{3} - 1\right)a^2 > 8 \cdot 10^{14}$ when $a > 134000$, violating Lemma 1.

So we can assume $\alpha = 1$. We define M_i, $1 \leq i \leq 6$ to be integers. From (68) we derive

$$\frac{a+1}{2} + \frac{a-1}{2}\left(1 + y_4(2a) + \frac{y_4(y_4-1)}{2}(2a)^2 + 2M_1 a^3\right) = M_2 a^{100}.$$

This simplifies to

$$(1 - e_1)a^2 = M_3 a^3.$$

Thus, $e_1 = 1$. Write $y_4 = 1 + a + e_2 a^2$. Then

$$\frac{a+1}{2} + \frac{a-1}{2}\left(1 + y_4(2a) + \frac{y_4(y_4-1)}{2}(2a)^2 + \frac{y_4(y_4-1)(y_4-2)}{6}(2a)^3\right.$$

$$\left. + \frac{y_4(y_4-1)(y_4-2)(y_4-3)}{24}(2a)^4 + 2M_4 a^5\right) = M_5 a^{100}.$$

This simplifies to

$$-e_2 a^3 + \frac{2}{3}a^4 = \frac{M_6}{3}a^5 \tag{70}$$

so that $e_2 = wa/3$ for some integer $w \geq 0$. If $w = 0$ then (70) becomes impossible. And if $w > 0$, $y_4 \geq 1 + a + \frac{a}{3}a^2 > 8 \cdot 10^{14}$ since $a > 134000$, violating Lemma 1.

This completes the proof of Lemma 16. □

Theorem 1 follows immediately from Lemmas 1, 8, 14, and 16.

Comment on a further use of Lemma 10: If (37) holds, then, by Lemma 10, we can assume without loss of generality that $y_1 \leq y_3 - y_1$. If $N > 3$, applying Lemma 2 we get

$$c \leq r + sb^{y_1} \leq r + sb^{y_3 - y_1} \leq (Z+1) + (Z+1)Z = (Z+1)^2. \tag{71}$$

It follows from the proof of Theorem 1.2 of [14] (Case 2, equation (4.19), which does not depend on the assumption $a > b$ used elsewhere in Case 2), that we must have

$$Z < 0.9 + \frac{\log(c)}{\log(2)} + 1.6901816335 \cdot 10^{10} \cdot \log(Z+1)\log(Z)\log(1.5eZ). \tag{72}$$

Using (71) in (72) we get $Z < 7.07 \cdot 10^{14}$, completing a simpler and shorter proof of Theorem 1.2 of [14].

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