OVERLAP CYCLES FOR PERMUTATIONS: NECESSARY AND SUFFICIENT CONDITIONS

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Abstract

Universal cycles are generalizations of de Bruijn cycles and Gray codes that were introduced originally by Chung, Diaconis, and Graham in 1992. They have been developed by many authors since, for various combinatorial objects such as strings, subsets, and designs. Certain classes of objects do not admit universal cycles without either a modification of either the object representation or a generalization of the listing structure. One such generalization of universal cycles, which require almost complete overlap of consecutive words, is s-overlap cycles, which relax such a constraint. In this paper we study permutations and some closely related classes of strings, namely juggling sequences and functions. We prove the existence of s-overlap cycles for these objects, as they do not always lend themselves to the universal cycle structure.

1. Introduction

Listing structures for combinatorial objects are quickly becoming useful in more and more interesting applications. Gray codes, first defined in 1947 by Frank Gray [6], are used in many different places from position encoders [11] to genetic algorithms [4]. More recently, universal cycles are being used in areas such as rank modulation for multilevel flash memories [12]. However overlap cycles are still being explored and have the potential to be useful in many applications.

An **s-overlap cycle**, or **s-ocycle**, is an ordering of a set of objects $\mathcal{C}$, each represented as a string of length n. The ordering requires that object $b = b_0b_1 \dots b_{n-1}$ follow object $a = a_0a_1 \dots a_{n-1}$ only if $a_{n-s}a_{n-s+1} \dots a_{n-1} = b_0b_1 \dots b_{s-1}$. Ocycles were introduced by Godbole, Knisley, and Norwood in 2010 [5]. **Universal cycles**, or **ucycles** are $(n-1)$-ocycles and were originally introduced in 1992 by Chung, Diaconis, and Graham [2].

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To find s-ocycles and ucycles on a set of strings, most proofs employ the same method. For a given string $X = x_1x_2 \dots x_n$, let $X^{s-} = x_1x_2 \dots x_s$ denote the s-prefix of X and $X^{s+} = x_{n-s+1}x_{n-s+1} \dots x_n$ denote the s-suffix of X. The first step in the proof is to construct the **transition digraph** for the set of strings as follows. Vertices represent s-prefixes and s-suffixes of strings (the overlaps), while each edge represents a string, traveling from its s-prefix to its s-suffix. Note that the transition digraph is a directed multigraph in which an Euler tour (a closed walk in which every edge is traversed exactly once) corresponds bijectively to an s-ocycle. To prove the existence of an Euler tour, we use the following well-known theorem.

Theorem 1.1. ([13], p. 60) *A directed graph G is Eulerian if and only if it is both balanced and weakly connected.*

In this paper, we consider using the ocycle listing structure for permutations, as well as functions and juggling sequences. We represent permutations of an n-set $\{0, 1, \dots, n-1\}$ as a string $\Pi = \pi_0\pi_1 \dots \pi_{n-1}$, where the functional representation is used, i.e., $\pi(0) = \pi_0$. Closely related to permutations, juggling sequences have been an active research area since the 1980's [1]. These sequences are used to determine in what patterns a fixed set of balls can be juggled, where only one ball may be caught and/or thrown at a time.

Definition 1.2. ([3]) A **juggling sequence** is a string $T = t_0t_1 \dots t_{n-1}$ where each t_i is nonnegative and such that

$$|\{i + t_i \pmod n \mid 0 \le i \le n-1\}| = n.$$

This sequence illustrates that at time i, we should throw a ball high enough that it is in the air for t_i beats, or to **height** t_i. The number of **balls** used in a given juggling sequence $T = t_0t_1 \dots t_{n-1}$ is given by

$$b = \frac{1}{n}\sum_{i=0}^{n-1} t_i.$$

The **period** of a juggling sequence $t_0t_1 \dots t_{n-1}$ is n, the length of the string.

An alternative definition considers the corresponding **permutation sequence** for a juggling sequence. Given a string $T = t_0t_1 \dots t_{n-1}$, the permutation sequence is the string $\Pi_T = \pi_0\pi_1 \dots \pi_{n-1}$ where $\pi_i = t_i + i \pmod n$. Then Π_T is a permutation of the n-set $\{0, 1, \dots, n-1\}$ if and only if T is a valid juggling sequence. From this definition it is clear that there is a very close relationship between juggling sequences and permutations, which is further illustrated by the similarity of the results explored in this paper. The permutation sequence clarifies the point that a juggler cannot catch or throw two objects simultaneously.

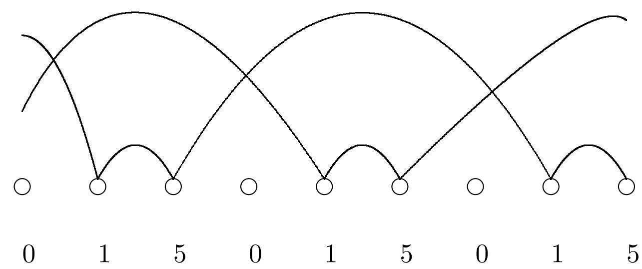

Figure 1: Juggling Diagram for Sequence 015

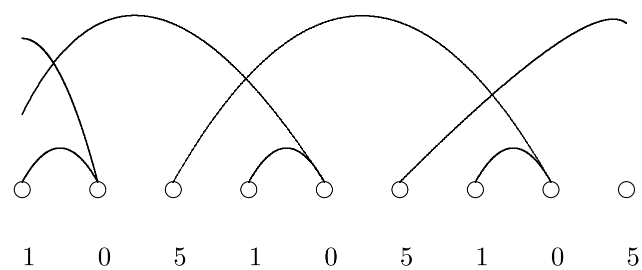

Figure 2: Juggling Diagram for Sequence 105

For example, when $n = 3$ and $b = 2$ the pattern 015 is valid. We can see this by drawing the **juggling diagram**, shown in Figure 1. However, the sequence 105 is not valid, as shown in Figure 2. In this second example, note that on the second beat we are required two catch two balls simultaneously - an operation that is not allowed.

In [3], Chung and Graham show that it is not always possible to find a single universal cycle that contains all juggling sequences of period n and at most b balls. However they do prove that we can use several disjoint universal cycles to cover each sequence exactly once. This result is closely related to the long studied problem of universal cycles for permutations. Since ucycles for permutations are not possible using the standard representation, a modification of the representation can often provide an effective solution [9, 10].

An alternative that has only recently been explored is to modify the listing structure rather than the object representation. By utilizing s-ocycles rather than ucycles, partial results for permutations have been obtained in [7]. However, we obtain a more complete solution in this paper. In Section 2 we establish a complete result on the existence of s-ocycles for permutations, and in Section 3 we explore similar results for functions and juggling sequences.

2. Permutations

We begin with a general lemma that will be used for both permutations and juggling sequences. For a string $X = x_0x_1 \ldots x_{n-1}$, define the **rotation function** as follows. A rotation by s is given by:

$$\rho^s(X) = x_s x_{s+1} \ldots x_{n-1} x_0 x_1 \ldots x_{s-1}.$$

The following lemma shows that for a string of length n, rotations by s partition the string into **blocks** of length $\gcd(n, s)$, and we can always perform repeated s-rotations to start X with any block desired.

Lemma 2.1. *Let $n, s \in \mathbb{Z}^+$ with $1 \leq s \leq n-1$ and $gcd(n, s) = d$. Consider a string $X = x_0x_1 \ldots x_{n-1}$, also written as $X = Y_0Y_1 \ldots Y_{m-1}$ where $n = md$ and $Y_i = x_{id}x_{id+1} \ldots x_{id+d-1}$. Then for any $i \in \{0, 1, \ldots, m-1\}$ there is some $j \in \{0, 1, \ldots, m-1\}$ such that:*

$$Y_iY_{i+1} \ldots Y_{m-1}Y_0Y_1 \ldots Y_{i-1} = \rho^{js}(X).$$

Proof. Suppose that $s = kd$. Then each rotation $\rho^s(X)$ advances through k blocks of X. Thus if we can show that $\gcd(m, k) = 1$, we are done. However we note that $\gcd(n, s) = d$ implies that there are integers p, q such that $pn + qs = d$. This implies the following.

$$\begin{aligned} pn + qs &= d \\ pmd + qkd &= d \\ pm + qk &= 1 \end{aligned}$$

Thus the same integers p, q provide confirmation that $\gcd(m, k) = 1$.

□

In [7], the following partial result was proven.

Theorem 2.2. [7] *Let $n, s \in \mathbb{Z}^+$ with $n \geq 2$ and let M be a multiset of size n. If we have either (1) $1 \leq s < \frac{n}{2}$, or (2) $gcd(s, n) = 1$ with $\frac{n}{2} \leq s < n-1$, then there exists an s-ocycle on the set of permutations of M.*

The following theorem offers a more complete solution the problem.

Theorem 2.3. *Let $n, s \in \mathbb{Z}^+$ with $1 \leq s \leq n-1$ and let M be a multiset of size n. There exists an s-ocycle on permutations of M if and only if $n - s > gcd(n, s)$.*

Proof. Define $\gcd(n, s) = d$, and suppose that $n - s = kd > d$. Construct the transition digraph D with vertices representing s-permutations of M and edges representing permutations of M. We will show that this graph is balanced and connected, and hence Eulerian. Recall that an Euler tour in this graph corresponds to an s-ocycle for permutations of M.

Balancedness: Let $X^{s-} = x_0x_1 \ldots x_{s-1}$ be an arbitrary vertex in the transition graph. For each possible out-edge corresponding to an n-permutation with prefix X^{s-}, such as $X = x_0x_1 \ldots x_{n-1}$, we also have the in-edge with suffix X^{s-}, namely $X' = x_sx_{s+1} \ldots x_{n-1}x_0x_1 \ldots x_{s-1}$. Thus there is a one-to-one correspondence between in- and out-edges at each vertex, and so D is balanced.

Connectedness: Consider an arbitrary vertex $X^{s-} = x_0x_1 \ldots x_{s-1}$, which is an s-prefix of some permutation $X = x_0x_1 \ldots x_{n-1}$. We will consider a partition of X into blocks of size d, such as $X = X_0X_1 \ldots X_{m-1}$ where we define $X_i = x_{id}x_{id+1} \ldots x_{(i+1)d-1}$. Our goal is to show that we can permute elements in any k consecutive blocks $X_iX_{i+1} \ldots X_{i+k-1}$, which illustrates that any adjacent transpositions are possible, and hence we can reach all permutations.

In D, rotations of X correspond to the following cycle, where subscripts are computed modulo n.

$$
\begin{aligned}
x_0x_1 \ldots x_{s-1} &\rightarrow x_{n-s}x_{n-s+1} \ldots x_{n-1} \\
&\rightarrow x_{n-2s}x_{n-2s+1} \ldots x_{n-s-1} \\
&\rightarrow x_{n-3s}x_{n-3s+1} \ldots x_{n-2s-1} \\
&\vdots \\
&\rightarrow x_{n-is}x_{n-is+1} \ldots x_{n-(i-1)s-1} \\
&\vdots \\
&\rightarrow x_0x_1 \ldots x_{s-1}
\end{aligned}
$$

From the vertex X^{s-} we can permute all elements in $\{x_s, x_{s+1}, \ldots, x_{n-1}\}$ to determine an out-edge, and at any point we can permute the $(n-s)$-suffix of a permutation. We would like to show that, through rotations, the $(n-s)$-suffix may contain any k consecutive blocks from $\{X_0, X_1, \ldots, X_{m-1}\}$. Consider the vertex $x_{n-is}x_{n-is+1} \ldots x_{n-(i-1)s-1}$. How does the starting index $n - is$

correspond to d? Note that:

$$\begin{aligned} n - is &\equiv in - is \pmod{n} \\ &\equiv i(n-s) \pmod{n} \\ &\equiv ikd \pmod{n}. \end{aligned}$$

Thus the vertices in this rotation cycle always begin with a multiple of kd. Our final step is to show that any multiple of d, say id, may be written as a multiple of kd modulo n, which is done by Lemma 2.1. Then we can rotate to start with any block desired, which is equivalent to pushing any k consecutive blocks to the $(n-s)$-suffix.

To summarize, we can perform adjacent transpositions $x_i \leftrightarrow x_{i+1}$ within X^{s-} by rotating X until x_i and x_{i+1} fall into the k blocks in the $(n-s)$-suffix and then transposing. Finally, by continuing along through rotations we will arrive at the vertex $x_0x_1 \ldots x_{i-1}x_{i+1}x_ix_{i+2} \ldots x_{s-1}$. Thus adjacent transpositions are always connected, hence all permutations can be reached.

For the converse, suppose that $n - s = d = \gcd(n, s)$. In this case, rotations of the permutation $X = x_0x_1 \ldots x_{n-1}$ provide an $(n-s)$-suffix of length d – just one block from $X = X_0X_1 \ldots X_{m-1}$. Thus we can always permute elements within each block, however the cyclic order of the blocks is fixed and we can perform the swap $x_i \leftrightarrow x_j$ if and only if x_i and x_j are in the same block. Thus we are able to permute elements within blocks, but not permute blocks (only rotate the block order). Hence the transition digraph connects only permutations with the same block order (rotations allowed). Permutations with block order that are not simple rotations are not connected, so the graph is disconnected and no Euler tour exists. □

Note that Theorem 2.3 agrees with the following well-known fact about universal cycles, or $(n-1)$-ocycles.

Corollary 2.4. *There is no universal cycle for permutations of M.*

Proof. In this case, we apply Theorem 2.3 setting $s = n-1$. Then it is clear that $\gcd(n, s) = 1 = n - s$, so no $(n-1)$-ocycle exists. □

3. Related Objects

3.1. Functions

Many times injective, surjective, and bijective functions are represented by permutations. We have the following facts about functions and theorems corresponding to their alternate representations. We will use the notation $[x] = \{1, 2, \ldots, x\}$.

It is a well-known fact that an injective function $f : [k] \to [n]$ may be represented by $x_1 x_2 \dots x_k$, the k-permutation of $[n]$ defined so that $x_i = f(i)$. Applying this, we have the following theorem.

Theorem 3.1. [7] *For all $n, s, k \in \mathbb{Z}^+$ with $1 \le s < k < n$, there is an s-ocycle for k-permutations of $[n]$.*

Similary, a surjective function $f : [n] \to [h]$ may be represented by $x_1 x_2 \dots x_n$, the string with ground set $[h]$ defined so that $x_i = f(i)$. In [8], it is shown that surjective functions are also represented by weak orders of $[n]$ with height $h - 1$. This observation is used for the following theorem.

Theorem 3.2. [8] *For all $n, s, h \in \mathbb{Z}^+$ with $1 \le s \le n - 2$, $gcd(s, n) = 1$, and $0 \le h \le n - 1$, there is an s-ocycle for $\mathcal{W}(n, h)$.*

We are able to improve this theorem as follows.

Theorem 3.3. *For all $n, s, h \in \mathbb{Z}^+$ with $1 \le s \le n - 2$ and $h \le n - 1$ there is an s-ocycle for strings with ground set $[h]$.*

Proof. We will show that the corresponding transition graph is Eulerian.

Balancedness: Consider a vertex $X^{s-} = x_1 x_2 \dots x_s$. X^{s-} is an s-prefix of the string $X = x_1 x_2 \dots x_n$ where X has ground set $[h]$. Since it is clear that $x_{s+1} x_{s+2} \dots x_n x_1 x_2 \dots x_s$ is also a string with ground set $[h]$, there is a bijection between in- and out-edges at X^{s-}. Hence the graph is balanced.

Connectedness: Define the minimum vertex V^{s-} to be the s-prefix of the permutation $V = 12 \cdots hh \cdots h$. Let $X = x_1 x_2 \dots x_n$ be an arbitrary multiset permutation with ground set $[h]$, and let X^{s-} be the s-prefix. We will show a path from X^{s-} to V^{s-} exists in the transition graph.

Compare X^{s-} and V^{s-}, and define i to be the least possible such that $X^{s-}(i) \ne V^{s-}(i)$. Note that since $h \le n - 1$, some element from $[h]$ must appear at least twice in X. We will refer to any element appearing more than once as a **duplicate**. We have two cases.

1. If the letter $x_i \in [h]$ appears twice in X:
   Let $d = \gcd(n, s)$, and rotate X until the d-block containing x_i is first. If we rotate X again by following an out-edge in the graph, we have arrived at a vertex A representing an s-substring of X without x_i. Since x_i also appears elsewhere in X, we can follow (backwards) the in-edge that is identical to A except with x_i replaced by v_i. Now we are at an edge corresponding to an s-substring of $x_1 x_2 \dots x_{i-1} v_i x_{i+1} \dots x_n$, so we are one step closer to the minimum vertex.

2. If the letter $x_i \in [h]$ appears exactly once in X:
   Since x_i is not a duplicate in X, some other letter $x_j \in [h]$ is a duplicate. In this case, we proceed as in case 1 to replace x_j with the letter x_i. Then x_i is a duplicate so we may follow case 1 again to replace x_i with v_i. At this point we have moved closer to the minimum vertex.

Continuing until we have transformed X^{s-} to V^{s-} produces a path from X^{s-} to the minimum vertex. Hence the graph is connected.

As the transition graph is balanced and connected, it is Eulerian by Theorem 1.1. □

Finally, one cannot discuss injective and surjective functions without considering bijective functions for completeness. A bijective function $f : [n] \to [n]$ may be represented by $x_1x_2 \dots x_n$, the permutation of $[n]$ defined so that $x_i = f(i)$. Using the results in the previous section, we know that there exists an s-ocycle on permutations of $[n]$ if and only if $n - s > \gcd(n, s)$.

3.2. Juggling Sequences

We begin with some lemmas that will help us to prove our main result.

Lemma 3.4. *Let $R = r_0r_1 \dots r_{n-1}$ be a string, and let $1 \le s \le n$. Then $R' = \rho^s(R) = r_{0+s}r_{1+s} \dots r_{n-1+s}$ (where addition is modulo n) is a valid juggling sequence if and only if R is a valid juggling sequence.*

Proof. We will check the corresponding permutation sequence for R' and show that it is valid. Suppose for a contradiction that there are $i, j \in \{0, 1, \dots, n-1\}$ with

$$r'_i + i \equiv r'_j + j \pmod{n}.$$

Then we have:

$$\begin{aligned} r'_i + i &\equiv r'_j + j \pmod{n} \\ r_{i-s} + i &\equiv r_{j-s} + j \pmod{n} \\ r_{i-s} + i - s &\equiv r_{j-s} + j - s \pmod{n} \\ r_k + k &\equiv r_\ell + \ell \pmod{n} \end{aligned}$$

Thus $\Pi_{R'}$ is valid if and only if Π_R is valid. □

Lemma 3.5. *Let $T = t_0t_1 \dots t_{n-1}$ be a juggling sequence, and let $0 \le i, s \le n-1$. Then*

$$\Pi_{\rho^s(T)}(i) = \rho^s(\Pi_T)(i) - s \pmod{n}.$$

Proof.

$$\begin{aligned}\Pi_{\rho^s(T)}(i) &= \rho^s(T)(i) + i \pmod{n} \\ &= T(i+s) + i \pmod{n} \\ &= T(i+s) + i + s - s \pmod{n} \\ &= \Pi_T(i+s) - s \pmod{n} \\ &= \rho^s(\Pi_T)(i) - s \pmod{n}\end{aligned}$$

□

Lemma 3.6. *Fix $n, s, b \in \mathbb{Z}^+$ with $1 \le s \le n-1$. Define D to be the s-ocycle transition digraph for juggling sequences of length n using at most b balls. From any vertex $v_0v_1 \dots v_{s-1}$, there exists a path to any vertex $v'_0v'_1 \dots v'_{s-1}$ whenever we have $v'_i \equiv v_i \pmod{n}$ for all $i \in \{0, 1, \dots, s-1\}$.*

Proof. Since adding/subtracting n to any digit in a juggling sequence of length n does not invalidate the sequence, if $v_i \ge n$ we can simply rotate some juggling sequence X with s-prefix $v_0v_1 \dots v_{s-1}$ until v_i is in the $(n-s)$-suffix, replace v_i with $v_i - n$, and then rotate back to our original s-prefix with v_i replaced by $v_i - n$. Repeating this eventually will find a path to $v'_1v'_2 \dots v'_s$. □

Theorem 3.7. *Fix $n, s, b \in \mathbb{Z}^+$ such that $1 \le s \le n-1$. There exists an s-ocycle for the set of juggling sequences with period n and at most b balls if and only if $n - s > gcd(n, s)$.*

Proof. We prove the forward direction by showing that the s-ocycle transition digraph D has an Euler tour. By Theorem 1.1 this is done by showing that the graph is balanced and connected.

Balancedness: Consider a vertex $R = r_0r_1 \dots r_{s-1}$ in D. We will show that any $(n-s)$-string $Q = q_0q_1 \dots q_{n-s-1}$ that is a valid s-suffix for R is also a valid s-prefix for R. Note that strings RQ and QR are simply rotations of each other, so by Lemma 3.4 either both strings are valid juggling sequences or neither string is. In this manner, there is a bijection between in- and out-edges, hence all vertices are balanced.

Connectedness: Consider an arbitrary vertex $T^{s-} = t_0t_1 \dots t_{s-1}$ that is an s-prefix to some juggling sequence $T = t_0t_1 \dots t_{n-1}$. First, by Lemma 3.6 we may assume that $t_i \in \{0, 1, \dots, n-1\}$ for all $i \in \{0, 1, \dots, n-1\}$. We will show that this arbitrary vertex is connected to the **minimum vertex**, which we define to be $V^{s-} = 0^s$ (a prefix of $V = 0^n$). In doing so, we will have shown that every vertex is connected to V^{s-} and hence the graph is connected.

Compare permutation sequences Π_T and Π_V corresponding to juggling sequences T and V, respectively. Note that $\Pi_V = 012\ldots(n-1)$, so suppose that for all $x \in \{0,1,\ldots,i-1\}$ we have $\Pi_T(x) = \Pi_V(x) = x$, but that $\Pi_T(i) = j$ for some $j \in \{i+1, i+2, \ldots, s-1\}$. We will find a path from the s-prefix of T to the s-prefix of some juggling sequence T' with permutation sequence $\Pi_{T'}$ that agrees with Π_V in the first $i+1$ positions. Repeating this procedure until $i = n$ will construct a path through D from T^{s-} to V^{s-}.

Assume $n - s = kd$ and $n = md$ for integers m, k where $d = \gcd(n, s)$, and let $T = Y_0Y_1\ldots Y_{m-1}$ be a partition of T into d-blocks, i.e., we define $Y_a = t_{ad}t_{ad+1}\ldots t_{ad+d-1}$. Suppose that $t_i \in Y_a$ and $t_j \in Y_b$. We have two cases.

1. If $b - a < k$:

   In this case, from Lemma 3.4 we may perform s-rotations on T so that we arrive at a vertex with both t_i and t_j in the $(n-s)$-suffix of T. Suppose that we performed a total rotation of size R, i.e., we are now at the vertex that represents the s-prefix of $\rho^R(T)$.

   At this point, the values t_i and t_j are located in positions $i-R$ and $j-R$ of $\rho^R(T)$, respectively. By Lemma 3.5 the corresponding permutation sequence entries are:

   $$\begin{aligned}\Pi_{\rho^R(T)}(i-R) &= \rho^R(\Pi_T)(i-R) - R \pmod{n}\\ &= \Pi_T(i) - R \pmod{n}\end{aligned}$$

   and

   $$\Pi_{\rho^R(T)}(j-R) = \Pi_T(j) - R \pmod{n}.$$

   We now note that the vertex $(\rho^R(T))^{s-}$, defined as the s-prefix of $\rho^R(T)$, is also the s-prefix of the juggling sequence $\rho^R(T')$ that is obtained from $\rho^R(T)$ by performing the swap $\Pi_{\rho^R(T)}(i-R) \leftrightarrow \Pi_{\rho^R(T)}(j-R)$ and adjusting the values $\rho^R(T)(i-R)$ and $\rho^R(T)(j-R)$ appropriately. Then rotating backwards by R we reach a juggling sequence T' with $\Pi_{T'}(x) = x$ for all $x \in \{0,1,\ldots,i-1,i\}$. This means that we have found a path from T^{s-} to T'^{s-}, which is one step closer to the minimum vertex.

2. If $b - a \geq k$:

   In this case the blocks containing t_i and t_j are more than k apart, so we cannot rotate to have both t_i and t_j in the $(n-s)$-suffix of T simultaneously. Instead, we pick a point t_z in the block Y_c that is k blocks preceding Y_b and follow case (1) with t_z in place of t_i. This produces a juggling sequence T' with $\Pi_{T'}(z) = i$, where $i \leq z < j$. Repeating, we will eventually have moved the permutation sequence value i to a block that is close enough to block Y_a to apply Case (1).

By repeating the above procedures, we will eventually have transitioned to the vertex corresponding to a juggling sequence with s-prefix $01\ldots(s-1)$. At this point we have reached the minimum vertex and we are done.

For the reverse direction, suppose that $n-s=\gcd(n,s)=d$. Recall that the number of balls $b \in \mathbb{Z}^+$ is given by:

$$b = \frac{1}{n}\sum_{i=0}^{n-1} t_i.$$

Thus juggling sequences of period n must always satisfy $\sum_{i=0}^{n-1} t_i \equiv 0 \pmod{n}$. Equivalently, when we partition a juggling sequence $T = t_0t_1\ldots t_{n-1}$ into blocks of length d, i.e., $T = Y_0Y_1\ldots Y_{m-1}$, where $n = md$, we must have:

$$\sum_{i=0}^{m-1} w(Y_i) = \sum_{i=0}^{m-1}\sum_{j=0}^{d} t_{id+j} = \sum_{i=0}^{n-1} t_i \equiv 0 \pmod{n},$$

where, for a given block Y_i, we call $w(Y_i) = \sum_{j=0}^{d} t_{id+j}$ the **weight** of block Y_i.

Now since $n-s=\gcd(n,s)$, from vertex $T^{s-} = t_0t_1\ldots t_{s-1}$ in D we may only move to vertices in which the $(n-s)$-suffix has weight equivalent to $w(Y_{m-1})$ mod n. Thus if we can show that, for any n, s, and b, there exists a juggling sequence with a block with weight $w \not\equiv 0 \pmod{n}$, then we are done. This is witnessed by the juggling sequence

$$T = \; d \quad 0 \quad 0 \quad \ldots \quad 0 \quad (n-d) \quad 0 \quad \ldots \quad 0,$$

with permutation sequence

$$\Pi_T = \; d \quad 1 \quad 2 \quad \ldots \quad (d-1) \quad 0 \quad (d+1) \quad (d+2) \quad \ldots \quad (n-1).$$

This juggling sequence utilizes one ball (recall we required $b \in \mathbb{Z}^+$) and the weight of the first block of length d is $d \not\equiv 0 \pmod{n}$. Thus this first block must always have weight equal to d modulo n if $n-s=d$, and so the vertex T^{s-} that represents the s-prefix of T is not connected to the vertex 0^s. Hence no Euler tour can exist and so no s-ocycle exists. □

While Theorem 3.7 completes the question of when s-ocycles for juggling sequences of period n and at most b balls exist, several variations remain open. For example, one might consider juggling sequences with:

- exactly b balls,
- at least b balls,
- fixed minimum period, or
- period n with no restriction on number of balls.

References

[1] J. Buhler, D. Eisenbud, R. Graham, and C. Wright, *Juggling drops and descents*, Amer. Math. Monthly **101**:6, 507-519.

[2] F. Chung, P. Diaconis, and R. Graham, *Universal cycles for combinatorial structures*, Discrete Math. **110** (1992), 43-59.

[3] F. Chung and R. Graham, *Universal juggling cycles*, Integers **7**:2 (2007), #A08.

[4] S. Forrest, *Genetic Algorithms: Principles of Natural Selection Applied to Computation*, Science **261**:5123 (1993), 872-878.

[5] A. Godbole, D. Knisley, and R. Norwood, *On α-overlap graphs*, Proceedings of the Forty-First Southeastern International Conference on Combinatorics, Graph Theory and Computing, Congr. Numer. **204** (2010), 161-171.

[6] F. Gray, *Pulse code communication*, March 17, 1953 (filed Nov. 1947), U.S. Patent 2,632,058.

[7] V. Horan and G. Hurlbert, *s-overlap cycles for permutations,* Bull. Inst. Combin. Appl. **69** (2013), 60-67.

[8] V. Horan and G. Hurlbert, *Universal cycles for weak orders,* SIAM J. Discrete Math. **27**:3 (2013), 1360-1371.

[9] G. Hurlbert and G. Isaak, *Equivalence class universal cycles for permutations*, Discrete Math. 149 (1996), 123-129.

[10] J. R. Johnson, *Universal cycles for permutations*, Discrete Math. 309 (2009), 5264-5270.

[11] E. A. Ragland, V. Nuys, and H. B. Scultheis, Jr., *Direction-sensitive binary code position control system*, Feb. 11, 1958 (filed Oct. 1953), U.S. Patent 2,823,345.

[12] Z. Wang and J. Bruck, *Partial rank modulation for flash memories*, IEEE Int. Symp. Inf. Theor. Proc. (2010), 864-868.

[13] D. B. West, *Introduction to Graph Theory*. Prentice Hall, Second Edition, 2001.