ON SEQUENCES WITHOUT GEOMETRIC PROGRESSIONS

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Abstract
An improved upper bound is obtained for the density of sequences of positive integers that contain no k-term geometric progression.

1. A Problem of Rankin

Let $k \geq 3$ be an integer. Let $r \neq 0, \pm 1$ be a real number. A *geometric progression of length k with common ratio r* is a sequence $(a_0, a_1, a_2, \ldots, a_{k-1})$ of nonzero real numbers such that

$$r = \frac{a_i}{a_{i-1}}$$

for $1, 2, \ldots, k-1$. For example, $(3/4, 3/2, 3, 6)$ and $(8, 12, 18, 27)$ are geometric progressions of length 4 with common ratios 2 and $3/2$, respectively. A *k-geometric progression* is a geometric progression of length k with common ratio r for some r. If the sequence $(a_0, a_1, a_2, \ldots, a_{k-1})$ is a k-geometric progression, then $a_i \neq a_j$ for $0 \leq i < j \leq k-1$.

A finite or infinite set of real numbers is *k-geometric progression free* if the set does not contain numbers $a_0, a_1, \ldots, a_{k-1}$ such that the sequence $(a_0, a_1, \ldots, a_{k-1})$ is a k-geometric progression. Rankin [3] introduced k-geometric progression free sets, and proved that there exist infinite k-geometric progression free sets with positive asymptotic density.[1] For example, the set Q of square-free positive integers, with asymptotic density $d(Q) = \pi^2/6$, contains no k-term geometric progression for $k \geq 3$.

[1] If $A(n)$ denotes the number of positive integers $a \in A$ with $a \leq n$, then the *upper asymptotic density* of A is $d_U(A) = \limsup_{n\to\infty} A(n)/n$, and the *asymptotic density* of A is $d(A) = \lim_{n\to\infty} A(n)/n$, if this limit exists.

Let A be a set of positive integers that contains no k-term geometric progression. Brown and Gordon [2] proved[2] that the upper asymptotic density of A, denoted $d_U(A)$, has the following upper bound:

$$d_U(A) \leq 1 - \frac{1}{2^k} - \frac{2}{5}\left(\frac{1}{5^{k-1}} - \frac{1}{6^{k-1}}\right).$$

Riddell [4] and Beiglböck, Bergelson, Hindman, and Strauss[1] proved that

$$d_U(A) \leq 1 - \frac{1}{2^k - 1}.$$

The purpose of this note is to improve these results.

2. An Upper Bound for Sets with No k-Term Geometric Progression

Theorem 1. *For integers $k \geq 3$ and $n \geq 2^{k-1}$, let $GPF_k(n)$ denote the set of subsets of $\{1, 2, \ldots, n\}$ that contain no k-term geometric progression. If $A \in GPF_k(n)$, then*

$$n - |A| \geq \left(\frac{1}{2^k - 1} + \frac{2}{5}\left(\frac{1}{5^{k-1}} - \frac{1}{6^{k-1}}\right) + \frac{4}{15}\left(\frac{1}{7^{k-1}} - \frac{1}{10^{k-1}}\right)\right) n + O\left(\frac{\log n}{k}\right).$$

Proof. Let

$$L = \left[\frac{\log 2n}{k \log 2}\right].$$

For $1 \leq \ell \leq L$ we have $2^{\ell k - 1} \leq n$. Let a be an odd positive integer such that

$$a \leq \frac{n}{2^{\ell k - 1}}.$$

The sequence

$$\left(2^{(\ell-1)k}a, 2^{(\ell-1)k+1}a, 2^{(\ell-1)k+2}a, \ldots, 2^{\ell k - 1}a\right)$$

is a geometric progression of length k with common ratio 2. If $A \in GPF_k(n)$, then A does not contain this geometric progression, and so at least one element in the set

$$X_\ell(a) = \left\{2^{(\ell-1)k}a, 2^{(\ell-1)k+1}a, 2^{(\ell-1)k+2}a, \ldots, 2^{\ell k - 1}a\right\}$$

is not an element of A. Because every nonzero integer has a unique representation as the product of an odd integer and a power of 2, it follows that, for integers $\ell = 1, \ldots, L$ and odd positive integers $a \leq 2^{1-\ell k}n$, the sets $X_\ell(a)$ are pairwise disjoint subsets of $\{1, 2, \ldots, n\}$.

[2] Brown and Gordon claimed a slightly stronger result, but their proof contains an (easily corrected) error.

For every real number $t \geq 1$, the number of odd positive integers not exceeding t is strictly greater than $(t-1)/2$. It follows that the cardinality of the set $\{1, 2, \ldots, n\} \setminus A$ is strictly greater than

$$\sum_{\ell=1}^{L} \frac{1}{2}\left(\frac{n}{2^{\ell k-1}} - 1\right) = \sum_{\ell=1}^{L}\left(\frac{n}{2^{\ell k}} - \frac{1}{2}\right) = n\sum_{\ell=1}^{L}\frac{1}{2^{\ell k}} + O\left(\frac{\log n}{k}\right)$$
$$= \frac{n}{2^k - 1} + O\left(\frac{\log n}{k}\right).$$

Note that if r is an odd integer and $r \in X_\ell(a)$, then $\ell = 1$ and $r = a$.

Let b be an odd integer such that

$$\frac{n}{6^{k-1}} < b \leq \frac{n}{5^{k-1}} \tag{1}$$

and b is not divisible by 5, that is,

$$b \equiv 1, 3, 7, \text{ or } 9 \pmod{10}. \tag{2}$$

We consider the following geometric progression of length k with ratio $5/3$:

$$(3^{k-1}b, 3^{k-2}5b, \ldots, 3^{k-1-i}5^i b, \cdots, 5^{k-1}b).$$

Every integer in this progression is odd, and

$$\frac{n}{2^{k-1}} < 3^{k-1}b < \cdots < 5^{k-1}b \leq n.$$

Let

$$Y(b) = \{3^{k-1}b, 3^{k-2}5b, \ldots, 3^{k-1-i}5^i b, \cdots, 5^{k-1}b\}.$$

It follows that $X_\ell(a) \cap Y(b) = \emptyset$ for all ℓ, a, and b. If the integers b and b' satisfy (1) and (2) with $b < b'$ and if $Y(b) \cap Y(b') \neq \emptyset$, then there exist integers $i, j \in \{0, 1, 2, \ldots, k-1\}$ such that $3^{k-1-i}5^i b = 3^{k-1-j}5^j b'$ or, equivalently,

$$5^{i-j}b = 3^{i-j}b'.$$

The inequality $b < b'$ implies that $0 \leq j < i \leq k-1$ and so $b' \equiv 0 \pmod 5$, which contradicts (2). Therefore, the sets $Y(b)$ are pairwise disjoint. The number of integers b satisfying inequality (1) and congruence (2) is

$$\frac{2}{5}\left(\frac{1}{5^{k-1}} - \frac{1}{6^{k-1}}\right)n + O(1).$$

Let c be an odd integer such that

$$\frac{n}{10^{k-1}} < c \leq \frac{n}{7^{k-1}} \tag{3}$$

and c is not divisible by 3 or 5, that is,

$$c \equiv 1, 7, 11, 13, 17, 19, 23, \text{ or } 29 \pmod{30}. \tag{4}$$

We consider the following geometric progression of length k with ratio $7/5$:

$$(5^{k-1}c, 5^{k-2}7c, \ldots, 5^{k-1-i}7^i c, \cdots, 7^{k-1}c).$$

Every integer in this progression is odd, and

$$\frac{n}{2^{k-1}} < 5^{k-1}c < \cdots < 7^{k-1}c \leq n.$$

Let

$$Z(c) = \{5^{k-1}c, 5^{k-2}7c, \ldots, 5^{k-1-i}7^i c, \cdots, 7^{k-1}c\}.$$

It follows that $X_\ell(a) \cap Z(c) = \emptyset$ for all ℓ, a, and c. If c and c' satisfy (3) and (4) with $c < c'$ and if $Z(c) \cap Z(c') \neq \emptyset$, then there exist integers $i, j \in \{0, 1, 2, \ldots, k-1\}$ such that $5^{k-1-i}7^i c = 5^{k-1-j}7^j c'$ or, equivalently,

$$7^{i-j}c = 5^{i-j}c'.$$

The inequality $c < c'$ implies that $0 \leq j < i \leq k-1$ and so $c \equiv 0 \pmod 5$, which contradicts (4). Therefore, the sets $Z(c)$ are pairwise disjoint.

If b and c satisfy inequalities (1) and (3), respectively, then $c < b$. If $Y(b) \cap Z(c) \neq \emptyset$, then there exist integers $i, j \in \{0, 1, \ldots, k-1\}$ such that $5^{k-1-i}7^i c = 5^{k-1-j}3^j b$ or, equivalently,

$$5^j 7^i c = 5^i 3^j b.$$

Because $bc \not\equiv 0 \pmod 5$, it follows that $i = j$ and so

$$7^i c = 3^i b.$$

Because $c < b$, we must have $i \geq 1$ and so $c \equiv 0 \pmod 3$, which contradicts congruence (4). Therefore, $Y(b) \cap Z(c) = \emptyset$ and the sets $X_\ell(a)$, $Y(b)$, and $Z(c)$ are pairwise disjoint. The number of integers c satisfying inequality (3) and congruence (4) is

$$\frac{4}{15}\left(\frac{1}{7^{k-1}} - \frac{1}{10^{k-1}}\right)n + O(1).$$

Because A contains no k-term geometic progression, at least one element from each of the sets $X_\ell(a)$, $Y(b)$, and $Z(c)$ is not in A. This completes the proof. □

Corollary 1. *If A_k is a set of positive integers that contains no k-term geometric progression, then*

$$d_U(A_k) \leq 1 - \frac{1}{2^k - 1} - \frac{2}{5}\left(\frac{1}{5^{k-1}} - \frac{1}{6^{k-1}}\right) - \frac{4}{15}\left(\frac{1}{7^{k-1}} - \frac{1}{10^{k-1}}\right).$$

Here is a table of upper bounds for $d_U(A)$ for various values of k:

k	3	4	5	6	7	10	17
$d_U(A_k) \leq$	0.84948	0.93147	0.96733	0.98404	0.99211	0.99902	0.99999

3. Open Problems

For every integer $k \geq 3$, let GPF_k denote the set of sets of positive integers that contain no k-term geometric progression. It would be interesting to determine precisely

$$\sup\{d_U(A) : A \in GPF_k\}$$

and

$$\sup\{d(A) : A \text{ has asymptotic density and } A \in GPF_k\}.$$

In the special case $k = 3$, Riddell [4, p. 145] claimed that if $A \in GPF_3$, then $d_U(A) < 0.8339$, but wrote, "The details are too lengthy to be included here."

An infinite sequence $A = (a_i)_{i=1}^{\infty}$ of positive integers is *syndetic* if it is strictly increasing with bounded gaps. Equivalently, A is syndetic if there is a number c such that $1 \leq a_{i+1} - a_i \leq c$ for all positive integers i. Beiglböck, Bergelson, Hindman, and Strauss [1] asked if every syndetic sequence must contain arbitrarily long finite geometric progressions.

References

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