#A32 INTEGERS 13 (2013)

AN INFINITE FAMILY OF RECURSIVE FORMULAS GENERATING POWER MOMENTS OF KLOOSTERMAN SUMS WITH TRACE ONE ARGUMENTS: $O(2N+1, 2^R)$ CASE

Dae San Kim[1]
Department of Mathematics, Sogang University, Seoul, Korea
dskim@sogong.ac.kr

Received: 7/14/12, Revised: 1/4/13, Accepted: 4/10/13, Published: 5/24/13

Abstract

In this paper, we construct an infinite family of binary linear codes associated with double cosets with respect to a certain maximal parabolic subgroup of the orthogonal group $O(2n+1, q)$. Here q is a power of two. Then we obtain an infinite family of recursive fomulas generating the odd power moments of Kloosterman sums with trace one arguments in terms of the frequencies of weights in the codes associated with those double cosets in $O(2n+1, q)$, and in the codes associated with similar double cosets in the symplectic group $Sp(2n, q)$. This is done via the Pless power moment identity and by utilizing the explicit expressions of exponential sums over those double cosets related to the evaluations of “Gauss sums” for the orthogonal group $O(2n+1, q)$.

1. Introduction

Let ψ be a nontrivial additive character of the finite field $\mathbb{F}_q$ with $q = p^r$ elements (p a prime). Then the Kloosterman sum $K(\psi; a)$ ([12]) is defined as

$$K(\psi; a) = \sum_{\alpha \in \mathbb{F}_q^*} \psi(\alpha + a\alpha^{-1}) \ (a \in \mathbb{F}_q^*).$$

For this, we have the Weil bound

$$|K(\psi; a)| \leq 2\sqrt{q}. \tag{1.1}$$

The Kloosterman sum was introduced in 1926([11]) to give an estimate for the Fourier coefficients of modular forms.

[1] This work was supported by National Research Foundation of Korea Grant funded by the Korean Government 2009-0072514.

For each nonnegative integer h, by $MK(\psi)^h$ we will denote the h-th moment of the Kloosterman sum $K(\psi;a)$. Namely, it is given by

$$MK(\psi)^h = \sum_{a \in \mathbb{F}_q^*} K(\psi;a)^h.$$

If $\psi = \lambda$ is the canonical additive character of $\mathbb{F}_q$, then $MK(\lambda)^h$ will be simply denoted by MK^h. Here we recall that $\lambda(x) = e^{2\pi i tr(x)/p}$ is the canonical additive character of $\mathbb{F}_q$, where $tr(x) = x + x^p + \cdots + x^{p^{r-1}}$ is the trace function $\mathbb{F}_q \to \mathbb{F}_p$.

Explicit computations on power moments of Kloosterman sums were begun with the paper [17] of Salié in 1931, where he showed, for any odd prime q,

$$MK^h = q^2 M_{h-1} - (q-1)^{h-1} + 2(-1)^{h-1} \ (h \geq 1).$$

Here $M_0 = 0$, and, for $h \in \mathbb{Z}_{>0}$,

$$M_h = |\{(\alpha_1, \cdots, \alpha_h) \in (\mathbb{F}_q^*)^h | \sum_{j=1}^{h} \alpha_j = 1 = \sum_{j=1}^{h} \alpha_j^{-1}\}|.$$

For $q = p$ odd prime, Salié obtained MK^1, MK^2, MK^3, MK^4 in [17] by determining M_1, M_2, M_3. On the other hand, MK^5 can be expressed in terms of the p-th eigenvalue for a weight 3 newform on $\Gamma_0(15)$(cf. [13], [16]). MK^6 can be expressed in terms of the p-th eigenvalue for a weight 4 newform on $\Gamma_0(6)$(cf. [3]). Also, based on numerical evidence, in [2] Evans was led to propose a conjecture which expresses MK^7 in terms of Hecke eigenvalues for a weight 3 newform on $\Gamma_0(525)$ with quartic nebentypus of conductor 105.

From now on, let us assume that $q = 2^r$. Carlitz[1] evaluated MK^h for $h \leq 4$. Recently, Moisio was able to find explicit expressions of MK^h, for $h \leq 10$ (cf.[15]). This was done, via Pless power moment identity, by connecting moments of Kloosterman sums and the frequencies of weights in the binary Zetterberg code of length $q+1$, which were known by the work of Schoof and Vlugt in [18].

In [7], the binary linear codes $C(SL(n,q))$ associated with finite special linear groups $SL(n,q)$ were constructed when n, q are both powers of two. Then we obtained a recursive formula for the power moments of multi-dimensional Kloosterman sums in terms of the frequencies of weights in $C(SL(n,q)$.

In order to describe our results, we introduce two incomplete power moments of Kloosterman sums, namely, the one with the sum over all a in $\mathbb{F}_q^*$ with $tr\ a$=0 and the other with the sum over all a in $\mathbb{F}_q^*$ with $tr\ a = 1$. For every nonnegative integer h, and ψ as before, we define

$$T_0K(\psi)^h = \sum_{a \in \mathbb{F}_q^*,\ tr a=0} K(\psi;a)^h,\ T_1K(\psi)^h = \sum_{a \in \mathbb{F}_q^*,\ tr a=1} K(\psi;a)^h, \qquad (1.2)$$

which will be respectively called the h-th moment of Kloosterman sums with "trace zero arguments" and those with "trace one arguments". Then, clearly we have

$$MK(\psi)^h = T_0K(\psi)^h + T_1K(\psi)^h. \tag{1.3}$$

If $\psi = \lambda$ is the canonical additive character of $\mathbb{F}_q$, then $T_0K(\lambda)^h$ and $T_1K(\lambda)^h$ will be respectively denoted by T_0K^h and T_1K^h, for brevity.

In this paper, we will show the main Theorem 1.1 giving an infinite family of recursive formulas generating the odd power moments of Kloosterman sums with trace one arguments. To do that, we construct binary linear codes $C(DC(n,q))$, associated with the double cosets $DC(n,q){=}P\sigma_{n-1}P$, for the maximal parabolic subgroup $P{=}P(2n+1,q)$ of the orthogonal group $O(2n+1,q)$, and express those power moments in terms of the frequencies of weights in the codes $C(DC(n,q))$ and $C(\widehat{DC}(n,q))$. Here $C(\widehat{DC}(n,q))$ is a binary linear code constructed similarly from certain double cosets $\widehat{DC}(n,q)$ in the sympletic group $Sp(2n,q)$. Then, thanks to our previous results on the explicit expressions of exponential sums over those double cosets related to the evaluations of "Gauss sums" for the orthogonal group $O(2n{+}1,q)$ [10], we can express the weight of each codeword in the dual of the codes $C(DC(n,q))$ in terms of Kloosterman sums. Then our formulas will follow immediately from the Pless power moment identity. Analogously to these, in [8](resp. [9]), for q a power of three, two(resp. eight) infinite families of ternary linear codes associated with double cosets in the symplectic group $Sp(2n,q)$(resp. orthogonal group $O^-(2n,q)$) were constructed in order to generate one (resp. four)infinite families of recursive formulas for the power moments of Kloosterman sums with square arguments and for the even power moments of those in terms of the frequencies of weights in those codes. We emphasize here that there have been only a few recursive formulas generating power moments of Kloosterman sums including the one in [15].

Theorem 1.1 in the following(cf. (1.6)-(1.8)) is the main result of this paper. Henceforth, we agree that the binomial coefficient $\binom{b}{a} = 0$ if $a > b$ or $a < 0$. To simplify notations, we introduce the following ones which will be used throughout this paper at various places.

$$A(n,q) = q^{\frac{1}{4}(5n^2-1)} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \Pi_{j=1}^{(n-1)/2}(q^{2j-1}-1), \tag{1.4}$$

$$B(n,q) = q^{\frac{1}{4}(n-1)^2}(q^n-1)\Pi_{j=1}^{(n-1)/2}(q^{2j}-1). \tag{1.5}$$

Here $\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{q^n-1}{q-1}$ is a q-binomial coefficient.

Theorem 1.1. *Let $q = 2^r$. Assume that n is any odd integer≥ 3, with all q, or n=1, with $q \geq 8$. Then, in the notations of (1.4) and (1.5), we have the following.*

For h=1,3,5,$\cdots$,

$$\begin{aligned} T_1K^h = &- \sum_{0\le l\le h-2,\ l\ odd} \binom{h}{l} B(n,q)^{h-l}T_1K^l \\ &+ qA(n,q)^{-h} \sum_{j=0}^{min\{N(n,q),h\}} (-1)^j D_j(n,q) \sum_{t=j}^{h} t!S(h,t)2^{h-t-1}\binom{N(n,q)-j}{N(n,q)-t}, \end{aligned} \tag{1.6}$$

where $N(n,q) = |DC(n,q)| = A(n,q)B(n,q)$, $D_j(n,q) = C_j(n,q) - \widehat{C}_j(n,q)$, *with* $\{C_j(n,q)\}_{j=0}^{N(n,q)}$, $\{\widehat{C}_j(n,q)\}_{j=0}^{N(n,q)}$ *respectively the weight distributions of the binary linear codes* $C(DC(n,q))$ *and* $C(\widehat{DC}(n,q))$ *given by: for* $j = 0,\cdots,N(n,q)$,

$$\begin{aligned} C_j(n,q) = &\sum \binom{q^{-1}A(n,q)(B(n,q)+1)}{\nu_1} \\ &\times \prod_{tr(\beta-1)^{-1}=0} \binom{q^{-1}A(n,q)(B(n,q)+q+1)}{\nu_\beta} \\ &\times \prod_{tr(\beta-1)^{-1}=1} \binom{q^{-1}A(n,q)(B(n,q)-q+1)}{\nu_\beta}, \end{aligned} \tag{1.7}$$

$$\begin{aligned} \widehat{C}_j(n,q) = &\sum \binom{q^{-1}A(n,q)(B(n,q)+1)}{\nu_0} \\ &\times \prod_{tr(\beta^{-1})=0} \binom{q^{-1}A(n,q)(B(n,q)+q+1)}{\nu_\beta} \\ &\times \prod_{tr(\beta^{-1})=1} \binom{q^{-1}A(n,q)(B(n,q)-q+1)}{\nu_\beta}. \end{aligned} \tag{1.8}$$

Here the first sum in (1.6) is 0 if $h = 1$ and the unspecified sums in (1.7) and (1.8) are over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta\in\mathbb{F}_q}$ satisfying $\sum_{\beta\in\mathbb{F}_q}\nu_\beta = j$ and $\sum_{\beta\in\mathbb{F}_q}\nu_\beta\beta = 0$. In addition, $S(h,t)$ is the Stirling number of the second kind defined by

$$S(h,t) = \frac{1}{t!}\sum_{j=0}^{t}(-1)^{t-j}\binom{t}{j}j^h. \tag{1.9}$$

2. $O(2n+1,q)$

For more details about this section, one is referred to the paper [10]. Throughout this paper, the following notations will be used:

$$\begin{aligned} q &= 2^r \ (r \in \mathbb{Z}_{>0}), \\ \mathbb{F}_q &= \ the\ finite\ field\ with\ q\ elements, \\ TrA &= \ the\ trace\ of\ A\ for\ a\ square\ matrix\ A, \\ {}^tB &= \ the\ transpose\ of\ B\ for\ any\ matrix\ B. \end{aligned}$$

Let θ be the nondegenerate quadratic form on the vector space $\mathbb{F}_q^{(2n+1)\times 1}$ of all $(2n+1)\times 1$ column vectors over $\mathbb{F}_q$, given by

$$\theta(\sum_{i=1}^{2n+1} x_i e^i) = \sum_{i=1}^{n} x_i x_{n+i} + x_{2n+1}^2,$$

where $\{e^1 = {}^t[10\cdots0], e^2 = {}^t[010\cdots0], \cdots, e^{2n+1} = {}^t[0\cdots01]\}$ is the standard basis of $\mathbb{F}_q^{(2n+1)\times 1}$.

The group $O(2n+1,q)$ of all isometries of $(\mathbb{F}_q^{(2n+1)\times 1}, \theta)$ consists of the matrices

$$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ g & h & 1 \end{bmatrix} (A, B, C, D\ n\times n, g, h\ 1\times n)$$

in $GL(2n+1,q)$ satisfying the relations:

$$\begin{aligned} &{}^tAC + {}^tgg\ is\ alternating \\ &{}^tBD + {}^thh\ is\ alternating \\ &{}^tAD + {}^tCB = 1_n. \end{aligned}$$

Here an $n\times n$ matrix (a_{ij}) is called alternating if

$$\begin{cases} a_{ii} \ = \ 0, & for\ 1 \le i \le n, \\ a_{ij} \ = \ -a_{ji} \ = \ a_{ji}, & for\ 1 \le i < j \le n. \end{cases}$$

Also, one observes, for example, that ${}^tAC + {}^tgg$ is alternating if and only if ${}^tAC = {}^tCA$ and $g = \sqrt{diag({}^tAC)}$, where $\sqrt{diag({}^tAC)}$ indicates the $1\times n$ matrix $[\alpha_1, \cdots, \alpha_n]$ if the diagonal entries of ${}^tAC$ are given by

$$({}^tAC)_{11} \ = \ \alpha_1^2, \cdots, ({}^tAC)_{nn} \ = \ \alpha_n^2,\ for\ \alpha_i \in \mathbb{F}_q.$$

As is well known, there is an isomorphism of groups

$$\iota:\ O(2n+1,q) \ \rightarrow \ Sp(2n,q)\ (\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ g & h & 1 \end{bmatrix} \ \mapsto \ \begin{bmatrix} A & B \\ C & D \end{bmatrix}). \qquad (2.1)$$

In particular, for any $w \in O(2n+1,q)$,

$$Trw = Tr\iota(w) + 1. \tag{2.2}$$

Let $P = P(2n+1,q)$ be the maximal parabolic subgroup of $O(2n+1,q)$ given by

$$P(2n+1,q) = \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & {}^tA^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1_n & B & 0 \\ 0 & 1_n & 0 \\ 0 & h & 1 \end{bmatrix} \middle| \begin{array}{c} A \in GL(n,q) \\ B + {}^thh \textit{ is alternating} \end{array} \right\}.$$

The Bruhat decomposition of $O(2n+1,q)$ with respect to $P = P(2n+1,q)$ is

$$O(2n+1,q) = \coprod_{r\,=\,0}^{n} P\sigma_r P,$$

where

$$\sigma_r \;=\; \begin{bmatrix} 0 & 0 & 1_r & 0 & 0 \\ 0 & 1_{n-r} & 0 & 0 & 0 \\ 1_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \;\in\; O(2n+1,q).$$

The symplectic group $Sp(2n,q)$ over the field $\mathbb{F}_q$ is defined as:

$$Sp(2n,q) \;=\; \{w \in GL(2n,q) |\ {}^twJw \;=\; J\},$$

with

$$J = \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix}.$$

Let $P^{'} = P^{'}(2n,q)$ be the maximal parabolic subgroup of $Sp(2n,q)$ defined by:

$$P^{'}(2n,q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix} \begin{bmatrix} 1_n & B \\ 0 & 1_n \end{bmatrix} \middle| A \in GL(n,q), {}^tB = B \right\}.$$

Then, with respect to $P^{'} = P^{'}(2n,q)$, the Bruhat decomposition of $Sp(2n,q)$ is given by

$$Sp(2n,q) = \coprod_{r\,=\,0}^{n} P^{'}\sigma_r^{'}P^{'},$$

where

$$\sigma_r^{'} = \begin{bmatrix} 0 & 0 & 1_r & 0 \\ 0 & 1_{n-r} & 0 & 0 \\ 1_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-r} \end{bmatrix} \in \ Sp(2n, q).$$

Put, for each r with $0 \leq r \leq n$,

$$\begin{aligned} A_r &= \{w \in P(2n+1, q) \mid \sigma_r w \sigma_r^{-1} \in P(2n+1, q)\}, \\ A_r^{'} &= \{w \in P^{'}(2n, q) \mid \sigma_r^{'} w (\sigma_r^{'})^{-1} \in P^{'}(2n, q)\}. \end{aligned}$$

Expressing them as the disjoint union of right cosets of maximal parabolic subgroups, the double cosets $P\sigma_r P$ and $P^{'}\sigma_r^{'}P^{'}$ can be written respectively as

$$P\sigma_r P = \ P\sigma_r(A_r \setminus P), \tag{2.3}$$

$$P^{'}\sigma_r^{'}P^{'} \ = \ P^{'}\sigma_r^{'}(A_r^{'} \setminus P^{'}). \tag{2.4}$$

The order of the general linear group $GL(n, q)$ is given by

$$g_n = \ \prod_{j=0}^{n-1}(q^n - q^j) = \ q^{\binom{n}{2}} \prod_{j=1}^{n}(q^j - 1).$$

For integers n,r with $0 \leq r \leq n$, the q-binomial coefficients are defined as:

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \prod_{j=0}^{r-1}(q^{n-j} - 1)/(q^{r-j} - 1).$$

The following results follow either from [10] or from [4] plus the observation that under the isomorphism ι in (2.1) P, A_r, σ_r are respectively mapped onto $P,^{'}$ $A_r^{'}, \sigma_r^{'}$:

$$\begin{aligned} |A_r| = |A_r^{'}| &= \ g_r g_{n-r} q^{\binom{n+1}{2}} q^{r(2n-3r-1)/2}, \\ |P(2n+1, q)| &= \ |P^{'}(2n, q)| = q^{\binom{n+1}{2}} g_n, \\ |A_r \setminus P(2n+1, q)| &= \ |A_r^{'} \setminus P^{'}(2n, q)| = q^{\binom{r+1}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q, \\ |P(2n+1, q)\sigma_r P(2n+1, q)| &= \ |P'(2n, q)\sigma_r^{'} P^{'}(2n, q)| \\ &= q^{n^2} \begin{bmatrix} n \\ r \end{bmatrix}_q q^{\binom{r}{2}} q^r \prod_{j=1}^{n}(q^j - 1) \\ &(= |P(2n+1, q)|^2 |A_r|^{-1} \\ &= |P^{'}(2n+q)|^2 |A_r^{'}|^{-1}). \end{aligned} \tag{2.5}$$

In particular, with

$$DC(n,q) = \ P(2n+1,q)\sigma_{n-1}P(2n+1,q),$$

$$|DC(n,q)| = q^{\frac{1}{2}n(3n-1)} \begin{bmatrix} n \\ 1 \end{bmatrix}_q \prod_{j=1}^{n}(q^j-1) = A(n,q)B(n,q) \qquad (cf.\ (1.4),(1.5)). \quad (2.6)$$

3. Exponential Sums Over Double Cosets of $O(2n+1,q)$

The following notations will be employed throughout this paper:

$$tr(x) = x + x^2 + \cdots + x^{2^{r-1}} \ the\ trace\ function\ \mathbb{F}_q \to \mathbb{F}_2,$$
$$\lambda(x) = (-1)^{tr(x)} \ the\ canonical\ additive\ character\ of\ \mathbb{F}_q.$$

Then any nontrivial additive character ψ of $\mathbb{F}_q$ is given by $\psi(x) = \lambda(ax)$, for a unique $a \in \mathbb{F}_q^*$.

For any nontrivial additive character ψ of $\mathbb{F}_q$ and $a \in \mathbb{F}_q^*$, the Kloosterman sum $K_{GL(t,q)}(\psi;a)$ for $GL(t,q)$ is defined as

$$K_{GL(t,q)}(\psi;a) = \sum_{w \in GL(t,q)} \psi(Trw + aTrw^{-1}).$$

Notice that, for $t=1$, $K_{GL(1,q)}(\psi;a)$ denotes the Kloosterman sum $K(\psi;a)$.

In [4], it is shown that $K_{GL(t,q)}(\psi;a)$ satisfies the following recursive relation: for integers $t \geq 2$, $a \in \mathbb{F}_q^*$,

$$K_{GL(t,q)}(\psi;a) = q^{t-1}K_{GL(t-1,q)}(\psi;a)K(\psi;a) + q^{2t-2}(q^{t-1}-1)K_{GL(t-2,q)}(\psi;a),$$

where we understand that $K_{GL(0,q)}(\psi,a) = 1$.

From [4] and [10], we have (cf. (2.2)-(2.5)):

$$\begin{aligned}
&\sum_{w \in P\sigma_r P} \psi(Trw) \\
&\qquad = |A_r \setminus P| \sum_{w \in P} \psi(Trw\sigma_r) \\
&\qquad = \psi(1)|A_r^{'} \setminus P^{'}| \sum_{w \in P} \psi(Tr\iota(w)\sigma_r^{'}) \\
&\qquad = \psi(1)|A_r^{'} \setminus P^{'}| \sum_{w \in P^{'}} \psi(Trw\sigma_r^{'}) \qquad (3.1) \\
&\qquad (= \psi(1) \sum_{w \in P^{'}\sigma_r^{'}P^{'}} \psi(Trw)) \\
&\qquad = \psi(1)q^{\binom{n+1}{2}}|A_r^{'} \setminus P^{'}|q^{r(n-r)}a_r K_{GL(n-r,q)}(\psi;1).
\end{aligned}$$

Here ψ is any nontrivial additive character of $\mathbb{F}_q$, $a_0 = 1$, and, for $r \in \mathbb{Z}_{>0}$, a_r denotes the number of all $r \times r$ nonsingular alternating matrices over $\mathbb{F}_q$, which is given by

$$a_r = \begin{cases} 0, & \text{if } r \text{ is odd,} \\ q^{\frac{r}{2}(\frac{r}{2}-1)} \prod_{j=1}^{\frac{r}{2}} (q^{2j-1} - 1), & \text{if } r \text{ is even} \end{cases} \tag{3.2}$$

(cf.[4], Proposition 5.1).
Thus we see from (2.5), (3.1), and (3.2) that, for each r with $0 \le r \le n$,

$$\sum_{w \in P\sigma_r P} \psi(Trw) = \begin{cases} 0, & \text{if } r \text{ is odd,} \\ \psi(1) q^{\binom{n+1}{2}} q^{rn - \frac{1}{4}r^2} \begin{bmatrix} n \\ r \end{bmatrix}_q & \\ \qquad \times \prod_{j=1}^{r/2} (q^{2j-1} - 1) K_{GL(n-r,q)}(\psi; 1), & \text{if } r \text{ is even.} \end{cases} \tag{3.3}$$

For our purposes, we need only one infinite family of exponential sums in (3.3) over $P(2n+1, q)\sigma_{n-1} P(2n+1, q) = DC(n, q)$, for $n = 1, 3, 5, \cdots$. So we state them separately as a theorem.

Theorem 3.1. *Let ψ be any nontrivial additive character of $\mathbb{F}_q$. Then in the notation of (1.4), we have*

$$\sum_{w \in DC(n,q)} \psi(Trw) = \psi(1) A(n, q) K(\psi; 1), for\ n = 1, 3, 5, \cdots . \tag{3.4}$$

Proposition 3.2. *([5]) For $n = 2^s (s \in \mathbb{Z}_{\ge 0})$, and λ the canonical additive character of $\mathbb{F}_q$,*

$$K(\lambda; a^n) = K(\lambda; a).$$

The next corollary follows from Theorem 3.1, Proposition 3.2 and a simple change of variables.

Corollary 3.3. *Let λ be the canonical additive character of $\mathbb{F}_q$, and let $a \in \mathbb{F}_q^*$. Then we have*

$$\sum_{w \in DC(n,q)} \lambda(aTrw) = \lambda(a) A(n, q) K(\lambda; a), for\ n = 1, 3, 5, \cdots \tag{3.5}$$

(cf. (1.4)).

Proposition 3.4. *([5]) Let λ be the canonical additive character of $\mathbb{F}_q$, $\beta \in \mathbb{F}_q$. Then*

$$\sum_{a \in \mathbb{F}_q^*} \lambda(-a\beta) K(\lambda; a) = \begin{cases} q\lambda(\beta^{-1}) + 1, & if\ \beta \neq 0, \\ 1, & if\ \beta = 0. \end{cases} \tag{3.6}$$

For any integer r with $0 \leq r \leq n$, and each $\beta \in \mathbb{F}_q$, we let

$$N_{P\sigma_r P}(\beta) = |\{w \in P\sigma_r P | Trw = \ \beta\}|.$$

Then it is easy to see that

$$qN_{P\sigma_r P}(\beta) = |P\sigma_r P| + \sum_{a \in \mathbb{F}_q^*} \lambda(-a\beta) \sum_{w \in P\sigma_r P} \lambda(aTrw). \tag{3.7}$$

For brevity, we write

$$n(\beta) = N_{DC(n,q)}(\beta). \tag{3.8}$$

Now, from (2.6) and (3.5)-(3.7), we have the following result.

Proposition 3.5. *With the notations in (1.4), (1.5), and (3.8), for* $n = 1, 3, 5, \cdots$,

$$n(\beta) = q^{-1}A(n,q)B(n,q) + q^{-1}A(n,q) \times \begin{cases} 1, & \beta = 1, \\ q+1, & tr(\beta-1)^{-1} = 0, \\ -q+1, & tr(\beta-1)^{-1} = 1. \end{cases} \tag{3.9}$$

Corollary 3.6. *For each odd* $n \geq 3$*, with all* q*,* $n(\beta) > 0$*, for all* β*; for* $n = 1$*, with all* q*,*

$$n(\beta) = \begin{cases} q, & \beta = 1, \\ 2q, & tr(\beta-1)^{-1} = 0, \\ 0, & tr(\beta-1)^{-1} = 1. \end{cases} \tag{3.10}$$

Proof. $n = 1$ case follows directly from (3.9). Let $n \geq 3$ be odd. Then, from (3.9), we see that, for any β, we have $n(\beta) \geq q^{-1}A(n,q)(B(n,q) - (q-1)) > 0$. □

4. Construction of Codes

Let

$$N(n,q) = |DC(n,q)| = A(n,q)B(n,q), \ for \ n = 1, 3, 5, \cdots \tag{4.1}$$

(cf. (1.4), (1.5), (2.6)).

Here we will construct one infinite family of binary linear codes $C(DC(n,q))$ of length $N(n,q)$ for all positive odd integers n and all q, associated with the double cosets $DC(n,q)$.

Let $g_1, g_2, \cdots, g_{N(n,q)}$ be a fixed ordering of the elements in $DC(n,q)$ ($n = 1, 3, 5, \cdots$). Then we put

$$v(n,q) = (Trg_1, Trg_2, \cdots, Trg_{N(n,q)}) \in \mathbb{F}_q^{N(n,q)}, \ for \ n = 1, 3, 5, \cdots.$$

Now, the binary linear code $C(DC(n,q))$ is defined as:

$$C(DC(n,q)) = \{u \in \mathbb{F}_2^{N(n,q)} | u \cdot v(n,q) = 0\}, \ for \ n = 1, 3, 5, \cdots, \tag{4.2}$$

where the dot denotes the usual inner product in $\mathbb{F}_q^{N(n,q)}$.

The following Delsarte's theorem is well-known.

Theorem 4.1. ([14]) *Let B be a linear code over $\mathbb{F}_q$. Then*

$$(B|_{\mathbb{F}_2})^\perp = tr(B^\perp).$$

In view of this theorem, the dual $C(DC(n,q))^\perp$ of the code $C(DC(n,q))$ is given by

$$\begin{aligned} &C(DC(n,q))^\perp \\ &\quad= \{c(a) = c(a;n,q) = (tr(aTrg_1)), \cdots, tr(aTrg_{N(n,q)})|a \in \mathbb{F}_q\} \end{aligned} \tag{4.3}$$

$(n = 1, 3, 5, \cdots)$.

Let $\mathbb{F}_2^+$, $\mathbb{F}_q^+$ denote the additive groups of the fields $\mathbb{F}_2$, $\mathbb{F}_q$, respectively. Then we have the following exact sequence of groups:

$$0 \to \mathbb{F}_2^+ \to \mathbb{F}_q^+ \to \Theta(\mathbb{F}_q) \to 0,$$

where the first map is the inclusion and the second one is the Artin-Schreier operator in characteristic two given by $\Theta(x) = x^2 + x$. So

$$\Theta(\mathbb{F}_q) = \{\alpha^2 + \alpha | \alpha \in \mathbb{F}_q\}, \ \textit{and} \quad [\mathbb{F}_q^+ : \Theta(\mathbb{F}_q)] = 2.$$

Theorem 4.2. ([5]) *Let λ be the canonical additive character of $\mathbb{F}_q$, and let $\beta \in \mathbb{F}_q^*$. Then*

$$\sum_{\alpha \in \mathbb{F}_q - \{0,1\}} \lambda(\frac{\beta}{\alpha^2 + \alpha}) = K(\lambda; \beta) - 1. \tag{4.4}$$

Theorem 4.3. *The map $\mathbb{F}_q \to C(DC(n,q))^\perp (a \mapsto c(a))$ is an $\mathbb{F}_2$-linear isomorphism for each odd integer $n \geq 1$and all q, except for $n = 1$ and $q = 4$.*

Proof. The map is clearly $\mathbb{F}_2$-linear and surjective. Let a be in the kernel of map. Then $tr(aTrg) = 0$, for all $g \in DC(n,q)$. If $n \geq 3$ is odd, then, by Corollary 3.6, $Tr : DC(n,q) \to \mathbb{F}_q$ is surjective and hence $tr(a\alpha) = 0$, for all $\alpha \in \mathbb{F}_q$. This implies that $a = 0$, since otherwise $tr : \mathbb{F}_q \to \mathbb{F}_2$ would be the zero map. Now, assume that $n = 1$. Then, by (3.10), $tr(a\beta) = 0$, for all $\beta \neq 1$, with $tr((\beta - 1)^{-1}) = 0$. Hilbert's theorem 90 says that $tr(\gamma) = 0 \Leftrightarrow \gamma = \alpha^2 + \alpha$, for some $\alpha \in \mathbb{F}_q$. This implies that $\lambda(a) \sum_{\alpha \in \mathbb{F}_q - \{0,1\}} \lambda(\frac{a}{\alpha^2+\alpha}) = q - 2$. If $a \neq 0$, then, invoking (4.4) and the Weil bound (1.1), we would have

$$q - 2 = \lambda(a) \sum_{\alpha \in \mathbb{F}_q - \{0,1\}} \lambda(\frac{a}{\alpha^2 + \alpha}) = \pm(K(\lambda; a) - 1) \leq 2\sqrt{q} + 1.$$

For $q \geq 16$, this is impossible, since $x > 2\sqrt{x} + 3$, for $x \geq 16$. On the other hand, for $q = 2, 4, 8$, one easily checks from (3.10) that the kernel is trivial for $q = 2, 8$ and is $\mathbb{F}_2$, for $q = 4$. □

5. Power Moments of Kloosterman Sums With Trace One Arguments

Here we will be able to find, via the Pless power moment identity, an infinite family of recursive formulas generating the odd power moments of Kloosterman sums with trace one arguments over all $\mathbb{F}_q$ in terms of the frequencies of weights in $C(DC(n,q))$ and $C(\widehat{DC}(n,q))$, respectively.

Theorem 5.1. (Pless power moment identity, [14]) *Let B be a q-ary $[n,k]$ code, and let B_i(resp. $B_i^{\perp}$) denote the number of codewords of weight i in B(resp. in $B^{\perp}$). Then, for $h = 0, 1, 2, \cdots$,*

$$\sum_{j=0}^{n} j^h B_j = \sum_{j=0}^{min\{n,h\}} (-1)^j B_j^{\perp} \sum_{t=j}^{h} t! S(h,t) q^{k-t} (q-1)^{t-j} \binom{n-j}{n-t}, \tag{5.1}$$

where $S(h,t)$ is the Stirling number of the second kind defined in (1.9).

Lemma 5.2. *Let $c(a) = (tr(aTrg_1), \cdots, tr(aTrg_{N(n,q)})) \in C(DC(n,q))^{\perp}(n = 1,3,5,\cdots)$, for $a \in \mathbb{F}_q^*$. Then the Hamming weight $w(c(a))$ is expressed as follows:*

$$w(c(a)) = \frac{1}{2}A(n,q)(B(n,q) - \lambda(a)K(\lambda;a))(cf.\ (1.4),\ (1.5)). \tag{5.2}$$

Proof. Here we recall that the Hamming weight of the codeword $c(a)$ is just the number of nonzero coordinates.

$$w(c(a)) = \frac{1}{2}\sum_{j=1}^{N(n,q)} (1 - (-1)^{tr(aTrg_j)}) = \frac{1}{2}(N(n,q) - \sum_{w \in DC(n,q)} \lambda(aTrw)).$$

Our result now follows from (3.5) and (4.1). □

Let $u = (u_1, \cdots, u_{N_{N(n,q)}}) \in \mathbb{F}_2^{N(n,q)}$, with ν_β 1's in the coordinate palces where $Tr(g_j) = \beta$, for each $\beta \in \mathbb{F}_q$. Then from the definition of the codes $C(DC(n,q))$ (cf.(4.2)) that u is a codeword with weight j if and only if $\sum_{\beta\in\mathbb{F}_q} \nu_\beta = j$ and $\sum_{\beta\in\mathbb{F}_q} \nu_\beta \beta = 0$(an identity in $\mathbb{F}_q$). As there are $\prod_{\beta\in\mathbb{F}_q} \binom{n(\beta)}{\nu_\beta}$ (cf. (3.8)) many such codewords with weight j, we obtain the following result.

Proposition 5.3. *Let $\{C_j(n,q)\}_{j=0}^{N(n,q)}$ be the weight distribution of $C(DC(n,q))$ $(n = 1,3,5,\cdots)$. Then*

$$C_j(n,q) = \sum \prod_{\beta\in\mathbb{F}_q} \binom{n(\beta)}{\nu_\beta}, \tag{5.3}$$

where the sum is over all the sets of integers $\{\nu_\beta\}_{\beta\in\mathbb{F}_q}$ $(0 \le \nu_\beta \le n(\beta))$, satisfying

$$\sum_{\beta\in\mathbb{F}_q} \nu_\beta = j,\ and \sum_{\beta\in\mathbb{F}_q} \nu_\beta\beta = 0. \tag{5.4}$$

Corollary 5.4. *Let* $\{C_j(n,q)\}_{j=0}^{N(n,q)}$ $(n=1,3,5,\cdots)$ *be as above. Then we have*

$$C_j(n,q) = C_{N(n,q)-j}(n,q),\ for\ all\ j,\ with\ 0 \le j \le N(n,q).$$

Proof. Under the replacements $\nu_\beta \to n(\beta) - \nu_\beta$, for each $\beta \in \mathbb{F}_q$, the first equation in (5.4) is changed to $N(n,q) - j$, while the second one in there and the summand in (5.3) is left unchanged. Here the second sum in (5.4) is left unchanged, since $\sum_{\beta \in \mathbb{F}_q} n(\beta)\beta = 0$, as one can see by using the explicit expressions of $n(\beta)$ in (3.9) and (3.10). □

The formula appearing in the next theorem and stated in (1.7) follows from the formula in (5.3), using the explicit value of $n(\beta)$ in (3.9).

Theorem 5.5. *Let* $\{C_j(n,q)\}_{j=0}^{N(n,q)}$ *be the weight distribution of* $C(DC(n,q))$ $(n = 1,3,5,\cdots)$. *Then, for* $j = 0,\cdots,N(n,q)$,

$$\begin{aligned} C_j(n,q) = \sum & \binom{q^{-1}A(n,q)(B(n,q)+1)}{\nu_1} \\ & \times \prod_{tr(\beta-1)^{-1}=0} \binom{q^{-1}A(n,q)(B(n,q)+q+1)}{\nu_\beta} \\ & \qquad \times \prod_{tr(\beta-1)^{-1}=1} \binom{q^{-1}A(n,q)(B(n,q)-q+1)}{\nu_\beta}, \end{aligned}$$

where the sum is over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta\in\mathbb{F}_q}$ *satisfying* $\sum_{\beta\in\mathbb{F}_q}\nu_\beta = j$ *and* $\sum_{\beta\in\mathbb{F}_q}\nu_\beta\beta = 0$.

The recursive formula in the next theorem follows from the study of codes associated with the double cosets $\widehat{DC}(n,q) = P^{'}(2n,q)\sigma_{n-1}^{'}P^{'}(2n,q)$ of the symplectic group $Sp(2n,q)$. It is slightly modified from its original version, which makes it more usable in below.

Theorem 5.6. *([6]) For each odd integer* $n \ge 3$, *with all* q, *or* $n = 1$, *with* $q \ge 8$,

$$\begin{aligned} & \frac{1}{2^h}A(n,q)^h \sum_{l=0}^{h}(-1)^l \binom{h}{l} B(n,q)^{h-l}MK^l \\ & = q \sum_{j=0}^{min\{N(n,q),h\}} (-1)^j \widehat{C}_j(n,q) \sum_{t=j}^{h} t!S(h,t)2^{-t}\binom{N(n,q)-j}{N(n,q)-t}\ (h=1,2,\cdots), \end{aligned} \tag{5.5}$$

where $N(n,q) = A(n,q)B(n,q)$, *and* $\{\widehat{C}_j(n,q)\}_{j=0}^{N(n,q)}$ *is the weight distribution of*

$C(\widehat{DC}(n,q))$ *given by*

$$\widehat{C}_j(n,q) = \sum \binom{q^{-1}A(n,q)(B(n,q)+1)}{\nu_0} \\ \times \prod_{tr(\beta^{-1})=0} \binom{q^{-1}A(n,q)(B(n,q)+q+1)}{\nu_\beta} \\ \times \prod_{tr(\beta^{-1})=1} \binom{q^{-1}A(n,q)(B(n,q)-q+1)}{\nu_\beta}.$$

Here the sum is over all the sets of nonnegative integers $\{\nu_\beta\}_{\beta\in\mathbb{F}_q}$ satisfying $\sum_{\beta\in\mathbb{F}_q}\nu_\beta = j$ and $\sum_{\beta\in\mathbb{F}_q}\nu_\beta\beta = 0$. In addition, $S(h,t)$ is the Stirling number of the second kind as in (1.9).

From now on, we will assume that n is any odd integer ≥ 3, with all q, or $n=1$, with $q \geq 8$. Under these assumptions, each codeword in $C(DC(n,q))^\perp$ can be written as $c(a)$, for a unique $a \in \mathbb{F}_q$ (cf. Theorem 4.3, (4.3)) and Theorem 5.6 in the above can be applied.

Now, we apply the Pless power moment identity in (5.1) to $B = C(DC(n,q))^\perp$(and hence $B_j^\perp = C_j(n,q)$), in order to get the result in Theorem 1.1 (cf. (1.6)-(1.8)) about recursive formulas. Below, "the sum over $tra = 0$(resp. $tra = 1$)" will mean "the sum over all nonzero $a \in \mathbb{F}_q^*$, with $tra = 0$ (resp. $tra = 1$)." The left-hand side of that identity in (5.1) is equal to

$$\sum_{a\in\mathbb{F}_q^*} w(c(a))^h,$$

with $w(c(a))$ given by (5.2). We have

$$\sum_{a\in\mathbb{F}_q^*} w(c(a))^h = \frac{1}{2^h}A(n,q)^h \sum_{a\in\mathbb{F}_q^*}(B(n,q)-\lambda(a)K(\lambda;a))^h \\ = \frac{1}{2^h}A(n,q)^h \sum_{tra=0}(B(n,q)-K(\lambda;a))^h + \frac{1}{2^h}A(n,q)^h \sum_{tra=1}(B(n,q)+K(\lambda;a))^h \tag{5.6}$$

$$
\begin{aligned}
&= \frac{1}{2^h} A(n,q)^h \sum_{tra=0} \sum_{l=0}^{h} (-1)^l \binom{h}{l} B(n,q)^{h-l} K(\lambda;a)^l \\
&\qquad + \frac{1}{2^h} A(n,q)^h \sum_{tra=1} \sum_{l=0}^{h} \binom{h}{l} B(n,q)^{h-l} K(\lambda;a)^l \\
&= \frac{1}{2^h} A(n,q)^h \sum_{l=0}^{h} (-1)^l \binom{h}{l} B(n,q)^{h-l} (MK^l - T_1 K^l) (\psi = \lambda \ case \ of \ (1.2),\ (1.3)) \\
&\qquad + \frac{1}{2^h} A(n,q)^h \sum_{l=0}^{h} \binom{h}{l} B(n,q)^{h-l} T_1 K^l \\
&= \frac{1}{2^h} A(n,q)^h \sum_{l=0}^{h} (-1)^l \binom{h}{l} B(n,q)^{h-l} MK^l \\
&\qquad + 2\frac{1}{2^h} A(n,q)^h \sum_{0 \le l \le h,\ l \ odd} \binom{h}{l} B(n,q)^{h-l} T_1 K^l \\
&= q \sum_{j=0}^{min\{N(n,q),h\}} (-1)^j \widehat{C}_j(n,q) \sum_{t=j}^{h} t! S(h,t) 2^{-t} \binom{N(n,q)-j}{N(n,q)-t} (cf.\ (5.5)) \\
&\qquad + 2\frac{1}{2^h} A(n,q)^h \sum_{0 \le l \le h,\ l \ odd} \binom{h}{l} B(n,q)^{h-l} T_1 K^l.
\end{aligned}
$$

On the other hand, the right hand side of the identity in (5.1) is given by:

$$
q \sum_{j=0}^{min\{N(n,q),h\}} (-1)^j C_j(n,q) \sum_{t=j}^{h} t! S(h,t) 2^{-t} \binom{N(n,q)-j}{N(n,q)-t}. \qquad (5.7)
$$

In (5.7), one has to note that $dim_{\mathbb{F}_2} C(DC(n,q))^{\perp} = r$. Our main result in (1.6) now follows by equating (5.6) and (5.7).

Acknowledgments I would like to thank the referee who read the manuscript carefully and made valuable comments and suggestions.

References

[1] L.Carlitz, *Gauss sums over finite fields of order* 2^n, Acta Arith. **15**(1969), 247–265.

[2] R.J.Evans, *Seventh power moments of Kloosterman sums*, Israel J. Math.**175** (2010), 349-362.

[3] K. Hulek, J. Spandaw, B. van Geemen and D.van van Straten, *The modulartiy of the Barth-Nieto quintic and its relatives*, Adv. Geom. **1** (2001), 263–289.

[4] D. S. Kim, *Gauss sums for symplectic groups over a finite field*, Mh. Math. **126** (1998), 55–71.

[5] D. S. Kim, *Codes associated with $O^{+}(2n, q)$ and power moments of Kloosterman sums*, Integers **11** (2011), A62, 19 pp.

[6] D. S. Kim, *Infinite families of recursive formulas generating power moments of Kloosterman sums: symplectic case*, submitted.

[7] D. S. Kim, *Codes associated with special linear groups and power moments of multi-dimensional Kloosterman sums*, Ann. Mat. Pura Appli. **190** (2011), 61-76.

[8] D. S. Kim, *Infinite families of recursive formulas generating power moments of ternary Kloosterman sums with square arguments arising from symplectic groups*, Adv. Math. Commun.**3** (2009), 167-178.

[9] D. S. Kim, *Infinite families of recursive formulas generating power moments of ternary Kloosterman sums with square arguments associated with $O^{-}(2n, q)$*, J. Korean Math. Soc.**48** (2011), 267-288.

[10] D. S. Kim and Y. H. Park, *Gauss sums for orthogonal groups over a finite field of characteristic two*, Acta Arith., **82** (1997), 331–357.

[11] H. D. Kloosterman, *On the representation of numbers in the form $ax^2 + by^2 + cz^2 + dt^2$*, Acta Math. **49** (1926), 407-464.

[12] R. Lidl and H. Niederreiter, *Finite Fields*, Encyclopedia Math. Appl. **20**, Cambridge University Pless, Cambridge, 1987.

[13] R. Livné, *Motivic orthogonal two-dimensional representations of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$*, Israel J. Math. **92** (1995), 149-156.

[14] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error Correcting Codes*, North-Holland, Amsterdam, 1998.

[15] M. Moisio, *The moments of a Kloosterman sum and the weight distribution of a Zetterberg-type binary cyclic code*, IEEE Trans. Inform. Theory. **53**(2007), 843–847.

[16] C. Peters, J. Top, and M. van der Vlugt, *The Hasse zeta function of a K3 surface related to the number of words of weight 5 in the Melas codes*, J. Reine Angew. Math. **432** (1992), 151-176.

[17] H. Salié, *Über die Kloostermanschen Summen $\mathcal{S}(u, v; q)$*, Math. Z. **34**(1931), 91-109.

[18] R. Schoof and M. van der Vlugt, *Hecke operators and the weight distributions of certain codes*, J. Combin. Theory Ser. A **57**(1991), 163-186.