BRUN MEETS SELMER

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Abstract

The most famous 2-dimensional continued fraction algorithm is the Jacobi algorithm. However, Brun and Selmer algorithms are also interesting 2-dimensional subtractive algorithms. Schratzberger shows that all these three algorithms are deeply related by a process similar to insertion and extension for continued fractions. In this note the basic ergodic properties of two mixtures of both maps are explored. Furthermore a digression to a quite different map is made which exhibits an "exotic" invariant measure.

1. Introduction

The most famous 2-dimensional continued fraction algorithm is the Jacobi algorithm. Last years saw an increasing interest in other 2-dimensional algorithms (see [9], chapters 6 and 7, and [2]). The Brun and the Selmer algorithms are remarkable examples of this type. In the first section we give a short description of both algorithms and look shortly on the flip-flop map built on both maps. It generalizes the 1-dimensional map

$$x \mapsto \frac{x}{1-x},\ 0 \le x \le \frac{1}{2}$$
$$x \mapsto \frac{1-x}{x},\ \frac{1}{2} \le x \le 1$$

to the set

$$B := \{(x_1, x_2) : \quad 0 \le x_2 \le x_1 \le 1\}.$$

The jump map (see [9], chapter 3) which avoids the critical point $(0, 0)$ leads to Garrity's triangle sequence (Assaf et al. [1]). The next section is devoted to the study of the composition of the Brun and the Selmer map. The set

$$D^- := \{(x_1, x_2) \in B : x_1 + x_2 \le 1\}$$

is transient for the Selmer map and therefore the study of its ergodic behaviour concentrates on the set

$$D^+ := \{(x_1, x_2) \in B : x_1 + x_2 \geq 1\}.$$

The Brun map expands this set D^+ onto the full set B. Therefore, the study of the interplay of these different dynamics may be of some interest.
In the last section a digression to a different map is made which exhibits an "exotic" invariant measure. "Exotic" means that it is possible to construct a fractal like set with positive Lebesgue measure and an invariant density.

2. The Brun, the Selmer Algorithm, and the Flip-flop Map

The Brun algorithm $T : B \to B$ is given by the matrices of its inverse branches

$$M_\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_\gamma = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

which correspond to a partition of B into three cells $B(\alpha) = M_\alpha B, B(\beta) = M_\beta B$, and $B(\gamma) = M_\gamma B = D^+$ (see Figure 1).

The Selmer algorithm $S : B \to B$ is defined by the matrices of its inverse branches

$$M_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

There is an important difference to be observed. $M_0 B$ is the triangle $B(0) = D^-$ with vertices $[1, 0, 0], [1, 1, 0]$ and $[2, 1, 1]$ but M_1 and M_2 are restricted to the triangle D^+. Then $M_1 D^+$ is the triangle $B(1)$ with vertices $[1, 1, 0], [2, 2, 1]$ and $[2, 1, 1]$. $M_2 D^+$ is the triangle $B(2)$ with vertices $[1, 1, 1], [2, 2, 1]$ and $[2, 1, 1]$ (see Figure 2).
The flip-flop map uses the matrices M_0 and M_γ. It gives the (forward) map

$$F(x_1, x_2) = \left(\frac{x_1}{1 - x_2}, \frac{x_2}{1 - x_2}\right); \, x \in B(0),$$

$$F(x_1, x_2) = \left(\frac{x_2}{x_1}, \frac{1 - x_1}{x_1}\right); \, x \in B(\gamma).$$

Although Pipping used a kind of mixture of both algorithms [6] this kind of a flip-flop between both algorithms seems not to be investigated. We show that this algorithm admits a σ-finite invariant measure but is related to Garrity's triangle sequence.

A product of n matrices M_η, $\eta \in \{0,\gamma\}$ gives a matrix $((B_{ij}^{(n)}))$, $0 \le i,j \le 2$ and the Jacobian of an inverse branch after n steps is given by

$$\omega(\eta_1,\ldots,\eta_n;x) = \frac{1}{(B_{00}^{(n)} + B_{01}^{(n)}x_1 + B_{02}^{(n)}x_2)^3}.$$

Therefore the measure of a cylinder of rank n is given by

$$\lambda(B(\eta_1,...,\eta_n)) = \frac{1}{2B_{00}^{(n)}(B_{00}^{(n)}+B_{01}^{(n)})(B_{00}^{(n)}+B_{01}^{(n)}+B_{02}^{(n)})}.$$

Theorem 1: *The function*

$$h(x_1,x_2) = \frac{1}{x_1x_2}$$

is the density of a σ-finite invariant measure.
This assertion is easily verified.

If we consider the jump map over the cylinder $B(0)$ we obtain a map with matrices

$$\begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & k & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

This algorithm is Garrity's triangle sequence (see e. g. [1, 4, 10]). Therefore the map F is ergodic. Since the segment $\overline{(0,0)(1,0)}$ is pointwise invariant it is no surprise that this algorithm does not converge everywhere. If $p^{(s)} = p(k_1,...,k_s)$ and $q^{(s)} = q(k_1,...,k_s)$ are the vertices of the cylinder $B(k_1,...,k_s)$ such that $F^sp^{(s)} = (0,0)$ and $F^sq^{(s)} = (1,0)$ then the segments $\overline{p(k_1,...,k_s,k_{s+1}),q(k_1,...,k_s,k_{s+1})}$ converge to the segment $\overline{p(k_1,...,k_s),q(k_1,...,k_s)}$ as $k_{s+1}\to\infty$. Then we choose a sequence $(k_1,k_2,k_3,...)$ such that

$$\frac{d(p(k_1,...,k_s,k_{s+1}),q(k_1,...,k_s,k_{s+1})}{d(p(k_1,...,k_s),q(k_1,...,k_s))} > \frac{k_s}{1+k_s}$$

and the infinite product $\prod_s \frac{k_s}{1+k_s}$ converges. More details can be found in Assaf et al. [1].

3. The Composition of Both Maps

We now consider the mixed map $(S\circ T)x = T(Sx)$. Since $SB(1) = SB(2) = D^+ = B(\gamma)$ the map $S\circ T$ can be described by the five matrices

$$M_{0\alpha} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_{0\beta} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, M_{0\gamma} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$M_{1\gamma} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, M_{2\gamma} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

These five matrices give a partition of B into five cylinders (see Figure 3).

Lemma 1: *The set*

$$E = \{x : (S \circ T)^j x \in B(0\alpha) \cup B(0\beta) \text{ for all } j \geq 0\}$$

has measure $\lambda(E) = 0$.

Proof. The product of N matrices $M_{0\alpha}$ and $M_{0\beta}$ has the form

$$M^{(N)} = \begin{pmatrix} B_{00}^{(N)} & B_{01}^{(N)} & B_{02}^{(N)} \\ B_{10}^{(N)} & B_{11}^{(N)} & B_{12}^{(N)} \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore $x = (x_1, x_2)$ is mapped onto

$$(x_1^{(N)}, x_2^{(N)}) = \left(\frac{B_{10}^{(N)} + B_{11}^{(N)} x_1 + B_{12}^{(N)} x_2}{B_{00}^{(N)} + B_{01}^{(N)} x_1 + B_{02}^{(N)} x_2}, \frac{x_2}{B_{00}^{(N)} + B_{01}^{(N)} x_1 + B_{02}^{(N)} x_2} \right).$$

This implies $\lim_{N \to \infty} x_2^{(N)} = 0$. □

Lemma 2: *We have* $B_{02}^{(N)} \leq B_{00}^{(N)} + B_{01}^{(N)}$.

Proof. For $N = 1$ this is verified by inspection. Then we use induction. Let 0α or 0β be the N-th digit. Then

$$B_{02}^{(N+1)} = B_{00}^{(N)} + B_{02}^{(N)} \leq B_{00}^{(N)} + B_{00}^{(N)} + B_{01}^{(N)} = B_{00}^{(N+1)} + B_{01}^{(N+1)}.$$

If $\varepsilon_N \in \{0\gamma, 1\gamma, 2\gamma\}$ the assertion is immediate.
Now we consider the jump transformation $R : B \to B$ which leaves out the digits 0α and 0β. This means we define

$$Rx := (S \circ T)^n x$$

if $x \in B(\varepsilon_1, \ldots, \varepsilon_n)$, $\varepsilon_1, \ldots, \varepsilon_{n-1} \in \{0\alpha, 0\beta\}$ but $\varepsilon_n \in \{0\gamma, 1\gamma, 2\gamma\}$. Lemma 1 implies that R is defined almost everywhere. □

Lemma 3: *R satisfies a Rényi condition.*

Proof. Let

$$\omega(\varepsilon_1, \ldots, \varepsilon_N; x) = \frac{1}{(B_{00}^{(N)} + B_{01}^{(N)} x_1 + B_{02}^{(N)} x_2)^3}$$

be the Jacobian of an inverse branch of R. We have to compare $B_{00}^{(N)}$ with $B_{00}^{(N)} + B_{01}^{(N)} + B_{02}^{(N)}$. Since $\varepsilon_N \in \{0\gamma, 1\gamma, 2\gamma\}$ we see that

$$B_{00}^{(N)} \geq B_{00}^{(N-1)} + B_{01}^{(N-1)}$$

but

$$B_{00}^{(N)} + B_{01}^{(N)} + B_{02}^{(N)} \leq 3B_{00}^{(N-1)} + 2B_{01}^{(N-1)} + B_{02}^{(N-1)} \leq 4B_{00}^{(N-1)} + 3B_{01}^{(N-1)}$$

by Lemma 2. □

Lemma 4: *If the sequence $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)$ contains one of the digits $0\gamma, 1\gamma$, or 2γ infinitely often then $\lim_{n\to\infty}$ diam $B(\varepsilon_1, \ldots, \varepsilon_n) = 0$.*

Proof. We describe the vertices of the cylinders we consider as the pictures of points in projective coordinates (see Figure 4) and suppress the upper index of the relevant matrix

$$\beta = \beta(\varepsilon_1, \ldots, \varepsilon_n) = \begin{pmatrix} B_{00} & B_{01} & B_{02} \\ B_{10} & B_{11} & B_{12} \\ B_{20} & B_{21} & B_{22} \end{pmatrix}.$$

We look for triangles which lie inside the triangle $B(\varepsilon_1, \ldots, \varepsilon_n)$ and contain the triangle $B(\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+1})$ or in some cases the triangle $B(\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+1}, \varepsilon_{n+2})$. If the points $[a, b, c]$, $[a', b', c']$, and $[a'', b'', c'']$ are collinear such that

$$\lambda[a, b, c] + [a', b', c'] = [a'', b'', c'']$$

we will estimate the ratio

$$\frac{d(\beta[a, b, c], \beta[a', b', c'])}{d(\beta[a, b, c], \beta[a'', b'', c''])} = \frac{B_{00}a'' + B_{01}b'' + B_{02}c''}{B_{00}a' + B_{01}b' + B_{02}c'}.$$

We further use that for $\alpha < \delta$ the function $f(t) = \frac{\alpha+t}{\delta+t}$ is increasing on $0 \leq t$.

$\boxed{\varepsilon_{n+1} = 0\alpha}$

$$\frac{d(\beta[1, 0, 0], \beta[2, 1, 0])}{d(\beta[1, 0, 0], \beta[1, 1, 0])} = \frac{B_{00} + B_{01}}{2B_{00} + B_{01}}.$$

$$\frac{d(\beta[1, 0, 0], \beta[3, 1, 1])}{d(\beta[1, 0, 0], \beta[1, 1, 1])} = \frac{B_{00} + B_{01} + B_{02}}{3B_{00} + B_{01} + B_{02}} \leq \frac{B_{00} + B_{01}}{2B_{00} + B_{01}}.$$

Since the periodic point $\overline{0\alpha}$ shrinks to the point $(0,0)$ we can additionally assume that $\varepsilon_n \in \{0\beta, 0\gamma, 1\gamma, 2\gamma\}$. Then the recursion relations show $B_{01} \le 2B_{00}$ and we obtain

$$\frac{B_{00}+B_{01}}{2B_{00}+B_{01}} \le \frac{3}{4}.$$

$\boxed{\varepsilon_{n+1} = 2\gamma}$

In a similar way as before we find the ratios

$$\frac{d(\beta[1,1,1],\beta[2,2,1])}{d(\beta[1,1,1],\beta[1,1,0])} = \frac{B_{00}+B_{01}}{2B_{00}+2B_{01}+B_{02}} \le \frac{1}{2}.$$

$$\frac{d(\beta[1,1,1],\beta[2,1,1])}{d(\beta[1,1,1],\beta[1,0,0])} = \frac{B_{00}}{2B_{00}+B_{01}+B_{02}} \le \frac{1}{2}.$$

$\boxed{\varepsilon_{n+1} = 0\beta}$

Here we use the additional points $\beta[3,2,1]$ and $\beta[2,1,1]$ which lie outside on the line which joins $\beta[1,1,0]$ and$\beta[2,1,1]$.

$\boxed{\varepsilon_{n+2} = 0\beta, 0\gamma, 1\gamma}$

$$\frac{d(\beta[2,1,0],\beta[3,2,0])}{d(\beta[2,1,0],\beta[1,1,0])} = \frac{B_{00}+B_{01}}{3B_{00}+2B_{01}} \le \frac{1}{2}.$$

$$\frac{d(\beta[2,1,0],\beta[5,3,1])}{d(\beta[2,1,0],\beta[3,2,1])} = \frac{3B_{00}+2B_{01}+B_{02}}{5B_{00}+3B_{01}+B_{02}} \le \frac{3}{4}.$$

$$\frac{d(\beta[2,1,0],\beta[4,2,1])}{d(\beta[2,1,0],\beta[2,1,1])} = \frac{2B_{00}+B_{01}+B_{02}}{4B_{00}+2B_{01}+B_{02}} \le \frac{2}{3}.$$

$$\frac{d(\beta[2,1,0],\beta[5,2,1])}{d(\beta[2,1,0],\beta[3,1,1])} = \frac{3B_{00}+B_{01}+B_{02}}{5B_{00}+2B_{01}+B_{02}} \le \frac{2}{3}.$$

$\boxed{\varepsilon_{n+1} = 0\gamma}$

Here we use the additional points $\beta[3,2,0]$, $[2,1,0]$, and $\beta[1,0,0]$.

$\boxed{\varepsilon_{n+2} = 0\gamma, 1\gamma}$

$$\frac{d(\beta[2,1,1],\beta[5,3,1])}{d(\beta[2,1,1],\beta[3,2,0])} = \frac{3B_{00}+2B_{01}}{5B_{00}+3B_{01}+B_{02}} \le \frac{2}{3}.$$

$$\frac{d(\beta[2,1,1],\beta[4,2,1])}{d(\beta[2,1,1],\beta[2,1,0])} = \frac{2B_{00}+B_{01}}{4B_{00}+2B_{01}+B_{02}} \le \frac{1}{2}.$$

$$\frac{d(\beta[2,1,1],\beta[5,2,2])}{d(\beta[2,1,1],\beta[1,0,0])} = \frac{B_{00}}{5B_{00}+2B_{01}+2B_{02}} \le \frac{1}{5}.$$

$\boxed{\varepsilon_{n+1} = 1\gamma}$

Only the case $\overline{1\gamma}$ remains; however, the sequence of associated triangles shrinks to the point $(\lambda-1, \lambda^2-\lambda-1)$, where $\lambda > 1$ is the greatest root of $\lambda^3 = \lambda^2+2\lambda-1$. □

Lemmas 1-4 provide the necessary machinery to deduce the following:

Theorem 2: *$S \circ T$ is ergodic and admits a σ-finite invariant measure $\mu \sim \lambda$.*

Remark: The map $(T \circ S)(x) = S(Tx)$ divides B into nine cells. Since $S \circ (T \circ S) = (S \circ T) \circ S$ their ergodic behaviors are equivalent.

4. A Split Algorithm

The next algorithm is not directly related to the Brun or the Selmer algorithm but shows that the “exotic” behaviour which was first detected with the Parry-Daniels map is quite common (see [5]).
The starting point are the three matrices

$$\beta(1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \beta(2) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \beta(3) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

These matrices form a 2-dimensional continued fraction on the basic set $(\mathbb{R}^+)^2$ with the three inverse branches

$$\begin{aligned} V(1)(u,v) &= (1+u, 1+v) \\ V(2)(u,v) &= \left(\frac{1+u}{1+v}, \frac{1}{1+v}\right) \\ V(3)(u,v) &= \left(\frac{1}{1+u}, \frac{1+v}{1+u}\right) \end{aligned}$$

and the basic partition is

$$B(1) = \{(u,v) : 1 \le u, 1 \le v\}$$

$$B(2) = \{(u,v) : 0 \le v \le u, v \le 1\}$$

$$B(3) = \{(u,v) : 0 \le u \le v, u \le 1\}.$$

The dual map is given given as

$$V^{\#}(1)(x,y) = \left(\frac{x}{1+x+y}, \frac{y}{1+x+y}\right)$$

$$V^{\#}(2)(x,y) = (\frac{x}{1+x+y}, \frac{1}{1+x+y})$$
$$V^{\#}(3)(x,y) = (\frac{1}{1+x+y}, \frac{y}{1+x+y})$$

which may be compared with the 2-dimensional Farey-Brocot algorithm which was considered in Schweiger [10]. This algorithm sits on a set E with $\lambda(E) = 0$ but the function

$$g(x_1, x_2) = \frac{1}{x_1 x_2}$$

behaves formally as an invariant density. It would be nice to explore if in some limiting sense the integral

$$\int_E \frac{dx_1 dx_2}{x_1 x_2}$$

is finite.

Let

$$E_{12} = \{(u,v) : T^s(u,v) \in B(1) \cup B(2), s \geq 0\}$$

and

$$E_{13} = \{(u,v) : T^s(u,v) \in B(1) \cup B(3), s \geq 0\}.$$

We will show that $\lambda(E_{12}) = \lambda(E_{13}) > 0$ and calculate an invariant density for the map T restricted to E_{12}.
We consider the first return map on the set on the set $B(2)$ of the restriction of T to E_{12}. This map is given as $R(u,v) = T^k(u,v)$ if $(u,v) \in B(2), T^j(u,v) \in B(1), 1 \leq j \leq k-1, T^k(u,v) \in B(2)$. The associated matrices are given as

$$\beta(2)\beta(1)^k =: \gamma(a) = \begin{pmatrix} a & 0 & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where $a = k+1$. These matrices are related to continued fractions! If

$$\gamma(a_1)...\gamma(a_s) = \begin{pmatrix} q_s & 0 & q_{s-1} \\ r_s & 1 & r_{s-1} \\ p_s & 0 & p_{s-1} \end{pmatrix}$$

then as usual $q_s = a_s q_{s-1} + q_{s-2}$, $p_s = a_s p_{s-1} + p_{s-2}$ but $r_s = a_s r_{s-1} + r_{s-2} + a_s$. The last recursion can be written as $r_s + 1 = a_s(r_{s-1}+1) + r_{s-2} + 1$ which shows that $q_s \leq r_s \leq 2q_s$.

Theorem 3: $\lambda(E_{12}) > 0$.

Proof. We transport the map T into the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$ by using the map $\psi(u,v) = (\frac{u}{1+u+v}, \frac{v}{1+u+v})$. The quotient of the measure of the cylinder $B(a_1, ..., a_s)$ and the length of the associated continued fraction interval $I(a_1, ...a_s)$ is bounded from below. Therefore we find $\lambda(E_{12}) > 0$. □

Theorem 4:*Let* $\theta = [a_1, a_2, ...]$ *be a regular continued fraction and define* $\Gamma(\theta) = \sum_{n=0}^{\infty}(\prod_{j=0}^{n} T^j\theta)a_{n+1}$. *Then the function*

$$h(u, v) = \frac{1}{(1 + v)(u - \Gamma(v))}$$

is an invariant density for the map T *restricted to the set* E_{12}.

Proof. We first remark

$$\Gamma(\theta) = \Gamma(\frac{1}{a + \theta})(a + \theta) - a.$$

Then we calculate

$$\sum_{a=1}^{\infty} h(\frac{a + u}{a + v}, \frac{1}{a + v})\frac{1}{(a + v)^3} = \sum_{a=1}^{\infty} \frac{1}{(a + 1 + v)(a + v)(a + u - \Gamma((a + v)^{-1})(a + v))}$$

$$= \frac{1}{u - \Gamma(v)} \sum_{a=1}^{\infty} \frac{1}{(a + v)(a + 1 + v)} = \frac{1}{(1 + v)(u - \Gamma(v))}.$$

Remark: The dual map defined by

$$\gamma^{\#}(a) = \begin{pmatrix} a & a & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

formally has the invariant density

$$f(x_1, x_2) = \frac{1}{x_1} \int_0^1 \frac{dv}{(1 + x_1\Gamma(v) + x_2 v)^2}.$$

We verify this by direct calculation:

$$\sum_{a=1}^{\infty} f(\frac{x_1}{a + ax_1 + x_2}, \frac{1}{a + ax_1 + x_2})\frac{1}{(a + ax_1 + x_2)^3}$$

$$= \frac{1}{x_1} \sum_{a=1}^{\infty} \int_0^1 \frac{dv}{(a + (a + \Gamma(v))x_1 + x_2 + v)^2}$$

$$= \frac{1}{x_1} \sum_{a=1}^{\infty} \int_{\frac{1}{a+1}}^{\frac{1}{a}} \frac{dw}{(1 + \Gamma(w))x_1 + x_2 w)^2} = F(x_1, x_2).$$

This follows from $w = \frac{1}{a+v}$ and the equation $\Gamma(v) + a = \Gamma(w)(a + v)$. □

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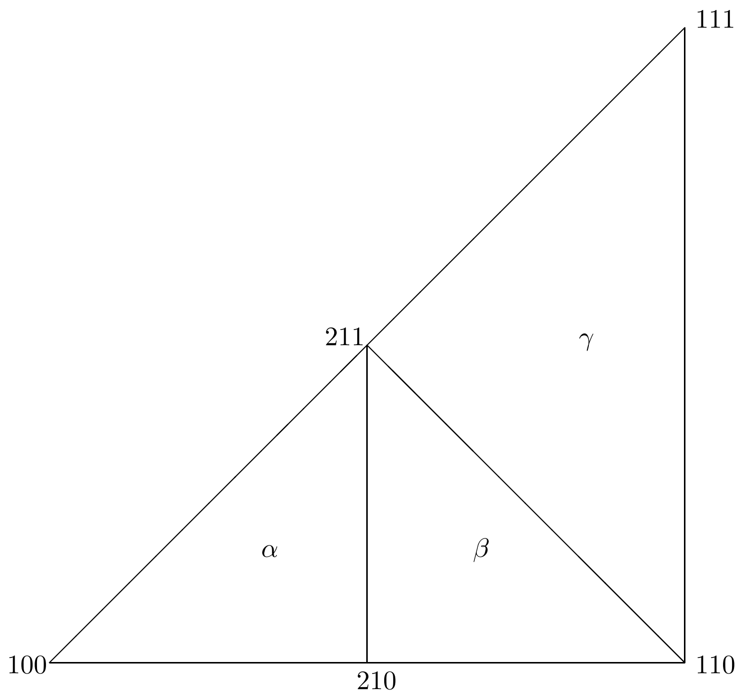

Figure 1

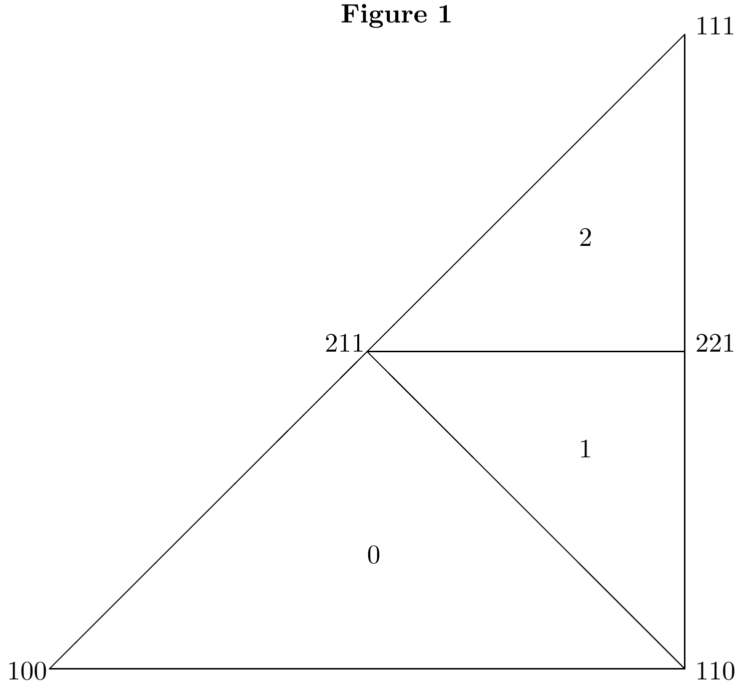

Figure 2

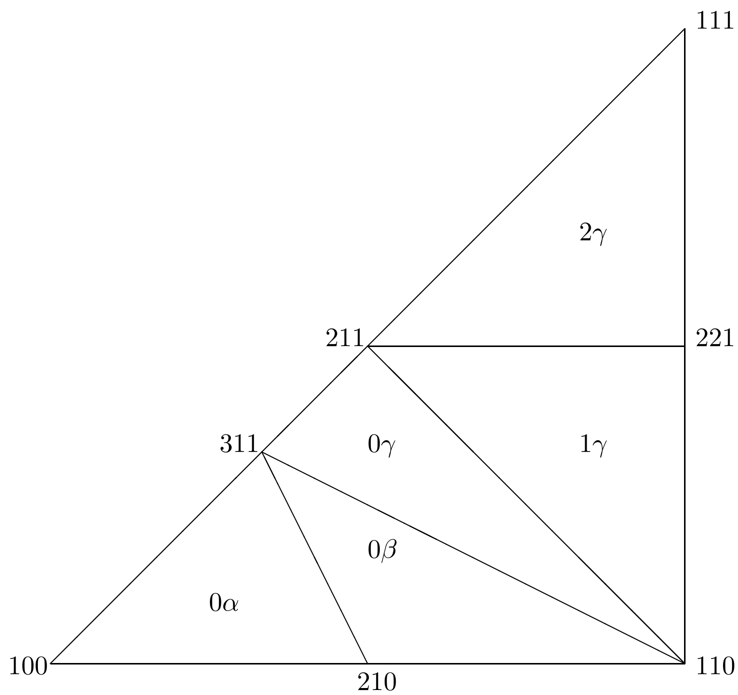

Figure 3

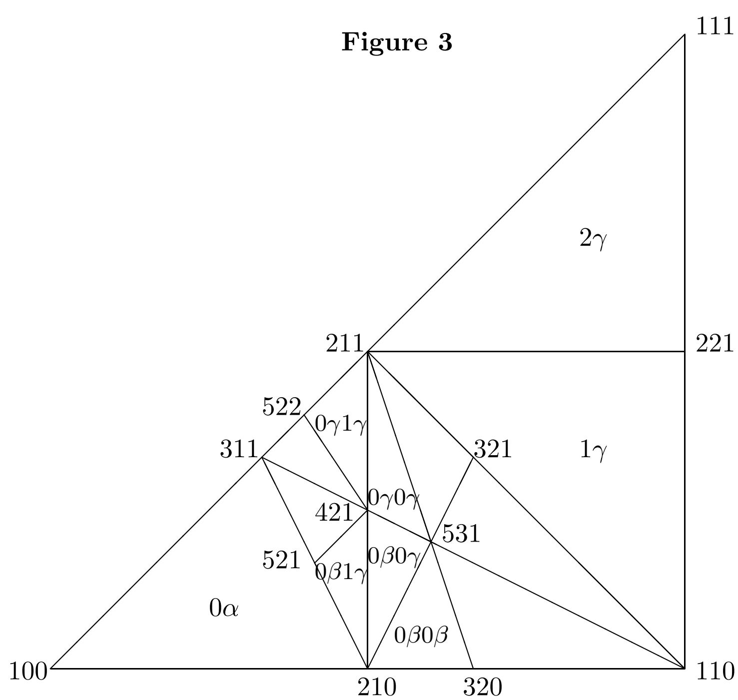

Figure 4