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BIJECTIVE PROOFS OF VAJDA'S NINETIETH FIBONACCI NUMBER IDENTITY AND RELATED IDENTITIES

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Abstract
This article provides the first bijective proof for a previously "uncounted" Fibonacci number identity of Vajda. Bijections on similar sets that illustrate a related family of Fibonacci number identities are also considered.

1. Introduction

The Fibonacci numbers F_n are defined for $n \geq 0$ by the recurrence

$$F_n = \begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_{n+2} = F_{n+1} + F_n \quad n \geq 0 \end{cases}.$$

Many surprising identities involving the Fibonacci numbers are known. In [1], Benjamin and Quinn provide bijective proofs of many such identities. They also provide a list of identities for which bijective proofs are not known. Many of these identities were drawn from a list created by Vajda [2], including the following identity:

$$-1 + \sum_{k=2}^{n} \frac{1}{F_{2^k}} = -\frac{F_{2^n-1}}{F_{2^n}}. \tag{$V90$}$$

An equivalent form, which we find more convenient to prove, is the following, which results from clearing the denominators, multiplying both sides by F_2 (which is equal to 1), and rearranging the result slightly:

$$F_{2^n-1} \prod_{1 \leq j \leq n-1} F_{2^j} + \sum_{k=1}^{n-1} \prod_{\substack{1 \leq j \leq n \\ j \neq k+1}} F_{2^j} = \prod_{1 \leq j \leq n} F_{2^j}. \tag{1}$$

Identity (V90) is easily proved by induction. Using the techniques from [3] for translating proofs by induction into bijective proofs, we were able to find a bijective proof of (V90). In Section 2, we introduce sets whose cardinalities are Fibonacci numbers. In Section 3, we describe a "tail-swap maneuver," inspired by the tail-swapping bijections highlighted in [1], which serves as the building block of the bijective proof. In Section 4, we describe the bijection itself. In Section 5, we describe a family of identities which we believe to be new, and which can be proved by bijections very similar to those which we used to prove (V90).

2. Combinatorial Interpretation of Fibonacci Numbers

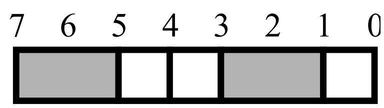

Figure 1: A tiling of length 7

Following [1], we note that the set of tilings, using only squares and dominoes, of a 1-by-n strip, has cardinality F_{n+1}. We will write T_n to denote the set of such tilings, and also write $f_n = |T_n|$. Such a tiling (where $n = 7$) is illustrated in Figure 1.

We generally identify the rightmost end of a tiling as "position 0," with positions $1, \ldots, n$ numbered from right to left. (This is different notation than is used in our sources, but it is more convenient for our purposes.) We say that a tile is at position k if its rightmost edge is at position k. For example, in Figure 1 there are dominoes at positions 1 and 5, and squares at positions 0, 3, and 4. We say that a tiling has a *fault* at position k, with $0 \leq k \leq n$, unless it has a domino at position $k-1$. For example, the tiling in Figure 1 has faults at all positions except 2 and 6.

When we consider multiple tilings at once, we consider them to be aligned vertically in some way. In case their rightmost edges are not aligned, we will consider position 0 to be the rightmost edge of the tiling with the rightmost edge located furthest to the right. Given two tilings written one above the other, we say they have a *common fault* at position k unless at least one of them has a domino at position $k-1$.

3. The Tail-Swap Maneuver

In our bijective proof of (1) we make use of a *tail-swap maneuver* (see [1]), illustrated in Figure 2.

Given integers $m, n \geq 0$, define

$$\tau_{m,n} : T_m \times T_n \to (T_m \cup T_{m+1}) \times (T_n \cup T_{n-1})$$

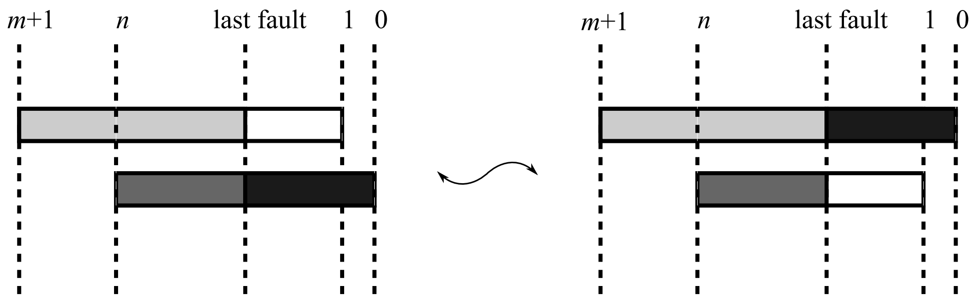

Figure 2: The tail-swap maneuver

as follows. Given two tilings t_1 and t_2, of lengths m and n respectively, we write t_1 above t_2 so that the rightmost block of t_2 protrudes 1 unit beyond the rightmost block of t_1. (Thus, the rightmost edge of t_1 is at position 1, and the rightmost edge of t_2 is at position 0.) Then we locate the rightmost fault common to both tilings, if it exists. Any tiles to the right of this fault are swapped to the other tiling. After this process, we have a tiling t_1' of length $m+1$ and a tiling t_2' of length $n-1$. We write $\tau_{m,n}(t_1,t_2) = (t_1', t_2')$ and say that the tail-swap succeeded. If no common fault exists, we write $\tau_{m,n}(t_1,t_2) = (t_1,t_2)$. In this case, we say that the tail-swap failed.

Remark 1. For brevity, when the lengths of the tilings are known, we write τ in place of $\tau_{m,n}$. We will also refer to the two tilings that are elements of $\tau(t_1,t_2)$ as the first and second *resultant tilings* of the tail-swap.

Proposition 2. *Suppose t_1 and t_2 are tilings of lengths m and n, respectively, with $m \le n$. If a tail-swap of t_1 and t_2 fails, then m is even and t_1 consists only of dominoes. Furthermore, the rightmost $m/2+1$ tiles of t_2 are all dominoes.*

Proof. If t_1 contains a square, then it has two adjacent faults. Tiling t_2 must have a fault at one of these positions (if it did not, then it would have dominoes at two adjacent positions, which is impossible), so t_1 and t_2 have a common fault. Therefore, a tail-swap of t_1 and t_2 does not fail. This proves that if a tail-swap of t_1 and t_2 fails, then t_1 consists only of dominoes. In particular, m is even. Furthermore, there are no common faults between t_1 and t_2, so the rightmost $m/2+1$ tiles of t_2 must also be dominoes. $\square$

The following proposition, which is apparent from Figure 2 , states that when a tail-swap succeeds, it can be reversed by performing a second tail-swap.

Proposition 3. *Given two tilings t_1 and t_2, if $\tau(t_1,t_2) = (t_1', t_2')$ is a successful tail-swap, then $\tau(t_2', t_1') = (t_2, t_1)$.*

Proof. Because the tail-swap succeeded, after the tails of t_1 and t_2 are swapped, the result is that tiling t_1' is aligned 1 block to the right of t_2'. The rightmost common fault is still at the same location, so performing $\tau(t_2', t_1')$ simply swaps the tails back to their original positions. □

4. The Bijection

We begin by describing sets whose cardinalities are the right- and left-hand sides of (1). Let

$$S_1 = \prod_{1 \le j \le n} T_{2^j - 1};$$

then the cardinality of S_1 is clearly

$$\prod_{1 \le j \le n} F_{2^j},$$

the right-hand side of (1). Let

$$S_2 = T_{2^n - 2} \times \left(\prod_{2 \le j \le n-1} T_{2^j - 1} \cup \bigcup_{k=2}^{n} \prod_{\substack{2 \le j \le n \\ j \ne k}} T_{2^j - 1} \right). \tag{2}$$

(The index j may begin at 2 because $|T_1| = 1$.) We note that all of the unions in (2) are disjoint. Clearly the cardinality of S_2 is the left-hand-side of (1). We will establish a bijection between S_1 and S_2.

Before describing the bijection, we describe an iterative version of the tail-swap process, called a multiple tail-swap, that takes place on a sequence of $k \ge 2$ tilings $t_1, \dots, t_k$ of lengths $l_1, \dots, l_k$. Define $\tau(t_1, \dots, t_k)$ as follows. First find $\tau(t_1, t_2)$. If this tail-swap fails, let $\tau(t_1, \dots, t_k)$ be $(t_1, \dots, t_k)$ If it succeeds, replace t_1 and t_2 with the first and second resultant tilings, respectively. Then find $\tau(t_2, t_3)$. If this tail-swap fails, let $\tau(t_1, \dots, t_k)$ be $(t_1, \dots, t_k)$. If it succeeds, replace t_2 and t_3 with the first and second resultant tilings and find $\tau(t_3, t_4)$. Continue in this way until every tiling has participated in a tail-swap, or until one of the tail-swaps has failed. If any tail-swap $\tau(t_r, t_{r+1})$ fails, then we say that the *failure index* of the multiple tail-swap is r and the process stops. If all of the successive tail-swaps succeed, the failure index is k. Notice that the failure index can be determined by comparing the lengths of the resultant tilings compared to their original lengths. In particular, if $2 \le I \le k$ and t_I has been shortened by one unit, then the failure index is I, and if all tilings still have their original lengths, then the failure index is 1.

An example of a multiple tail-swap, with $k = 3$, can be seen in Figure 3.

Notice that if $k = 2$, this is merely the original tail-swap operation. We now consider the reversibility of the multiple tail-swap.

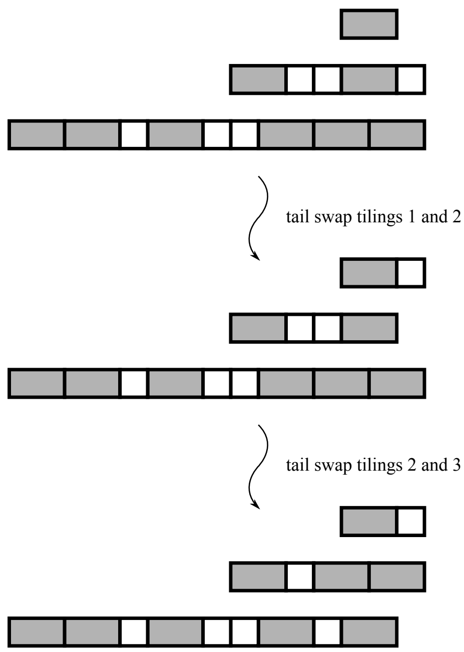

Figure 3: A multiple tail-swap on 3 tilings, in which the failure index is 3.

Proposition 4. *If $\tau(t_1, \ldots, t_n) = (t'_1, \ldots, t'_n)$ is a multiple tail-swap with failure index $I > 1$, then $\tau(t'_I, \ldots, t'_1) = (t_I, \ldots, t_1)$ is a successful multiple tail-swap.*

Proof. Because the failure index is I, the first $I - 1$ tail-swaps of the multiple tail-swap succeed, and then the Ith either fails or, if $I = n$, is not performed because the last tiling has been swapped. Failed tail-swaps do not change any of the tilings, so the original multiple tail-swap leaves the tilings in the state they are in after the $(I-1)$th tail-swap. Performing $\tau(t'_I, \ldots, t'_1)$ reverses these $(I-1)$ tail-swaps, which restores the sequence $(t_I, \ldots, t_1)$ by Proposition 2. $\square$

Remark 5. Proposition 3 shows that

$$\tau : \prod_{k=1}^{n} T_{l_k} \to \left(\prod_{k=1}^{n} T_{l_k}\right) \cup \bigcup_{I=2}^{n-1} \left(T_{l_1+1} \times \prod_{1\le k\le I-1} T_{l_k} \times T_{l_I-1} \times \prod_{I+1\le k\le n} T_{l_k}\right)$$

is injective.

We next describe $\mathrm{img}(\tau)$ when the lengths of the tilings satisfy a certain condition.

Theorem 6. *Suppose l_1 is even and $l_2, \ldots, l_n$ are all odd, and that $l_1, \ldots, l_n$ is* *increasing. Given*

$$(t'_1,\dots,t'_n)\in\left(\prod_{k=1}^{n}T_{l_k}\right)\cup\bigcup_{I=2}^{n-1}\left(T_{l_1+1}\times\left(\prod_{1\le k\le I-1}T_{l_k}\right)\times T_{l_I-1}\times\left(\prod_{I+1\le k\le n}T_{l_k}\right)\right),$$

where $t'_1,\dots,t'_n$ *have lengths* $l'_1,\dots,l'_n$ *respectively, there exists* $(t_1,\dots,t_n)\in\prod_{k=1}^{n}T_{l_k}$ *such that* $\tau(t_1,\dots,t_n)=(t'_1,\dots,t'_n)$ *if and only if one of the following is true:*
`Case 1`: *All the* l'_i *are equal to the* l_i*, and* t'_1 *consists entirely of dominoes, and the rightmost* $l'_1/2$ *tiles of* t'_2 *are dominoes, or*
`Case 2`: $l'_I=l_I-1$ *for some* $I\ge 2$*,* t'_I *consists entirely of dominoes, and the rightmost* $(l'_I)/2$ *tiles of* t'_{I+1} *are dominoes.*

Proof. If $(t_1,\dots,t_n)$ exist, then consider the failure index of $\tau(t_1,\dots,t_n)$. If the failure index is 1, then Case 1 holds, by Proposition 1. If the failure index is $I\ge 2$, then Case 2 holds by Proposition 1. So one of the two cases must hold.

Now, suppose that one of the cases holds. We will show that the appropriate choice of $(t_1,\dots,t_n)$ exists. First, suppose that Case 1 holds. Then $(t_1,\dots,t_n)=(t'_1,\dots,t'_n)$ is an appropriate choice for $(t_1,\dots,t_n)$.

Otherwise, Case 2 holds. Note that $l'_1,\dots,l'_{I-1}$ are all odd, and l'_I is even. Also, we have $l'_I\ge l'_{I-1}\ge\cdots\ge l_1$. Therefore, by Proposition 1, the tail-swap

$$\tau(t'_I,t'_{I-1},\dots,t'_1)=(t_I,t_{I-1},\dots,t_1)$$

succeeds. Now, by Proposition 3, $\tau(t_1,\dots,t_I)=(t'_1,\dots,t'_I)$. Furthermore, if we let $t_{I+1}=t'_{I+1}$, then $\tau(t_I,t_{I+1})$ fails. So

$$\tau(t_1,\dots,t_I,t'_{I+1},t'_{I+2},\dots,t'_n)=(t'_1,\dots,t'_n),$$

as desired. □

Corollary 7. *The tail-swap procedure* τ *is bijective from* $\mathrm{dom}(\tau)$ *to* $\mathrm{img}(\tau)$*, where* $\mathrm{img}(\tau)$ *is characterized in Theorem 1.*

Proof. By Propositions 3 and 4 (see Remark 2), τ is injective. Therefore, it is bijective onto its image. □

Remark 8. The condition on the lengths $l_1,\dots,l_n$ is satisfied by the sequence $0,3,7,\dots,2^n-1$. The sequence of the lengths of the tilings in S_1 (for the Vajda identity) is $1,3,7,\dots,2^n-1$. However, the number of tilings of length 0 is the same as the number of tilings of length 1 (namely, there is one of each). Thus, we prove Vajda's identity while assuming that the length of the first tiling is 0, rather than 1.

We are now ready to describe the bijection $f : S_1 \to S_2$. Let $\mathbf{t} \in S_1$ be $(t_1, t_2, \ldots, t_n)$. The lengths of these tilings are $2^1 - 1, 2^2 - 1, \ldots, 2^n - 1$. Let $(t'_1, \ldots, t'_n) = \tau(\emptyset, t_2, t_3, \ldots, t_n)$, and let i be the failure index. Now, if $i = n$, then let $f(\mathbf{t}') = (t'_1, \ldots, t'_n)$. Note that $f(\mathbf{t}) \in S_2$ in this case, because the lengths of $t'_1, t'_2, \ldots, t'_n$ are $2^1 - 1, 2^2 - 1, \ldots, 2^{n-1} - 1, 2^n - 2$, respectively.

On the other hand, if $i < n$, then the ith tail-swap failed. Therefore, by Proposition 1, t'_{i+1} ends with 2^{i-2} dominoes. Let t^*_{i+1} be the result of removing those 2^{i-2} dominoes from t^*_i. Then let $f(\mathbf{t}) = (t'_1, \ldots, t'_{i-1}, t^*_{i+1}, t'_{i+2}, t'_{i+3}, \ldots, t'_n)$. Note that the lengths of the tilings here are

$$2^1 - 1, 2^2 - 1, \ldots, 2^{i-1} - 1, 2^i - 1, 2^{i+2} - 1, 2^{i+3} - 1, \ldots, 2^n - 1,$$

so $f(\mathbf{t})$ is once again in S_2.

Proposition 9. *The map f is injective.*

Proof. If $f(t_1, \ldots, t_n) = (u_1, \ldots, u_n)$ (that is, if the result has n tilings), then f was just a multiple tail-swap, so by Corollary 1, it is injective.

On the other hand, suppose $f(t_1, \ldots, t_n) = (u_1, \ldots, u_{n-1})$ (that is, there are only $n-1$ tilings). then the set of the lengths of $u_1, \ldots, u_{n-1}$ is equal to

$$\{1, 3, \ldots, 2^n - 1\} \backslash \{2^k - 1\}$$

for some $k \geq 2$. How did this happen? Well, first of all, we know that the $(k-1)$th tail-swap failed. Using the notation from above, this means that t'_{k-1} had length $2^{k-1} - 2$. Furthermore, by Proposition 1, it consisted only of dominoes. Also, we know that $u_{k-1} = t^*_k$, so we can reconstruct t'_k by appending $2^{k-2} - 1$ dominoes to the end of u_{k-1}. Finally, for $1 \leq j \leq k-2$, we have $u_j = t'_j$, while for $k \leq j \leq n-1$, we have $u_j = t'_{j+1}$. Thus, we can reconstruct all the values $t'_1, t'_2, \ldots, t'_n$. By Corollary 1, the values $t_1, t_2, \ldots, t_n$ are uniquely determined. □

Remark 10. The above proposition demonstrates the existence of an inverse map $f^{-1} : S_2 \to S_1$ for f.

Proposition 11. *The map f^{-1} is injective.*

Proof. Suppose $f^{-1}(u_1, \ldots, u_l) = f^{-1}(v_1, \ldots, v_m)$, where $(u_1, \ldots, u_l)$ and $(v_1, \ldots, v_m)$ are both in S_2 and $l, m \in \{n-1, n\}$.

`Case 1:` $l = m = n$. In this case, f^{-1} is just a multiple tail swap, which is a bijection by Corollary 1. In particular, it must be the case that $(u_1, \ldots, u_l) = (v_1, \ldots, v_m)$.

`Case 2:` $l = n - 1$ and $m = n$, or vice versa. Without loss of generality, let $l = n-1$, and let the lengths of the tilings $u_1, \ldots, u_l$ be $\{1, 3, \ldots, 2^n - 1\} \backslash \{2^k - 1\}$.

Let $u'_1, \ldots, u'_n$ be the result of the reconstruction process described above; that is,

$$u'_j = \begin{cases} u_j & 1 \leq j \leq k-2 \\ 2^{n-2} - 1 \text{ dominoes} & j = k-1 \\ u_{k-1} \text{with } 2^{n-2} - 1 \text{ dominoes appended} & j = k \\ u_{k-1} & k+1 \leq j \leq n \end{cases}.$$

Now, $u'_1, \ldots, u'_n$ is the result of a multiple tail-swap with failure index $k-1$. Note that $v_1, \ldots, v_m$ is the result of a multiple tail-swap with failure index n (since $m = n$). Therefore, applying τ^{-1} to $u'_1, \ldots, u'_n$ and $v_1, \ldots, v_m$ yields different results, by Corollary 1. But this is a contradiction, since

$$\tau^{-1}(u'_1, \ldots, u'_n) = f^{-1}(u_1, \ldots, u_l) = f^{-1}(v_1, \ldots, v_m) = \tau^{-1}(v_1, \ldots, v_m).$$

So Case 2 cannot occur.
`Case 3:` $l = m = n-1$. Let the lengths of $u_1, \ldots, u_{n-1}$ be

$$\{1, 3, \ldots, 2^n - 1\} \backslash \{2^{k_u} - 1\},$$

and let the lengths of $v_1, \ldots, v_{n-1}$ be

$$\{1, 3, \ldots, 2^n - 1\} \backslash \{2^{k_v} - 1\}.$$

As above, let $u'_1, \ldots, u'_n$ be the result of the reconstruction process applied to $u_1, \ldots, u_{n-1}$, and let $v'_1, \ldots, v'_n$ be the reconstruction of $v_1, \ldots, v_{n-1}$. Now, $u'_1, \ldots, u'_n$ is the result of a multiple tail-swap with failure index $k_u - 1$, and $v_1, \ldots, v'_n$ is the result of a multiple tail-swap with failure index $k_v - 1$. As above, if $k_u \neq k_v$, we have a contradiction, so we can assume that $k_u = k_v$. Then we have

$$(u'_1, \ldots, u'_n) = \tau(f^{-1}(u_1, \ldots, u_l)) = \tau(f^{-1}(v_1, \ldots, v_m)) = (v'_1, \ldots, v'_n).$$

But if $(u'_1, \ldots, u'_n) = (v'_1, \ldots, v'_n)$, then $(u_1, \ldots, u_{n-1}) = (v_1, \ldots, v_{n-1})$, as claimed.

So f^{-1} is injective. □

Theorem 12. *The function f is a bijection.*

Proof. Propositions 5 and 6 show that both f and f^{-1} are injective. □

In our proof, we used the form (1) of the identity rather than the form (V90). There are three differences between (1) and (V90). The first is that both sides have been multiplied by F_2. This makes no combinatorial difference, but merely clarifies the bijection. The second difference is that some terms have been moved from one side to the other. Again, this does not make any combinatorial difference. The second is that both sides have been multiplied by $F_4 F_8 \cdots F_{2^n}$. Therefore, to understand the identity in the form (V90), we use the following probabilistic reasoning. Suppose we are given a sequence of tilings

$$(t_1, \ldots, t_n) \in S_1.$$

Replace the tiling of length 1 with the empty tiling and then perform a multiple tail-swap. There are several mutually exclusive possible outcomes: we could have any failure index in $\{0, 1, \ldots, n-1\}$. Our bijection demonstrates that the number of $t \in S_1$ with failure index $k \geq 1$ is exactly

$$\prod_{\substack{1 \leq j \leq n-1 \\ j \neq k+1}} F_{2^j},$$

and hence the probability of a randomly chosen element of S_1 having failure index $k \geq 1$ is $\frac{1}{F_{2^{k+1}}}$. Similarly, the probability of having failure index 0 is $\frac{F_{2^n-1}}{F_{2^n}}$. Since these are the only possible outcomes and they are mutually exclusive, we have

$$\sum_{k=1}^{n-1} \frac{1}{F_{2^{k+1}}} + \frac{F_{2^n-1}}{F_{2^n}} = 1,$$

which is (V90). So the identity (V90) has both a *probabilistic form* and a *combinatorial form*.

5. Further Identities

There is nothing unique about the values $\left\{2^k\right\}_{k=1}^n$. Given any increasing sequence of even integers $\{a_n\}_{k=1}^n$ with $a_1 = 2$, we can use the multiple tail-swap procedure to give a bijective proof of an identity about products of Fibonacci numbers:

$$\prod_{k=1}^{n} F_{a_k} = F_{a_n-1} \prod_{k=1}^{n-1} F_{a_k} + \sum_{1 \leq k \leq n-1} F_{a_{k+1}-a_k} \prod_{\substack{j \notin \{k,k+1\} \\ 1 \leq j \leq n}} F_{a_j}. \qquad (*)$$

The proof of this general formula proceeds exactly as in the proof of the specific case of Vajda's 90th identity. Like Vajda's 90th identity, this general identity has a nice probabilistic form:

$$\sum_{1 \leq k \leq n-1} \frac{F_{a_{k+1}-a_k}}{F_{a_k} F_{a_{k+1}}} + \frac{F_{a_n-1}}{F_{a_n}} = 1. \qquad (**)$$

Notice that when $a_n = 2^n$, the values $F_{a_{k+1}-a_k}$ in the numerator and F_{a_k} in the denominator cancel, accounting for the elegance of (V90).

As an example, we provide the special cases of $(*)$ for the sequences $\{2k\}_{k=1}^n$, $\{4k-2\}_{k=1}^n$, and $\left\{3^k - 1\right\}_{k=1}^n$. Using these sequences in $(*)$, we get the identities

$$\prod_{k=1}^{n} F_{2k} = F_{2n-1} \prod_{k=1}^{n-1} F_{2k} + \sum_{1\le k\le n-1} \prod_{\substack{j\notin\{k,k+1\}\\ 1\le j\le n}} F_{2j}$$

$$\prod_{k=1}^{n} F_{4k-2} = F_{4n-3} \prod_{k=1}^{n-1} F_{4k-2} + 3 \sum_{1\le k\le n-1} \prod_{\substack{j\notin\{k,k+1\}\\ 1\le j\le n}} F_{4j-2}$$

$$\prod_{k=1}^{n} F_{3^k-1} = F_{3^k-2} \prod_{k=1}^{n-1} F_{3^k-1} + F_{2\cdot 3^k} \prod_{\substack{j\notin\{k,k+1\}\\ 1\le j\le n}} F_{3^k-1}.$$

The equivalent probabilistic forms, given by $(**)$, are

$$\sum_{k=1}^{n-1} \frac{1}{F_{2k}F_{2k+2}} + \frac{F_{2n-1}}{F_{2n}} = 1,$$

$$\sum_{k=1}^{n-1} \frac{3}{F_{4k-2}F_{4k+2}} + \frac{F_{4n-3}}{F_{4n-2}} = 1,$$

and

$$\sum_{k=1}^{n-1} \frac{F_{2\cdot 3^k}}{F_{3^k-1}F_{3^{k+1}-1}} + \frac{F_{3^n-2}}{F_{3^n-1}} = 1.$$

Some of these identities are reasonably attractive; in particular, when $F_{a_{k+1}-a_k}$ is constant, the probabilistic form of the identity is relatively simple. This happens when the sequence $\{a_k\}$ is arithmetic.

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