ODD REPDIGITS TO SMALL BASES ARE NOT PERFECT

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Abstract
We demonstrate, by considering each base in the range 2 through 9, that *no* odd repdigit with a base in that range is a perfect number.

1. Introduction

Let $g \geq 2$. We say a natural number n is a *repdigit* to base g if there is an integer a with $1 \leq a < g$ and $m \geq 0$ such that $n = a + ag + ag^2 + \cdots ag^m$. If $a = 1$ then n is called a *repunit*. If $\sigma(n)$ is the usual sum of divisors function then n is called *perfect* if $\sigma(n) = 2n$.

Interest in the relationships between repdigits and perfect numbers was initiated by Pollack [12] who showed that for a given base g there are only a finite number of perfect repdigits to that base, and that the set of all such numbers is effectively computable. He also showed that in base 10 the only perfect repdigit is $n = 6$.

Base 10 has been of special interest since it has been shown by Oblàth [11], Bugeaud and Mignotte [3] that the only perfect powers which are also repdigits are 1,4,8 and 9.

Work in showing infinite classes of natural numbers which are not perfect or multiperfect has been developed by Luca, who has shown that no Fibonacci number is perfect [6], that no element of a Lucas sequence with odd parameters is multiperfect [7] and, with Broughan, González, Lewis, Huguet and Togbé, that no Fibonacci number is multiperfect [2].

In this paper we continue the work of Pollack, showing in Theorem 1 that there are no odd perfect repdigits in bases 2 through 9. This is done by using a mixture of techniques, including Pollack's original method, a method developed by Pollack and Luca, Thue equations with no solutions, properties of rings of integers of the quadratic fields $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{6})$, and $\mathbb{Q}(\sqrt{7})$, quadratic reciprocity, linear forms in logarithms (LFL) and the known comprehensive sets of solutions for Pellian, Nagel-Ljunggren and Catalan equations.

The following notations are used. If n is a natural number then $P(n)$ is the maximum prime divisor of n, $v_p(n)$ is the maximum power of the prime p which divides n and (a, b) is the greatest common divisor (GCD) of the integers a and b. If $g \geq 2$ and $n \in \mathbb{N}$ let $U_n := (g^n - 1)/(g - 1)$ and $V_n := g^n + 1$. The expression s^2 in equations represents an integer which is a square or possibly 1, and which can take different values, even in the same context.

Theorem 1. *Let g satisfying $2 \leq g \leq 10$ be the base, a with $1 \leq a < g$ a digit and aU_n a repdigit to base g which is odd. Then aU_n is not a perfect number.*

We also settle the issue of perfect even repunits for all g in terms of Mersenne primes, and describe the perfect even repdigits up to base $g = 100$.

2. Preliminary Results

Lemma 2. *Let n be a composite natural number, x an integer with $x \geq 2$ and $p := P(n)$. Then*

(a) *If $p > 2$ is odd, $p^2 \mid n$ and $p \mid x - 1$ then*

$$g := \left(\frac{x^n - 1}{x^p - 1}, \frac{x^p - 1}{x - 1} \right) = p.$$

(b) *If $p = 2$ and $4 \mid n = 2^m$ then*

$$\left(\frac{x^{2^m} - 1}{x^2 - 1}, \frac{x^2 - 1}{x - 1} \right) = 2^{\min(m-1,\ v_2(x+1))}.$$

In all other cases the GCD is 1.

Proof. (a) Let p be odd with $p \mid x - 1$ and $p^2 \mid n$. Let $k = n/p$. Then

$$\frac{x^n - 1}{x^p - 1} = (x^p)^{k-1} + \cdots + x^p + 1 \equiv k \pmod{p} \equiv 0 \pmod{p}.$$

Thus $p \mid (x^n - 1)/(x^p - 1)$. Also

$$\frac{x^p - 1}{x - 1} = x^{p-1} + \cdots + x + 1 \equiv 0 \pmod{p},$$

and therefore

$$p \mid \left(\frac{x^n - 1}{x^p - 1}, \frac{x^p - 1}{x - 1} \right).$$

By [13, P1.2(iv)], if q is an odd prime and $q \mid x-1$ then

$$v_q\left(\frac{x^n-1}{x-1}\right) = v_q(n).$$

Thus, letting $q = n = p$ we have $p^2 \nmid (x^p-1)/(x-1)$; and if $q \neq p$ then $v_q((x^p-1)/(x-1)) = 0$. Hence, the only positive divisors of the GCD other than p are powers of 2. But $(x^p-1)/(x-1)$ is odd. Therefore, in this case,

$$\left(\frac{x^n-1}{x^p-1}, \frac{x^p-1}{x-1}\right) = p.$$

(b) Now let $n = 2^m$ with $m \geq 2$. We can write

$$\left(\frac{x^{2^m}-1}{x^2-1}, \frac{x^2-1}{x-1}\right) = \left(\prod_{i=2}^{m} \Phi_{2^i}(x), \Phi_{2^1}(x)\right)$$

and use the fact that [13, P1.9], with $i \geq 2$, we have $(\Phi_{2^i}(x), \Phi_2(x)) = 2$ if x is odd and 1 if x is even, and the given formula follows immediately.

(c) Let $p \nmid (x-1)$ and be odd and suppose that a prime $q \mid g$. Since p is odd, so is q. Let e be the order of x modulo q. Because $q \mid x^p - 1$ we must have $e \mid p$ so that $e = 1$ or $e = p$. In the first case $q \mid (x-1)$ and we can apply [13, P1.2(iv)] to get

$$1 \leq v_q\left(\frac{x^p-1}{x-1}\right) = v_q(p) \leq 1.$$

Hence $p = q \mid (x-1)$, which is false, since we are assuming $p \not| x-1$. Therefore $e = p$ so $p \mid q-1$ and hence $p < q$. But q is odd and $q \mid (x^n-1)/(x^p-1)$ and $q \mid x^p - 1$ so $q \mid n$, a contradiction since $p = P(n)$. Therefore the GCD $g = 1$.

(d) Let p be odd, $p \mid (x-1)$ and $p^2 \nmid n$. If an odd prime q divides the GCD, and e is the order of x modulo q, then since $x^p \equiv 1 \pmod{q}$ we have $e = 1$ or $e = p$. But the former is equivalent to $q \mid (x-1)$ so, as above, $v_q(p) = 1$ so $p = q$ and thus $p \mid (x^n-1)/(x^p-1)$. But, as in (a), this ratio is equivalent to n/p modulo p which is non zero. In the former case where $e = p$ we have $p \mid q-1$ so $p < q$. But $1 \leq v_q((x^n-1)/(x^p-1)) = v_q(n/p)$ implies $q \mid n$, contradicting $p = P(n)$. Hence in this case the GCD g is 1. □

We note that $U_n \equiv 1 \pmod{4}$ in exactly the following distinct situations.

(1) $g \equiv 0 \pmod 4$,

(2) $g \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$, or

(3) $g \equiv 3 \pmod 4$ and n is odd.

We note also that $U_n \equiv 3 \pmod 4$ in exactly the following distinct situations.

(4) $g \equiv 1 \pmod 4$ and $n \not\equiv 1 \pmod 4$,

(5) $g \equiv 2 \pmod 4$, or

(6) $g \equiv 3 \pmod 4$ and n is even.

We will use these properties without comment below.

Lemma 3. *Let the base be $g \geq 2$ and suppose the prime $p \geq \max\{7, g\}$. Then the repunit U_p is not perfect.*

Proof. We follow the proof of [12, Lemma 10]. Then

$$\frac{\sigma(U_p)}{U_p} \leq \exp\left(\frac{1}{p}\right) \exp\left(\frac{\log(2p\log p)}{p}\right)$$

and this right-hand side is less than 2 when $p \geq 7$. □

We need the following classical result.

Lemma 4. (Ljunggren [5]) *The only integer solutions $(x,\ n,\ y)$ with $|x| > 1,\ n > 2,\ y > 0$, to the equation $(x^n - 1)/(x-1) = y^2$ are $(7,\ 4,\ 20)$ and $(3,\ 5,\ 11)$, i.e.,*

$$\frac{7^4 - 1}{7 - 1} = 20^2 \ \textit{and}\ \frac{3^5 - 1}{3 - 1} = 11^2.$$

For bases 5 and 9 we also need a result of Laurent, Mignotte and Nesterenko [4], which we refer to as the method of linear forms in logarithms. To describe this theorem we need the notion of "logarithmic height" for an algebraic number η, namely if $f(x) = a_0 \prod_{i=1}^{d}(x - \eta^{(i)})$ is the factored form of its minimal polynomial over $\mathbb{Z}$, then its height is replaced by

$$h(\eta) := \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \log\left(\max(|\eta^{(i)}|, 1)\right)\right).$$

Theorem 5. (Corollary 2, Laurent et al. [4]) *Let $b_1,\ b_2$ be positive integers and η_1 , η_2 algebraic numbers which are real, positive and multiplicatively independent. Let*

$$\Lambda = b_1 \log \eta_1 - b_2 \log \eta_2$$

be non-zero. Let $D = [\mathbb{Q}(\eta_1, \eta_2) : \mathbb{Q}]$. Let real numbers A_j satisfy, for $j = 1,\ 2$

$$A_j \geq \max(Dh(\eta_j),\ |\log \eta_j|,\ 1).$$

and define $b' := A_1/b_2 + A_2/b_1$. Then

$$\log|\Lambda| \geq -24.34 D^2 \left(\max\left\{\log b' + 0.14,\ \frac{21}{D},\ \frac{1}{2}\right\}\right)^2 A_1 A_2.$$

3. Perfect Even Repdigits

Lemma 6. *A repunit U_n to base $g > 1$ is even and perfect if and only if $n = 2$ and $g = 2^{p-1}(2^p - 1) - 1$ where p and $2^p - 1$ are prime.*

Proof. Suppose that U_n is even and perfect. Since every repunit in base 2 is odd, we have $g > 2$. We are able to write $U_n = 2^{p-1}(2^p - 1)$ where p is prime and and $2^p - 1$ is prime. Since U_n is even then $U_n = 1 + g + \cdots + g^{n-1}$ implies g is odd and n is even, say $n = 2m$. Then, if $V_m := 1 + g^m$, we have $V_m U_m = U_n = 2^{p-1}(2^p - 1)$.

If m is even then V_m, as the sum of two squares, has a prime divisor $q \equiv 1 \pmod 4$, which is a contradiction since necessarily $q = 2^p - 1 \equiv 3 \pmod 4$. Hence m is odd and so U_m is also odd since g is odd. Since $2^p - 1$ is prime we must have $U_m = 2^p - 1$ so $V_m = 2^{p-1}$. But we can write $V_m = g^m - (-1)^m = (g + 1)(g^{m-1} - g^{m-2} + g^{m-3} - \cdots + 1)$ and the second factor is odd so must be 1. Therefore $V_m = g + 1$ which implies $m = 1$ and thus $n = 2$. But this means $U_n = 1 + g = 2^{p-1}(2^p - 1)$. □

Example 7. By exhaustive search using Lemma 7, Pollack [12], we found that for $g \in \{2, 3, 4\}$ there are no perfect even repdigits to base g. For $g = 5$, 6 is the only repdigit and for $g = 6$, 28 is the only even repdigit. For $g \geq 7$ each even perfect number n with $n < g$ is an even repdigit. In addition, 28 is an even repdigit to base 13. Up to base $g = 100$ the maximum number of even perfect repdigits is 3, namely 6, 28 and 496, and these occur when $g = 30$ and when $g = 61$.

4. Perfect Odd Repdigits

Lemma 8. *To base 2 there are no odd perfect repdigits.*

Proof. If U_n is an odd perfect repdigit, using Euler's structure theorem for odd perfect numbers we can write $2^n - 1 = rs^2$ for some prime $r \equiv 1 \pmod 4$ and $n \geq 2$ so $3 \equiv 1 \pmod 4$. Thus there are none. □

Lemma 9. (Luca and Pollack [9]) *Let $U_n = U_n(g) = (g^n - 1)/(g - 1)$ where n is an odd positive integer and $g \geq 2$ are fixed. If the prime decomposition of n is written $n = p_1 \cdots p_k$ with $p_1 \leq p_2 \leq \cdots \leq p_k$ and we define the partial products $n_1 = n$, $n_{i+1} = n_i/p_i$ $1 \leq i \leq k$, with $n_{k+1} = 1$, then in the factorization*

$$U_n = \frac{U_{n_1}}{U_{n_2}} \cdot \frac{U_{n_2}}{U_{n_3}} \cdots \frac{U_{n_k}}{U_{n_{k+1}}},$$

for all $1 \leq i < j \leq k$ if a prime p divides the GCD of the ith and jth terms then

$$p \mid \left(\frac{U_{n_i}}{U_{n_{i+1}}}, \frac{U_{n_j}}{U_{n_{j+1}}} \right),$$

then $p \mid g - 1$ and $p = p_i = p_{i+1} = \cdots = p_j$.

Lemma 10. *To base 3 there are no odd perfect repdigits.*

Proof. We need only consider $a = 1$. If U_n is an odd perfect repunit to base 3 we have $3^n - 1 = 2rs^2$ where $rs^2 \equiv 1 \pmod 4$ so n is odd.

Let p be an odd prime and define $U_p := (3^p - 1)/(3 - 1)$. Then, by Lemma 3, if U_p is perfect we can assume $p \le 13$. It is easy to show directly that none of these repunits is perfect.

Now let n be composite and suppose that U_n is perfect. Let $p = P(n)$ and write

$$U_n = \left(\frac{3^n - 1}{3^p - 1}\right)\left(\frac{3^p - 1}{3 - 1}\right) = rs^2$$

where r is an odd prime. Since p is odd, cases (a) and (b) of Lemma 2 do not apply so the GCD

$$\left(\frac{3^n - 1}{3^p - 1}, \frac{3^p - 1}{3 - 1}\right) = 1.$$

Therefore $(3^n - 1)/(3^p - 1) = s^2$ or $(3^p - 1)/2 = s^2$, where each of these squares is odd.

If $(3^n - 1)/(3^p - 1) = s^2$ then by Lemma 4 (in Pollack [12]), since n is odd, $U_n = s^2$ and $U_p = s^2$, the former by Lemma 4 in this paper implying $n = 5$, which is not possible since n is composite.

Finally we consider $3^p - 1 = 2s^2$ where p is an odd prime. By Lemma 4 in this paper we must have $p = P(n) = 5$ so there exists non-negative integers a, b with $n = 3^a 5^b$. If $3 \mid n$ then $13 = U_3 \mid U_n$ so $13 \| U_n$ or $13^2 \mid U_n$. But the order of 3 modulo 13^2 is 39 so $13 \mid n$ which is false. Hence $13 \| U_n$ so, since $\sigma(U_n) = 2U_n$, $7 \mid U_n$ and therefore 6, the order of 3 modulo 7, divides n which again is false. Therefore $3 \nmid n$.

So we can let $n = 5^b$ and assume, after a simple numerical check, that $b \ge 3$. Then U_n is odd and let us assume U_n is also perfect. Write

$$U_n = \frac{U_{5^b}}{U_{5^{b-1}}} \cdot \frac{U_{5^{b-1}}}{U_{5^{b-2}}} \cdots \frac{U_{5^2}}{U_{5^1}} \cdot \frac{U_{5^1}}{U_{5^0}}.$$

By Lemma 9 the greatest common divisor of any two of the factors on the right-hand side is supported by primes dividing $g - 1 = 2$. But each of the U_{5^j} is odd so each of the greatest common divisors is 1 and the b factors are pairwise coprime. Since, by Lemma 4 again, each of the factors on the right-hand side of this expression is not a square, except the last which is $U_5 = 11^2$, there must be at least $b - 1$ prime factors of U_n which appear to an odd power. Therefore $\nu_2(\sigma(U_n)) \ge b - 1$. But then

$$\nu_2(\sigma(U_n)) = \nu_2(2U_n) = 1 \ge b - 1 \ge 2$$

is false. Hence U_n is not perfect. $\square$

Lemma 11. *To base 4 there are no odd perfect repdigits.*

Proof. We need only consider $a = 1, 3$. First let $a = 3$ so $aU_n = rs^2$, where r is an odd prime, implies $4^n - 1 = rs^2$ so $(2^n - 1)(2^n + 1) = rs^2$. Since the GCD $(2^n - 1, 2^n + 1) = 1$ we must have $2^n + 1 = s^2$ or $2^n - 1 = s^2$. In the first case, for some natural number x we have $2^n = (x-1)(x+1)$ giving $x = 3$ and $2^3 + 1 = 3^2$ as the only solution. But $aU_n = 63$ is not perfect.

If $2^n - 1 = s^2$ and $n \geq 2$ then $3 \equiv 1 \pmod 4$, which is false. Thus $n = 1$ is the only solution, but this gives the number $aU_n = 3$, which is not perfect.

Now let $a = 1$, so that $(2^n - 1)(2^n + 1) = 3rs^2$. If n is even, $3 \mid 2^n - 1$ so

$$\left(\frac{2^n - 1}{3}\right)(2^n + 1) = rs^2,$$

and therefore either $2^n + 1 = s^2$ giving, as we saw before, $2^3 + 1 = 3^2$ as the only solution, or $2^n - 1 = 3s^2$. In this latter case let $n = 2m$ so $(2^m - 1)(2^m + 1) = 3s^2$ giving $2^m - 1 = s^2$ or $2^m + 1 = s^2$. The latter implies $n = 6$ but then $aU_n = 1365$ which is not perfect. If $2^m - 1 = s^2$, $n = 2$, so we fail to obtain an odd perfect number.

If, however, n is odd we have

$$\left(\frac{2^n + 1}{3}\right)(2^n - 1) = rs^2,$$

so either $2^n - 1 = s^2$ or $2^n + 1 = 3s^2$, but as we have seen the former requires $n = 1$ and the latter, taken modulo 4, also requires $n = 1$, so we fail once more. This exhausts all possibilities. $\square$

Lemma 12. *The diophantine equation $5^m + 1 = 6s^2$ has just one solution $m = 1,\ s^2 = 1$.*

Proof. First let $5^m + 1 = 6s^2 = 6x^2$ for some odd $m > 1$ and $x \in \mathbb{N}$, with $x > 1$. Then $5^m = (\sqrt{6}x + 1)(\sqrt{6}x - 1)$. Now the class number $h(\mathbb{Q}(\sqrt{6})) = 1$ so the ring of integers $R := \mathcal{O}_{\mathbb{Q}(\sqrt{6})}$ has unique factorization, and since $6 \not\equiv 1 \pmod 4$, its integers have the form $u + v\sqrt{6}, u, v \in \mathbb{Z}$. The fundamental unit is $\epsilon = 5 + 2\sqrt{6}$.

Now in R we factor $5 = (\sqrt{6} + 1)(\sqrt{6} - 1)$ into primes, giving

$$(\sqrt{6} + 1)^m (\sqrt{6} - 1)^m = (\sqrt{6}x + 1)(\sqrt{6}x - 1).$$

Note that the GCD of the two factors on the right is 1, since necessarily x is odd, so therefore

$$\begin{aligned}
\sqrt{6}x + 1 &= \epsilon^{\pm l}(\sqrt{6} + 1)^m,\ \sqrt{6}x - 1 = \epsilon^{\mp l}(\sqrt{6} - 1),\ \text{or} \\
\sqrt{6}x + 1 &= \epsilon^{\pm l}(\sqrt{6} - 1)^m,\ \sqrt{6}x - 1 = \epsilon^{\mp l}(\sqrt{6} + 1).
\end{aligned}$$

If $l = 0$ then subtracting each pair of equations clearly leads to $m = 1$ as the only solution. Again subtracting, the case $\sqrt{6}x+1 = \epsilon^l(\sqrt{6}+1)^m$, $\sqrt{6}x-1 = \epsilon^{-l}(\sqrt{6}-1)$ with $l > 0$ has no solution since $2 < 5 \cdot (\sqrt{6}+1)^m - (\sqrt{6}-1)^m$ for all $m \geq 1$. In this manner we are left with the only possible situation, with $l > 0$,

$$2 = (5-2\sqrt{6})^l(\sqrt{6}+1)^m - (5+2\sqrt{6})^l(\sqrt{6}-1)^m \tag{1}$$

and

$$(5-2\sqrt{6})^l(\sqrt{6}+1)^m = 1+\sqrt{6}x = 1+\sqrt{1+5^m}. \tag{2}$$

Next we derive an inequality relating l and m. Considering the right-hand side of Equation (2) we get $(5-2\sqrt{6})^l(\sqrt{6}+1)^m > 5^{\frac{m}{2}}$ so therefore

$$m\log\left(\frac{\sqrt{6}+1}{\sqrt{5}}\right) > l\log(5+2\sqrt{6}).$$

Also $(\sqrt{6}+1)^m < 2\cdot 5^{\frac{m}{2}}(5+2\sqrt{6})^l$ and therefore

$$m\log\left(\frac{\sqrt{6}+1}{\sqrt{5}}\right) < \log 2 + l\log(5+2\sqrt{6}).$$

These inequalities imply $0.189m - 0.303 < l < 0.189m$.

Now let $\Lambda := m\log\left(\frac{\sqrt{6}+1}{\sqrt{5}}\right) - l\log(5+\sqrt{6})$. We derive the bounds, valid for all $m \geq 1$,

$$\frac{1}{4\cdot 5^m} < |\Lambda| < \frac{2}{5^{\frac{m}{2}}}. \tag{3}$$

By Equation (2) we have $\log|\Lambda| = \log((1+\sqrt{1+5^m})/5^{m/2})$ and the inequality follows using the relation $x/2 < \log(1+x) < x$ valid for $0 < x < 1$.

Now we apply Theorem 5 to Equation (2). Let $b_1 = m,\ b_2 = l, \eta_1 = (\sqrt{6}+1)/\sqrt{5}$ and $\eta_2 = 5+2\sqrt{6}$. Then the minimal polynomial for η_1 is $5x^4 - 14x^2 + 5$ and the degree, D, of the number field is 4. We calculate the lower bound $\log|\Lambda| \geq -15087$. Using the right-hand side of inequality (3) gives

$$-15087 < \log|\Lambda| < \log 2 - \frac{m}{2}\log 5 \implies m < 18750.$$

Finally we tested m in the range $2 \leq m \leq 18749$ numerically. To do this we solved Equation (2) numerically for each value of m in the given range for l to twenty decimal places and checked to see whether l was within $1/10^{10}$ of an integer, and found no such integer l. This completes the proof. □

Lemma 13. *To base 5 there are no odd perfect repdigits.*

Proof. First consider $a = 1$ and suppose $U_n = rs^2$ for an odd prime $r \equiv 1 \pmod 4$. By the note above we can assume n is odd. If n is prime then, by Lemma 3, U_n is not odd and perfect for $n \geq 7$. Since $U_2 = 6$ is not odd and $U_5 = 31$ and $U_7 = 19531$ not perfect, we can take n to be odd and composite.

Let $p = P(n)$ so $p \geq 3$. Then using Pollack's method we write

$$\frac{5^n - 1}{5^p - 1} \cdot \frac{5^p - 1}{5 - 1} = rs^2$$

and note that, by Lemma 2,

$$\left(\frac{5^n - 1}{5^p - 1}, \frac{5^p - 1}{5 - 1} \right) = 1.$$

Therefore

$$\frac{5^n - 1}{5^p - 1} = s^2 \text{ or } \frac{5^p - 1}{5 - 1} = s^2,$$

but both of these are impossible by Lemma 4.

Now let $a = 3$ so we can assume $3U_n = rs^2$. Since the prime $r \equiv 1 \pmod 4$ we must have $3 \nmid r$ so $3 \mid 5^n - 1$, and thus $n = 2m$ is even. Writing

$$3 \left(\frac{5^m - 1}{2} \right) \left(\frac{5^m + 1}{2} \right) = rs^2$$

we get $5^m + 1 = 6s^2$ or $5^m - 1 = 2s^2$ when m is odd, and $5^m + 1 = 2s^2$ or $5^m - 1 = 6s^2$ when m is even. As we will show below, each of these four possibilities is impossible.

First let $5^m + 1 = 6s^2 = 6x^2$ for some odd $m > 1$ and $x \in \mathbb{N}$, with $x > 1$. Then, by Lemma 12 the equation has only one solution, $m = x = 1$, but $3U_2 = 18$ is not perfect or odd.

The cases $5^m - 1 = 2s^2$ or $5^m - 1 = 6s^2$ with $s^2 \equiv 1 \pmod 4$ are not possible since the right-hand side is congruent to 0 modulo 4.

Finally let $5^m + 1 = 2s^2$ with m even. The quadratic residue of the left-hand side modulo 5, $(5^m + 1|5) = 1$, but the residue $(2s^2|5) = (2|5) = -1$ so this fails also. □

Lemma 14. *There are no odd perfect repdigits to base 6.*

Proof. We need only consider the case where U_n is odd and $a \in \{1, 3, 5\}$. If $a = 1$ and $U_n = rs^2$ then $6^n - 1 = 5rs^2$ with $r \equiv 1 \pmod 4$, so if $n > 1$ $3 \equiv 1 \pmod 4$, while $U_1 = 1$ is not perfect. If $a = 3$ then $6^n - 1 = 15rs^2$ so $3 \mid 1$, so this case fails. If $a = 5$ then $6^n - 1 = rs^2$ so if $n > 1$, again $3 \equiv 1 \pmod 4$. □

Lemma 15. *There are no odd perfect repdigits to base 7.*

Proof. If $a = 1$ then $7^n - 1 = 6rs^2$. We have $U_n \equiv 1 \pmod 4$ and n must be odd, and we can assume also composite. Using Pollack's method, letting $p = P(n)$,

$$\left(\frac{7^n - 1}{7^p - 1}\right)\left(\frac{7^p - 1}{7 - 1}\right) = rs^2,$$

and, by Lemma 1, the GCD $((7^n-1)/(7^p-1), (7^p-1)/(7-1)) = 1$ unless $p = 3,\ 9 \mid n$, and in that case, the GCD is 3 and we must have

$$\left(\frac{7^n - 1}{3(7^3 - 1)}\right)\left(\frac{7^3 - 1}{3(7 - 1)}\right) = \left(\frac{7^n - 1}{3(7^3 - 1)}\right)19 = rs^2,$$

so r divides the first factor, but 19 is not a square. Hence we can assume the GCD is 1.

But then

$$\frac{7^n - 1}{7^p - 1} = s^2 \text{ or } \frac{7^p - 1}{7 - 1} = s^2.$$

By Lemma 4 both of these are impossible.

If $a = 3$ then $7^n - 1 = 2rs^2$. We have $U_n \equiv 3 \pmod 4$ and n must be even, $n = 2m$. But then the left-hand side is congruent to 0 modulo 4 and the right to 2, so this is impossible.

Finally if $a = 5$ if $5(7^n - 1) = 6rs^2$. We have $U_n \equiv 1 \pmod 4$ and n must be odd. If $r = 5$ we have

$$\frac{7^n - 1}{7 - 1} = s^2$$

which, by Lemma 4, is impossible unless $n \leq 2$, but $5U_1$ and $5U_2$ are not perfect. Thus we can assume $r > 5$ and write $7^n - 1 = 30rs^2$. We may also assume n is composite so, using Pollack's method, in case $p = P(n) = 3$ and $9 \mid n$ then

$$\left(\frac{7^n - 1}{3(7^3 - 1}\right)\left(\frac{7^3 - 1}{3(7 - 1)}\right) = \left(\frac{7^n - 1}{3(7^3 - 1)}\right)19 = 5rs^2,$$

where the two factors on the left are coprime. But this is impossible since $5r$ must divide the first factor and 19 is not a square. So we can assume $p > 3$ and in

$$\left(\frac{7^n - 1}{7^p - 1}\right)\left(\frac{7^p - 1}{7 - 1}\right) = 5rs^2,$$

that the two factors in the left are coprime. But this means we must have at least one of the following.

$$\frac{7^n - 1}{7^p - 1} = s^2,\ \frac{7^n - 1}{7^p - 1} = 5s^2,\ \frac{7^p - 1}{7 - 1} = s^2,\ \frac{7^p - 1}{7 - 1} = 5s^2.$$

The first and third of these are impossible by Lemma 4. The second and fourth are also not possible because the multiplicative order of 2 modulo 5 is 4 which does not divide n or p. $\square$

Lemma 16. *Let $m \geq 1$ be odd. Then*

$$a_m := \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m \equiv \begin{cases} 4 \pmod 8, & m \equiv 0 \pmod 3, \\ 1 \pmod 4, & m \equiv 1 \pmod 3, \\ 3 \pmod 4, & m \equiv 2 \pmod 3. \end{cases}$$

Proof. First increment the equations of this lemma by including the additional set with left-hand side denoted b_m:

$$\sqrt{5}\left(\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right) \equiv \begin{cases} 2 \pmod 8, & m \equiv 0 \pmod 3, \\ 1 \pmod 4, & m \equiv 1 \pmod 3, \\ 1 \pmod 4, & m \equiv 2 \pmod 3. \end{cases}$$

Let $m = 3n,\ m = 6n+1,\ m = 6n+5$ to model the three equivalence classes modulo 3 with odd values of m and then, within each class, we use induction on n, the base being $n = 0$ which is easily checked.

Let $m = 3n$. Then

$$\begin{aligned} a_{3(n+1)} &= (2+\sqrt{5})\left(\frac{1+\sqrt{5}}{2}\right)^{3n} + (2-\sqrt{5})\left(\frac{1-\sqrt{5}}{2}\right)^{3n} \\ &= 2a_{3n} + b_{3n} = 2(4+8x) + (2+8y), \text{ for some integers } x,\ y \\ &\equiv 2 \pmod 8. \end{aligned}$$

Let $m = 6n+1$. Then

$$\begin{aligned} a_{1+6(n+1)} &= (9+4\sqrt{5})\left(\frac{1+\sqrt{5}}{2}\right)^{6n+1} + (9-4\sqrt{5})\left(\frac{1-\sqrt{5}}{2}\right)^{6n+1} \\ &= 9a_{6n+1} + 4b_{6n+1} = 9(1+4x) + 4(1+4y), \text{ for some integers } x,\ y \\ &\equiv 1 \pmod 4. \end{aligned}$$

Finally let $m = 6n+5$. Then

$$\begin{aligned} a_{5+6(n+1)} &= (9+4\sqrt{5})\left(\frac{1+\sqrt{5}}{2}\right)^{6n+5} + (9-4\sqrt{5})\left(\frac{1-\sqrt{5}}{2}\right)^{6n+5} \\ &= 9a_{6n+5} + 4b_{6n+5} = 9(3+4x) + 4(1+4y), \text{ for some integers } x,\ y \\ &\equiv 3 \pmod 4. \end{aligned}$$

Thus, by induction on n, the given formula for a_m is true for all $m \geq 1$. □

Lemma 17. *For all $n \in \mathbb{N}$*

$$a_n := (16 - 6\sqrt{7})(127 + 48\sqrt{7})^n + (16 + 6\sqrt{7})(127 - 48\sqrt{7})^n = 2^5 m_n$$

where m_n is odd.

Proof. First let $b_n := (16-6\sqrt{7})(127+48\sqrt{7})^n - (16+6\sqrt{7})(127-48\sqrt{7})^n$. We will show by induction that $2^5 \| a_n$ and $2^2 \| b_n/\sqrt{7}$. Start the inductive proof with the calculation $a_1 = 2^5,\ b_1 = 2^2.3$. Then note that

$$\begin{aligned} a_{n+1} &= 127a_n + 48\sqrt{7}b_n, \\ b_{n+1} &= 127b_n + 48\sqrt{7}a_n, \end{aligned}$$

and the result follows immediately from these recurrences. □

Lemma 18. *There are no odd perfect repdigits to base 8.*

Proof. If $a = 1$ then $8^n - 1 = 7rs^2$. We have $U_n \equiv 1 \pmod 4$ and, by Lemma 3 we can assume n is composite.

First let n be even, $n = 2m$. Then

$$\left(\frac{2^{3m}-1}{7}\right)\left(2^{3m}+1\right) = rs^2$$

so $2^{3m} - 1 = 7s^2$ or $2^{3m} + 1 = s^2$. The latter is Catalan's equation with the single solution $m = 1$, [14, 10], leading to $8^2 - 1 = 7 \cdot 3^2$, but $U_2 = 9$ is not perfect.

For the former, if $m = 2l$ is even then

$$\left(\frac{2^{3l}-1}{7}\right)\left(2^{3l}+1\right) = s^2$$

so $2^{3l} - 1 = 7s^2$ and $2^{3l} + 1 = s^2$. Subtracting we can write $2 = x^2 - 7y^2$, a Pellian equation with solutions $(x_1, y_1) = (3, 1)$ and general solution $\{(x_j, y_j) : j \in \mathbb{N}\}$ where

$$\begin{aligned} x_n &= \frac{1}{2\sqrt{7}}\left((3-\sqrt{7})(8+3\sqrt{7})^n - (3+\sqrt{7})(8-3\sqrt{7})^n\right) \\ y_n &= \frac{1}{2}\left((3-\sqrt{7})(8+3\sqrt{7})^n + (3+\sqrt{7})(8-3\sqrt{7})^n\right). \end{aligned}$$

Therefore for each l there exists an n_l such that $2^{3l} + 1 = x_{n_l}^2$ and $2^{3l} - 1 = 7y_{n_l}^2$. Hence $x_{n_l}^2 + 7y_{n_l}^2 = 2^{3l+1}$. But for all $n \in \mathbb{N}$

$$x_n^2 + 7y_n^2 = (16-6\sqrt{7})(127+48\sqrt{7})^n + (16+6\sqrt{7})(127-48\sqrt{7})^n$$

and the maximum power of two dividing the integer which is the right-hand side is, by Lemma 17 always exactly 5. Since $3l + 1 \neq 5$ for any l this case does not occur.

So assume $2^{3m} - 1 = 7s^2$ where m is odd. But then, if p is the maximum prime dividing $3m$ and we write

$$\left(\frac{2^{3m}-1}{2^p-1}\right)\left(\frac{2^p-1}{2-1}\right) = 7s^2$$

we must have $2^p - 1 = s^2$ or $(2^{3m} - 1)/(2^p - 1) = s^2$, both of which equations are impossible by [12, Lemma 4] and the case $g = 2$ above.

Now let n be odd and we may assume it is not prime.

Let $p = P(n) = 7$ and suppose also that $7^2 \mid n$. Then by Lemma 2 we can write

$$\left(\frac{8^n - 1}{7(8^7 - 1)}\right)\left(\frac{8^p - 1}{7^2}\right) = rs^2$$

so

$$\frac{8^n - 1}{7(8^7 - 1)} = s^2 \text{ or } \frac{8^7 - 1}{7^2} = s^2.$$

Now the left-hand side of the latter equation is 127×337 and if we recast the former as the Thue equation $2x^2 - 7(8^7 - 1)y^2 = 1$, which has no integer solutions, we see that this case yields no solutions.

If $p = P(n) \neq 7$ or $7^2 \nmid n$ then again by Lemma 2 we have

$$\left(\frac{8^n - 1}{8^p - 1}, \frac{8^p - 1}{7}\right) = 1,$$

and since

$$\left(\frac{8^n - 1}{8^p - 1}\right)\left(\frac{8^p - 1}{7}\right) = rs^2$$

we must have at least one of the following.

$$\frac{8^n - 1}{8^p - 1} = s^2, \ \frac{8^p - 1}{7} = s^2.$$

If $(8^n - 1)/(8^p - 1) = s^2$ then writing

$$\frac{(8^p)^{\frac{n}{p}} - 1}{8^p - 1} = s^2$$

and consulting Lemma 4 we see there are no solutions in this case.

If $8^p - 1 = 7s^2$ then, following the case $a = 1$ we see there are no solutions.

Now let $a = 3$. The equation $3(8^n - 1) = 7rs^2$ is impossible because $1 \not\equiv 3 \pmod 4$.

If $a = 5$ then $5(8^n - 1) = 7rs^2$. If $r = 5$ we have $8^n - 1 = 7rs^2$, the same as the case $a = 1$, so we assume $r \neq 5$. This gives rise to a rather large number of diophantine equations as follows.

First assume $n = 2m$ is even. Then splitting as before we must have one of the cases (i)-(iv) as set out below.

`Case (i)` If $8^m + 1 = s^2$ then $2^{3m} = (x + 1)(x - 1)$ for some odd $x > 1$ so $m = 1$ giving $aU_n = 45$, which is not perfect.

`Case (ii)` If $8^m + 1 = 5s^2$, a check with Magma [1], using the Thue equation $2x^2 - 5y^2 = -1$, shows there are no solutions if m is odd. If $m = 2l$ is even write

$$1 = 5x^2 - 8^{2l} = (\sqrt{5}x + 8^l)(\sqrt{5}x - 8^l)$$

and note that $\sqrt{5}x+8^l$ is a unit in the ring $R = \mathcal{O}_{\mathbb{Q}(\sqrt{5})}$. In this ring the fundamental unit is $\eta := (1+\sqrt{5})/2$ with norm -1 so there exists an odd integer s such that $\pm\eta^s = \sqrt{5}x + 8^l$. But this implies

$$\left(\frac{1+\sqrt{5}}{2}\right)^s + \left(\frac{1-\sqrt{5}}{2}\right)^s = \pm 2 \cdot 8^l.$$

By Lemma 16 the left-hand side is either odd or congruent to 4 modulo 8, so $8^m + 1 = 5s^2$, with m even, has no solution.

`Case (iii)` Consider $8^m - 1 = 7s^2$. That this is impossible unless $s^2 = 1$ follows from the calculations done in case $a = 1$ above.

`Case (iv)` If $8^m - 1 = 35s^2$, a check with Magma, using the Thue equation $2x^2 - 35y^2 = 1$, shows there are no solutions if m is odd. If m is even, $m = 2l$, then either $8^l - 1 = 35s^2$ and $8^l + 1 = s^2$, so $x^2 - 35y^2 = 2$ which has no solutions, or $8^l - 1 = 7s^2$ and $8^l + 1 = 5s^2$ so $5x^2 - 7y^2 = 2$, which again has no solutions.

If n is odd, $p = P(n) = 7$ and $p^2 \mid n$, by Lemma 2 the GCD

$$\left(\frac{8^n - 1}{8^p - 1}\right)\left(\frac{8^p - 1}{7}\right) = 7,$$

we must have at least one of three cases given below.

Either $8^7 - 1 = 7^2 s^2$ or $8^7 - 1 = 5 \cdot 7^2 s^2$, both of which are false.

If $8^{7^2 m} - 1 = 7(8^7 - 1)s^2$, a check with Magma using the Thue equation $2x^2 - 7(8^7 - 1)y^2 = 1$ shows there are no solutions in this case.

If $8^{7^2 m} - 1 = 35(8^7 - 1)s^2$, a check with Magma using the Thue equation $2x^2 - 35(8^7 - 1)y^2 = 1$ shows there are no solutions in this case.

If $p \neq 7$ then the GCD is 1 so we must have at least one of the three cases given below.

If $8^p - 1 = 7s^2$ this is impossible by Case (iii) above.

If $8^p - 1 = 35s^2$ there are no solutions, by Case (iv).

If $8^n - 1 = (8^p - 1)s^2$, this is impossible since by Lemma 4

$$\frac{x^{n/p} - 1}{x - 1} = s^2$$

has no solution with $x = 8^p$ unless $n/p \leq 2$, but n is odd and composite.

If $8^n - 1 = 5(8^p - 1)s^2$, this is impossible since $5 \mid 8^n - 1$ implies $4 \mid n$, but n is odd.

Let $a = 7$. Then the equation $8^n - 1 = rs^2$ is impossible since $3 \not\equiv 1 \pmod 4$. □

Using the same approach as for Lemma 12, but working in the unique factorization domain which is the ring of integers of $\mathbb{Q}(\sqrt{7})$ we obtain the following.

Lemma 19. *The diophantine equation $3^m + 1 = 7s^2$ has just one solution $m = 3,\ s^2 = 4$.*

Lemma 20. *There are no odd perfect repdigits to base 9.*

Proof. If $a = 1$ then $9^n - 1 = 8rs^2$. We have $U_n \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$, $n = 4m + 1$. Then

$$\left(\frac{3^{4m+1}+1}{4}\right)\left(\frac{3^{4m+1}-1}{2}\right) = rs^2$$

and the GCD of the two factors on the left is 1, so either $3^{4m+1} + 1 = 4s^2$ or $3^{4m+1} - 1 = 2s^2$. By Lemma 4 the only solution to the latter equation is $m = 1$ or $n = 5$ and U_5 is not perfect. The former is Catalan's equation [10, 14, 13], so this has no solution for all m.

Let $a = 3$. If $3(9^n - 1) = 8rs^2$ then since $r \equiv 1 \pmod 4$ we have $3 \mid s^2$ so $9^n - 1 = 24rs^2$ which is not possible since $3 \nmid 1$.

If $a = 5$ then $5(9^n - 1) = 8rs^2$. We have $U_n \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$. If $r = 5$, $3^{2n} - 1 = 2s^2$ which, by Lemma 4 again is impossible, so assume $r > 5$. Then

$$\left(\frac{3^{4m+1}+1}{4}\right)\left(\frac{3^{4m+1}-1}{2}\right) = 5rs^2$$

and the factors on the left are coprime so we must have at least one of

$$3^{4m+1} + 1 = s^2,\ 3^{4m+1} + 1 = 5s^2,\ 3^{4m+1} - 1 = 2s^2,\ 3^{4m+1} - 1 = 10s^2.$$

The second and fourth are impossible, since $3^{4m+1} \pm 1$ is never a multiple of 5. The first is impossible since it is Catalan's equation. The only solution to the third is $m = 1$, but $5U_5 = 36905$ is not perfect.

If $a = 7$ then $7(9^n - 1) = 8rs^2$. We have $U_n \equiv 3 \pmod 4$ and $n \not\equiv 1 \pmod 4$. Then $7U_n = rs^2$ implies

$$\left(\frac{3^n+1}{4}\right)\left(\frac{3^n-1}{2}\right) = 7rs^2$$

so we must have at least one of

$$3^n - 1 = 2s^2,\ 3^n - 1 = 14s^2,\ 3^n + 1 = s^2,\ 3^n + 1 = 7s^2.$$

The first is impossible since $3^n - 1)/(3 - 1) = s^2$ implies, by Lemma 4, $n = 5$ or $n \le 2$, but $7U_5,\ 7U_2,\ 7U_1$ are not perfect.

The second gives $3^n \equiv 1 \pmod 7$, but 6 is the multiplicative order of 3 modulo 7 so $6 \mid n$ and n must be even, $n = 2m$. Then $(3^m + 1)(3^m - 1) = 14s^2$ so 4 divides the left-hand side, but the right-hand side is twice an odd number.

The third is Catalan's equation and has no solution. By Lemma 19 the fourth has just one solution with $n = 3$, but $7U_3 = 637$ is not perfect. □

The proof of Theorem 1, using Lemmas 8, 10, 11, 13, 14, 15, 18, and 20 is now complete.

5. Conjectures

Following the result of Pollack, this study, the theorem of Luca [7], that no member of a Lucas sequence with odd parameters (P, Q) is multiperfect, and the theorem of Luca and Pollack [9] that no odd repdigit to base 10 is multiperfect, it is natural to induce there should be no odd perfect repdigits to any base $g \geq 2$. This of course is too difficult since it implies there is no odd perfect number.

Conjecture U: There is no odd repunit which is perfect.

Conjecture R: There exists an infinite set of bases G such that for each $g \in G$ no odd repdigit to base g is multiperfect.

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