DISTRIBUTION FUNCTIONS OF THE SEQUENCE $\varphi(M)/M$, $M \in (K, K+N]$ AS K, N GO TO INFINITY

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Abstract

Let $\varphi(n)$ be the number-theoretic Euler's function. It is well-known that the sequence $\varphi(n)/n$, $n = 1, 2, 3, \ldots$ has a singular asymptotic distribution function $g_0(x)$ $(0 \le x \le 1)$. P. Erdős in 1946 found a sufficient condition on sequences of intervals $(k_m, k_m + N_m]$ (k_m, N_m tend to infinity with m), such that the sequence of step distribution functions $F_{(k_m,k_m+N_m]}(x) := \frac{\#\{n \in (k_m,k_m+N_m]\,;\, \varphi(n)/n<x\}}{N_m}$, also converges to $g_0(x)$. In this note, a necessary and sufficient condition is given to have such a convergence, and the Erdős result is refined by giving error terms. Also, H. Davenport in 1933 gave an explicit construction of $g_0(x)$. Using that, we obtain $g_0(x) \le g(x)$ for every limit distribution function $g(x)$ of $F_{(k,k+N]}(x)$. Finally, applying a result of A. Schinzel and Y. Wang (1958) asserting the density of $\left(\frac{\varphi(k+2)}{\varphi(k+1)}, \frac{\varphi(k+3)}{\varphi(k+2)}, \ldots, \frac{\varphi(k+N)}{\varphi(k+N-1)}\right)$, $k = 1, 2, 3, \ldots$ in $[0, +\infty)^{N-1}$, we show that such a limit distribution function $g(x)$ can have the form $\tilde{g}(x/\alpha)$, where $\tilde{g}(x)$ is an arbitrary distribution function and α is a related suitable constant.

1. Introduction

Many papers have been devoted to the study of the distribution of the sequence $\frac{\varphi(n)}{n}$, $n = 1, 2, 3, \ldots$, where φ denotes the classical Euler totient function. I. J. Schoenberg [19], [20] established, among other results, that this sequence has a continuous and strictly increasing asymptotic distribution function (basic properties of distribution functions can be found in [12, p. 53], [3, p. 138–157] and [21, p. 1–7]) and P. Erdős [6] showed that this function is singular (i.e., the derivative exists almost everywhere

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on $[0,1]$ and is zero, see [21, p. 2–191]). Recall that the asymptotic distribution function $g_0(x)$ of $\varphi(n)/n$, $n = 1, 2, 3\ldots$, is defined as

$$g_0(x) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_{[0,x)}\left(\frac{\varphi(n)}{n}\right), \qquad \text{for any } x \in [0,1],$$

where $c_{[0,x)}(t)$ denotes the characteristic function of the subinterval $[0,x)$ of $[0,1]$. An explicit construction of $g_0(x)$ can be found in B. A. Venkov [22]. For any interval $(k, k+N]$ define the step distribution function

$$F_{(k,k+N]}(x) = \frac{1}{N} \sum_{k<n\le k+N} c_{[0,x)}\left(\frac{\varphi(n)}{n}\right) \ (x \in [0,1)) \text{ and } F_{(k,k+N]}(1) = 1.$$

In this paper, convergence properties of $F_{(k_n,k_n+N_n]}$ are investigated for sequences of intervals $(k_m, k_m+N_m]$, $m = 1, 2, 3, \ldots$ using and mixing mainly two methods. The first one designed as the P. Erdős's approach introduces a parameter t to separate the prime divisors of integers into those greater that t and the others. The second one associated to the name of H. Davenport, takes also his foundation from the works of S. Ramanujan [16], P. Erdős [5, 8], B. A. Venkov [22], and many other people, is related to the notion of primitive x-abundant number introduced about the divisor function.

The initial source of this paper is the following result asserted by P. Erdős in [7] without providing details of the proof: if

$$\lim_{m\to\infty} \frac{\log\log\log k_m}{N_m} = 0$$

(for given increasing subsequences k_m and N_m of integers) then

$$\lim_{m\to\infty} F_{(k_m,k_m+N_m]}(x) = g_0(x), \quad \text{for every } x \in [0,1]\,. \tag{1}$$

As the Referee point out to the authors, a complete proof of (1) derives from the work of Galambos and I. Kátai in [11] where the method of characteristic functions is exploited in a somewhat more general setting. In the opposite direction, P. Erdős completed his theorem by constructing sequences k_m and N_m such that $\lim_m \frac{\log\log\log k_m}{N_m} = \frac{1}{2}$ and the sequence of distribution functions $F_{(k_m,k_m+N_m]}$ does not converge in distribution to g_0.

In the sequel, for short, the index m will be omitted but keeping in mind that N_m and k_m both go to infinity. In that case we write simply $k, N \to \infty$ if the constraints on these sequences are unambiguous.

In Part 2, a necessary and sufficient condition to have (1) is given, that depends on divisors d of n, $d > N$, with $n \in (k, k+N]$. In Part 3, we analyze the Erdős approach and improve his result by exhibiting some error terms. In Part 4, examples of sequences of intervals $(k, k+N]$ $(k, N \to \infty)$ are given such that $\lim_{N\to\infty} \frac{\log\log\log k}{N} = +\infty$ but (1) still holds. Next, in Part 5, we analyze the H. Davenport's method and find a necessary and sufficient condition such that $F_{(k,k+N]}(x)$ converges to a given distribution function $g(x)$ (as $N \to \infty$). Finally, applying Schinzel–Wang's Theorem [18] in Part 6, we show that asymptotic distribution $g(x)$ of $F_{(k,k+N]}(x)$ $(k, N \to \infty)$, can have the form $g(x) = \tilde{g}\left(\frac{x}{\alpha}\right)$ $(x \in [0,1])$, where $\tilde{g}(x)$ is an arbitrary given distribution function and α is a related constant depending on $\tilde{g}(x)$.

2. A Necessary and Sufficient Condition

Theorem 1. *For any two increasing sequences of natural numbers N_m and k_m, the limit (1) holds if and only if for every positive integer s,*

$$\lim_{m\to\infty} \frac{1}{N_m} \sum_{k_m < n \le k_m + N_m} \sum_{\substack{d > N_m \\ d \mid n}} \Phi_s(d) = 0,$$

where Φ_s is given by $\Phi_s(1) := 1$,

$$\Phi_s(d) := \prod_{\substack{p \mid d \\ (p\ prime)}} \left(\left(1 - \frac{1}{p}\right)^s - 1 \right)$$

for any square-free integer d and $\Phi_s(d) := 0$ otherwise.

Proof. By applying Weyl's limit relation (see [21, p. 1–12, Th. 1.8.1.1]) we get (1) if and only if, for all positive integers s,

$$\lim_{m\to\infty} \frac{1}{N_m} \sum_{k_m < n \le k_m + N_m} \left(\frac{\varphi(n)}{n}\right)^s = \int_0^1 x^s \mathrm{d}g_0(x).$$

Notice that $\Phi_s(\cdot)$ is a multiplicative arithmetic function (*i.e.,* $\Phi_s(1) = 1$ and $\Phi_s(mn) = \Phi_s(m)\Phi_s(n)$ if m, n are coprime integers). From a result of I. Schur, reported by Schoenberg in [19], page 194 (see [4], page 214 and also a general theorem of H. Delange ([2, Théorème 2])) one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(\frac{\varphi(n)}{n}\right)^s = \prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right). \tag{2}$$

Now we use the easy equality

$$\sum_{d \mid n} \Phi_s(d) = \left(\frac{\varphi(n)}{n}\right)^s$$

to expand $\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)^s$. To this aim, we write

$$\begin{aligned}\sum_{k<n\le k+N}\sum_{d|n}\Phi_s(d) &= \sum_{d=1}^{k+N}\Phi_s(d)\left(\left\lfloor\frac{k+N}{d}\right\rfloor - \left\lfloor\frac{k}{d}\right\rfloor\right)\\ &= \sum_{d=1}^{k+N} N\frac{\Phi_s(d)}{d} + \sum_{d=1}^{k+N}\Phi_s(d)\left(\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\}\right),\end{aligned}$$

where $\lfloor x\rfloor$ denotes the integer part of x and $\{x\}$ the fractional part of x. Since

$$\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\} = \begin{cases} -\left\{\frac{N}{d}\right\} & \text{if } \left\{\frac{k}{d}\right\} + \left\{\frac{N}{d}\right\} < 1,\\ 1-\left\{\frac{N}{d}\right\} & \text{otherwise,}\end{cases} \tag{3}$$

the summation up to $k+N$ can be reduced to N to get

$$\begin{aligned}\frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n)}{n}\right)^s &= \sum_{d=1}^{N}\frac{\Phi_s(d)}{d} + \frac{1}{N}\sum_{d=1}^{N}\Phi_s(d)\left(\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\}\right)\\ &\quad + \frac{1}{N}\sum_{\substack{N<d\le k+N\\ \left\{\frac{k}{d}\right\}+\frac{N}{d}\ge 1}}\Phi_s(d). \end{aligned}\tag{4}$$

Let us prove that

$$\sum_{\substack{N<d\le k+N\\ \left\{\frac{k}{d}\right\}+\frac{N}{d}\ge 1}}\Phi_s(d) = \sum_{j=1}^{N}\sum_{\substack{d|k+j\\ d>N}}\Phi_s(d)$$

for any positive integers s, k and N by using the following lemma:

Lemma 2. *Let $d > N$, then $\left\{\frac{k}{d}\right\} + \frac{N}{d} \ge 1$ if and only if there exists $1 \le j \le N$ such that*

$$d \mid k+j,$$

and in that case, j is unique.

Proof. The unicity is clear due to $d > N$ and k can be assumed non negative and strictly less than d. Now the inequality $\left\{\frac{k}{d}\right\} + \frac{N}{d} \geq 1$ means that $k + N \geq d$ which is equivalent to $d \mid k + j$ for $j = d - k$ with $1 \leq j \leq N$ as required. □

Applying Lemma 2 in (4) we obtain the following basic equality:

$$\frac{1}{N} \sum_{k<n\leq k+N} \left(\frac{\varphi(n)}{n}\right)^s = \sum_{d=1}^{N} \frac{\Phi_s(d)}{d} + \frac{1}{N} \sum_{d=1}^{N} \Phi_s(d) \left(\left\{\frac{k}{d}\right\} - \left\{\frac{k+N}{d}\right\}\right) + \frac{1}{N} \sum_{k<n\leq k+N} \sum_{d>N,\, d \mid n} \Phi_s(d)\,. \tag{5}$$

Clearly, $|\Phi_s(d)| \leq \frac{s^{\omega(d)}}{d}$, if d is square free, where $\omega(d)$ denotes the number of different primes which divide d and successively, from A. G. Postnikov [15, p. 361–363 or English trans. p. 264–266],

$$\sum_{d=1}^{N} |\Phi_s(d)| \leq (1 + \log N)^s, \tag{6}$$

$$\sum_{d=N+1}^{\infty} \frac{|\Phi_s(d)|}{d} \leq \frac{3^s(1 + \log N)^{s-1}}{N}, \tag{7}$$

$$\sum_{d=1}^{\infty} \frac{\Phi_s(d)}{d} = \prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right).$$

Consequently, Theorem 1 follows from (2), (5) and the above relations. □

Remark 3. Using (5) and

$$\frac{1}{N} \sum_{k<n\leq k+N} \sum_{\substack{d>N \\ d \mid n}} \Phi_s(d) + \frac{1}{N} \sum_{k<n\leq k+N} \sum_{\substack{d\leq N \\ d \mid n}} \Phi_s(d) = \frac{1}{N} \sum_{k<n\leq k+N} \left(\frac{\varphi(n)}{n}\right)^s$$

we obtain

$$\frac{1}{N} \sum_{k<n\leq k+N} \sum_{\substack{d\leq N \\ d \mid n}} \Phi_s(d) = \sum_{d=1}^{N} \frac{\Phi_s(d)}{d} + \mathcal{O}\left(\frac{(1 + \log N)^s}{N}\right),$$

the error term being independent of k and thus, when the integer N goes to infinity, the left-hand side of this equality converges to $\prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right)$ uniformly with respect to k.

3. The Erdős Approach

For any positive integer n and real number $t \geq 2$, set

$$n(t) := \prod_{\substack{p \mid n \\ p \leq t}} p, \quad n'(t) := \prod_{\substack{p \mid n \\ p > t}} p, \text{ and } P(t) := \prod_{p \leq t} p, \tag{8}$$

where p are primes and the empty product is 1. P. Erdős in [7] proved the following lemma but without any explicit error term and only for $s = 1$:

Lemma 4. *For all positive integers k, N and for $t = N$, the equality*

$$\frac{1}{N} \sum_{k < n \leq k+N} \left(\frac{\varphi(n(t))}{n(t)} \right)^s = \frac{1}{N} \sum_{n=1}^{N} \left(\frac{\varphi(n)}{n} \right)^s + \mathcal{O}\left(\frac{3^s (1 + \log N)^s}{N} \right) \tag{9}$$

holds for all integers $s \geq 1$ and $N \geq 2$, the constant involved in the big $\mathcal{O}$ being absolute.

Proof. As above, from the definition of Φ_s, we have for any $t \geq 2$

$$\begin{aligned} \sum_{k < n \leq k+N} \left(\frac{\varphi(n(t))}{n(t)} \right)^s &= \sum_{k < n \leq k+N} \sum_{d \mid n(t)} \Phi_s(d) \\ &= \sum_{d \mid P(t)} \Phi_s(d) \left(\left\lfloor \frac{k+N}{d} \right\rfloor - \left\lfloor \frac{k}{d} \right\rfloor \right) \\ &= N \sum_{d \mid P(t)} \frac{\Phi_s(d)}{d} + \sum_{d \mid P(t)} \Phi_s(d) \left(\left\{ \frac{k}{d} \right\} - \left\{ \frac{k+N}{d} \right\} \right). \end{aligned}$$

Observe that

$$\sum_{d \mid P(t)} |\Phi_s(d)| \leq \sum_{d \mid P(t)} \frac{s^{\omega(d)}}{d} = \prod_{p \leq t} \left(1 + \frac{s}{p} \right)$$

and using the classical estimate $\left(\prod_{p \leq t} \left(1 - \frac{1}{p} \right) \right)^{-1} \leq (e^{\gamma} \log t)\big(1 + c(\log t)^{-2}\big)$ with an absolute constant $c > 0$ (see [17] for explicit value of c) we get

$$\begin{aligned} \prod_{p \leq t} \left(1 + \frac{s}{p} \right) &\leq \prod_{p \leq t} \left(1 - \frac{1}{p^2} \right)^s \prod_{p \leq t} \left(1 - \frac{1}{p} \right)^{-s} \\ &\leq (3/4)^s e^{s(\gamma + c(\log t)^{-2})} (\log t)^s . \end{aligned}$$

In particular, there exists an integer $t_0 \geq 2$ (which is explicit, in fact $t_0 = 286$ works well) such that

$$\sum_{d|P(t)} |\Phi_s(d)| \leq 3^s (\log t)^s \,.$$

for any $t \geq t_0$ and $s \geq 1$.

Now, due to the multiplicativity of $n \mapsto \Phi_s(n)/n$,

$$\sum_{d|P(t)} \frac{\Phi_s(d)}{d} = \prod_{p \leq t} \left(1 + \frac{\left(1 - \frac{1}{p}\right)^s - 1}{p}\right)$$

and from [15, p. 363, or English trans. p. 264 and p. 265] one has the quantitative form of the above result of Schur

$$\frac{1}{N} \sum_{n=1}^{N} \left(\frac{\varphi(n)}{n}\right)^s = \prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right) + \mathcal{O}\left(\frac{3^s(1+\log N)^s}{N}\right)$$

where the constant involved by the big O is absolute and also (see (7)),

$$\left|1 - \prod_{p>N} \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right)\right| \leq \sum_{n>N} \frac{|\Phi_s(n)|}{n} \leq \frac{3^s(1+\log N)^{s-1}}{N} \,.$$

Consequently, for all integers $s \geq 1$ and $N \geq 2$,

$$\Big| \prod_{p \leq N} \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right) - \prod_p \left(1 - \frac{1}{p} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^s\right) \Big|$$

$$\leq (3/4)\, \frac{3^s(1+\log N)^{s-1}}{N} \,.$$

Taking into account all these bounds leads to (9). □

In his work, Erdős used implicitly the following theorem:

Theorem 5. *For every two increasing sequences of integers k_m and N_m and for $t = N_m$ if*

$$\lim_{m\to\infty} \left(\prod_{k_m < n \leq k_m + N_m} \frac{\varphi(n'(t))}{n'(t)}\right)^{\frac{1}{N_m}} = 1$$

then

$$\lim_{m\to\infty} F_{(k_m, k_m+N_m]}(x) = g_0(x)$$

holds for all $x \in [0,1]$.

Proof. We claim that for any integer $s \geq 1$, the assumption means that for all ε in $(0,1]$ there exists an integer M_s such that the inequality $m \geq M_s$ implies

$$\#\{n \in \mathbf{N};\, k_m < n \leq k_m + N_m \text{ and } x_n^s \leq 1-\varepsilon\} \leq \varepsilon N_m \tag{10}$$

with $x_n = \frac{\varphi(n'(t))}{n'(t)}$ $(t = N_m)$. This result is a consequence of the following elementary lemma:

Lemma 6. *Let $y_1 \ldots, y_N$ be a finite sequence of nonnegative real numbers and assume that*

$$\sum_{n=1}^{N} y_n \leq \eta_1 \eta_2 N$$

for positive real numbers η_1 and η_2. Then

$$\#\{n \in \mathbf{N}; 1 \leq n \leq N \text{ and } y_n > \eta_2\} < \eta_1 N\,.$$

The proof is straightforward.

The assumption of Theorem 5, by taking the logarithm, leads to

$$\sum_{k_m < n \leq K_m + N_m} -s \log\Big(\frac{\varphi(n'(t))}{n'(t)}\Big) \leq \log(1-\varepsilon)\log(1-\varepsilon/2) N_m$$

for m large enough. Consequently, (10) follows from Lemma 6 with $N = N_m$, $k_m < n \leq k_m + N_m$, $\eta_1 = -\log(1-\frac{\varepsilon}{2})$ and $\eta_2 = -\log(1-\varepsilon)$. This proves our claim.

Now we assume $m \geq M_s$ in order to have (10) and define

$$A(m,\varepsilon) := \{n \in \mathbf{N};\, k_m < n \leq k_m + N_m : \text{and } x_n^s \leq 1-\varepsilon\}\,.$$

Using

$$\frac{\varphi(n)}{n} = \frac{\varphi(n(t))}{n(t)} \frac{\varphi(n'(t))}{n'(t)}$$

we obtain on one side

$$\frac{1}{N_m}\sum_{n=k_m+1}^{k_m+N_m}\Big(\frac{\varphi(n)}{n}\Big)^s \geq (1-\varepsilon)\Big(\frac{1}{N_m}\sum_{n=k_m+1}^{k_m+N_m}\Big(\frac{\varphi(n(t))}{n(t)}\Big)^s\Big) - (1-\varepsilon)\frac{\#A(m,\varepsilon)}{N_m} \tag{11}$$

and, on the other side,

$$\frac{1}{N_m}\sum_{n=k_m+1}^{k_m+N_m}\Big(\frac{\varphi(n)}{n}\Big)^s \leq \frac{1}{N_m}\sum_{n=k_m+1}^{k_m+N_m}\Big(\frac{\varphi(n(t))}{n(t)}\Big)^s.$$

Lemma 4 implies

$$(1-\varepsilon)\int_0^1 x^s \mathrm{d}g_0(x) \le \lim_{m\to\infty}\frac{1}{N_m}\sum_{k_m<n\le k_m+N_m}\Big(\frac{\varphi(n)}{n}\Big)^s \le \int_0^1 x^s \mathrm{d}g_0(x)\,,$$

proving Theorem 5. □

Notice that $\lim_{m\to\infty}\frac{1}{N_m}\sum_{k_m<n\le k_m+N_m}\frac{\varphi(n'(t))}{n'(t)}=1$ is equivalent to the assumption of Theorem 5. In other words,

Proposition 7. *For any two increasing sequences of integers k_m and N_m, if*

$$\lim_{m\to\infty}\frac{1}{N_m}\sum_{k<n\le k_m+N_m}\frac{\varphi(n'(N_m))}{n'(N_m)}=1$$

then

$$\lim_{m\to\infty}F_{(k_m,k_m+N_m]}(x)=g_0(x)$$

holds for all $x\in[0,1]$.

Remark 8. The converse of Theorem 5 is not true. In fact, replacing in Equation (11) the right-hand side by the following more accurate expression

$$(1-\varepsilon)\Big(\frac{1}{N}\sum_{k<n\le k+N}\Big(\frac{\varphi(n(t))}{n(t)}\Big)^s\Big)-(1-\varepsilon)\Big(\frac{1}{N}\sum_{\substack{k<n\le k+N\\ n\in A(m,\varepsilon)}}\Big(\frac{\varphi(n(t))}{n(t)}\Big)^s\Big),$$

it may appear that simultaneously $\lim_{m\to\infty}\Big(\frac{1}{N}\sum_{\substack{k<n\le k+N\\ n\in A(m,\varepsilon)}}\Big(\frac{\varphi(n(t))}{n(t)}\Big)^s\Big)=0$ and $\lim_{m\to\infty}\frac{\#A(m,\varepsilon)}{N_m}=\delta$ with $\delta>0$.

Finally, Erdős proved the following theorem but we give here a more readable proof for the convenience of the reader.

Theorem 9. *For any increasing sequences of integers k_m and N_m such that*

$$\lim_{m\to\infty}\frac{\log\log\log k_m}{N_m}=0$$

one has

$$\lim_{m\to\infty}F_{(k_m,k_m+N_m]}(x)=g_0(x)$$

for all $x\in[0,1]$.

Proof. The basic fact is that for $t = N$ the integers $n'(t)$ such that $k < n \leq k + N$ are pairwise relatively prime, because the interval $(k, k + N]$ cannot contain two different integers divisible by the same prime number $p > N$. Set

$$M'(k,N,t) := \prod_{k<n\leq k+N} n'(t) \tag{12}$$

but use notation $M'(t)$ for short and let $x = x(k, N)$ be defined such that the number of prime numbers p, $N < p \leq x$, is equal to $\omega(M'(t))$, where $t = N$. From the classical Mertens' formula

$$\prod_{p\leq y}\Big(1-\frac{1}{p}\Big) = \frac{e^{-\gamma}}{\log y}\Big(1+\mathcal{O}\Big(\frac{1}{\log y}\Big)\Big)$$

(see [14, p. 259, VII. 29] for example) we get

$$\frac{\varphi(M'(t))}{M'(t)} \geq \prod_{N<p\leq x}\Big(1-\frac{1}{p}\Big) \geq c_1\frac{\log N}{\log x}$$

for a constant $c_1 > 0$. Therefore, for any increasing sequences k_m and N_m, if $\big(\frac{\log N_m}{\log x(k_m,N_m)}\big)^{1/N_m}$ converges to 1 then the corresponding sequence $\big(\frac{\varphi(M'(N_m))}{M'(N_m)}\big)^{1/N_m}$ also converges to 1. Having in mind the Landau inequalities

$$\log 2 \leq \liminf_{x\to\infty}\frac{1}{x}\sum_{p\leq x}\log p \leq \limsup_{x\to\infty}\frac{1}{x}\sum_{p\leq x}\log p \leq 2\log 2 \tag{13}$$

(see [13, p. 83]) we conclude there exist suitable absolute positive constants c_2, c_3 such that

$$e^{c_2 x(k,N)-c_3 N} \leq \prod_{N<p\leq x(k,N)} p$$

and, after considering the obvious inequalities

$$\prod_{N<p\leq x(k,N)} p \leq (k+1)(k+2)\dots(k+N) < (k+N)^N,$$

we obtain $x(k, N) < c_4 N \log(k + N)$ with $c_4 > 0$.

Consequently, if the sequence $\big(\frac{\log N_m}{\log(N_m\log(k_m+N_m))}\big)^{1/N_m}$ converges to 1, the same is true for the sequence $\big(\frac{\log N_m}{\log x(k_m,N_m)}\big)^{1/N_m}$, hence the corresponding sequence $\big(\frac{\varphi(M'(t))}{M'(t)}\big)^{1/N_m}$ also converges to 1 and so, $F_{(k_m,k_m+N_m]}(x)$ converges to $g_0(x)$ for all $x \in [0, 1]$ by Theorem 5. The proof ends after noticing that

$$\lim_{m\to\infty}\frac{1}{N_m}\Big(\log\frac{\log N_m}{\log(N_m\log(k_m+N_m))}\Big) = 0$$

if, and only if,

$$\lim_{m\to\infty} \frac{\log\log\log k_m}{N_m} = 0.$$

□

Remark 10. Assume that $P(t) \mid k$, where $P(t) = \prod_{p\le t} p$ and $t = N$. As in (8), we introduce for divisors d of n the integers

$$d(t) = \prod_{\substack{p \mid d\\ p\le t}} p \quad \text{and} \quad d'(t) = \prod_{\substack{p \mid d,\\ p>t}} p.$$

Since $d(t) \mid n$, $n = k + j$ with $j \le N$ and $d(t) \mid k$, it follows that $d(t) \le N$. Hence, if $d > N$ one has $d'(t) > 1$. Therefore

$$\begin{aligned} \sum_{\substack{d>N\\ d \mid n}} \Phi_s(d) &= \sum_{d \mid n(t)} \Phi_s(d) \sum_{\substack{d' \mid n'(t)\\ d'\ne 1}} \Phi_s(d') \\ &= \left(\frac{\varphi(n(t))}{n(t)}\right)^s \left(\left(\frac{\varphi(n'(t))}{n'(t)}\right)^s - 1\right) \end{aligned}$$

leading to

$$\left|\sum_{\substack{d>N\\ d \mid n}} \Phi_s(d)\right| \le 1 - \left(\frac{\varphi(n'(t))}{n'(t)}\right)^s.$$

Thus, for all $s = 1, 2, 3, \dots$ one has

$$\lim_{m\to\infty} \frac{1}{N_m} \sum_{k_m<n\le k_m+N_m} \left(\frac{\varphi(n'(t))}{n'(t)}\right)^s = 1$$

for a given subsequence of integers k_m and for N_m with $P(N_m) \mid k_m$. By Theorem 1, we may conclude (1), but in fact Proposition 7 gives the same conclusion without such a constraint on k_m.

Notice that due to $\frac{\varphi(M'(k,N,t))}{M'(k,N,t)} \le \frac{\varphi(n'(t))}{n'(t)}$ for $k < n \le k + N$ (with $M'(k, N, t) = \prod_{k<n\le k+N} n'(t)$ as above in (12)) one obtains

Corollary 11. *If the sequence* $\frac{\varphi(M'(k_m,N_n,N_m))}{M'(k_m,N_n,N_m)}$ *converges to 1 for increasing sequences of integers* k_m *and* N_m*, then the sequence of distribution functions* $F_{(k_m,k_m+N_m]}$ *converges to the distribution function* g_0*.*

To end this section we prove the following quantitative version of Theorem 1.

Theorem 12. *For any positive integers k, N and s,*

$$\frac{1}{N}\sum_{k<n\le k+N}\sum_{\substack{d>N\\ d\mid n}}\Phi_s(d) = \frac{1}{N}\sum_{k<n\le k+N}\Big(\frac{\varphi(n)}{n}\Big)^s - \frac{1}{N}\sum_{n=1}^{N}\Big(\frac{\varphi(n)}{n}\Big)^s + \mathcal{O}\Big(\frac{(1+\log N)^s}{N}\Big) \tag{14}$$

and the constant in the big $\mathcal{O}$ can be chosen equal to 2.

Proof. Let $t = N$. Notice that

$$\sum_{\substack{d>N\\ d\mid n}}\Phi_s(d) = \sum_{\substack{d>N\\ d\mid n(t)}}\Phi_s(d) + \sum_{\substack{d\mid n(t)n'(t)\\ d'(t)\neq 1}}\Phi_s(d)$$

and the second sum is equal to $\Big(\frac{\varphi(n(t)}{n(t)}\Big)^s\Big(\Big(\frac{\varphi(n'(t))}{n'(t)}\Big)^s - 1\Big)$. Summing from $k+1$ to $k+N$ gives

$$\frac{1}{N}\sum_{k<n\le k+N}\sum_{\substack{d>N\\ d\mid n}}\Phi_s(d) = \frac{1}{N}\sum_{k<n\le k+N}\sum_{\substack{d>N\\ d\mid n(t)}}\Phi_s(d) + \frac{1}{N}\sum_{k<n\le k+N}\Big(\frac{\varphi(n)}{n}\Big)^s - \frac{1}{N}\sum_{k<n\le k+N}\Big(\frac{\varphi(n(t))}{n(t)}\Big)^s. \tag{15}$$

Now, successively

$$\begin{aligned}
\frac{1}{N}\sum_{k<n\le k+N}\sum_{d\mid n(t)}\Phi_s(d) &= \frac{1}{N}\sum_{k<n\le k+N}\Big(\frac{\varphi(n(t))}{n(t)}\Big)^s\\
&= \frac{1}{N}\sum_{k<n\le k+N}\sum_{\substack{d\le N\\ d\mid n(t)}}\Phi_s(d) + \frac{1}{N}\sum_{k<n\le k+N}\sum_{\substack{d>N\\ d\mid n(t)}}\Phi_s(d)\\
&= \sum_{d=1}^{N}\frac{\Phi_s(d)}{d} + \frac{1}{N}\sum_{d=1}^{N}\Phi_s(d)\Big(\Big\{\frac{k}{d}\Big\} - \Big\{\frac{k+N}{d}\Big\}\Big)\\
&\quad + \frac{1}{N}\sum_{k<n\le k+N}\sum_{\substack{d>N\\ d\mid n(t)}}\Phi_s(d)\\
&= \sum_{d=1}^{N}\frac{\Phi_s(d)}{d} + \frac{1}{N}\sum_{k<n\le k+N}\sum_{\substack{d>N\\ d\mid n(t)}}\Phi_s(d) + \mathcal{O}\Big(\frac{(1+\log N)^s}{N}\Big)
\end{aligned}$$

and after inserting

$$\sum_{d=1}^{N}\frac{\Phi_s(d)}{d}=\sum_{n=1}^{N}\left(\frac{\varphi(n)}{n}\right)^s+\mathcal{O}\Big(\frac{(1+\log)^s}{N}\Big),$$

which can be obtained from (5) with $k = 0$, we get

$$\begin{aligned}\frac{1}{N}\sum_{k<n\le k+N}\sum_{\substack{d>N\\ d|n(t)}}\Phi_s(d) \qquad &= \frac{1}{N}\sum_{k<n\le k+N}\left(\frac{\varphi(n(t))}{n(t)}\right)^s-\sum_{n=1}^{N}\left(\frac{\varphi(n)}{n}\right)^s\\ &+\mathcal{O}\Big(\frac{(1+\log N)^s}{N}\Big).\end{aligned}$$

Inserting this equality in (15) gives (14). Finally, notice that the error term comes from the bound (6) used twice. □

4. Examples

To show that his assumption in Theorem 3 is optimal, Erdős gave the following example.

Example 13. Take t large enough to write $P(t) = \prod_{p\le t} p$ as the product of N numbers $A_1, A_2, \ldots, A_N$ such that

(i) A_i, $i = 1, \ldots, N$, are relatively prime,

(ii) $\frac{\varphi(A_i)}{A_i} < \frac{1}{2}$ for $i = 1, \ldots, N$,

(iii) if p is the maximal prime in A_i, then for $A'_i = A_i/p$ one has $\frac{\varphi(A'_i)}{A'_i} > \frac{1}{2}$.

Part (iii) implies $\frac{\varphi(A_i)}{A_i} > \frac{1}{4}$ and thus

$$\left(\frac{1}{4}\right)^N<\prod_{p\le t}\left(1-\frac{1}{p}\right)=\frac{\varphi(A_1)}{A_1}\cdots\frac{\varphi(A_N)}{A_N}<\left(\frac{1}{2}\right)^N.$$

From that, applying (12), we find $N < c_1 \log\log t$. By the Chinese remainder theorem there exists $k_0 < A_1 \ldots A_N$ such that $k_0 \equiv -i (\mathrm{mod}\ A_i)$ for $i = 1, \ldots, N$. Put $k = k_0 + A_1 \ldots A_N$; then

$$e^{c_2 t} < P(t) = A_1 \ldots A_N < k$$

which implies $t < c_3 \log k$ and $\log\log t < c_4 \log\log\log k$. Thus

$$\frac{\log\log\log k}{N} > \frac{1}{c_1 c_4}\frac{\log\log t}{\log\log t}.$$

Furthermore, for these k and N, the sequence of distribution functions $F_{(k,k+N]}(x)$ does not converge to $g_0(x)$ due to (ii), that gives

$$\frac{1}{N}\sum_{k<n\leq k+N}\frac{\varphi(n)}{n} < \frac{1}{2} < \frac{1}{N}\sum_{n=1}^{N}\frac{\varphi(n)}{n} = \frac{6}{\pi^2} + \mathcal{O}\left(\frac{\log N}{N}\right).$$

Example 14. In Example 1, replace in (ii) the ratio $1/2$ by $1/N$ and use the corresponding definition of the A_n as above. Then, by the Chinese remainder theorem, for every N we can find k such that $A_n|k+n$, $n = 1,\ldots,N$, and consequently

$$\frac{1}{N}\sum_{k<n\leq k+N}\left(\frac{\varphi(n)}{n}\right) \leq \frac{1}{N}\sum_{n=1}^{N}\left(\frac{\varphi(A_n)}{A_n}\right) \leq \frac{1}{N}.$$

Now select sequences of such integers k and N, but with a distribution function $g(x)$ such that $\lim_{k,N\to\infty} F_{(k,k+N]}(x) = g(x)$ a.e. in $[0,1]$. With this construction we obtain $\int_0^1 x\mathrm{d}g(x) = 0$. Therefore, $g(x)$ is the Heaviside distribution function (jump 1 at $x = 0$).

In the next example we construct sequences of integers k, N, for which (1) holds but $\lim_{N\to\infty}\frac{\log\log\log k}{N} = +\infty$.

Example 15. For any integer $N \geq 1$, let $x = x(N)$ be a real number, $x > N$, that will be chosen later but very large with respect to N (like $x(N) = e^{e^{e^N}}$ for example). Let $k := \prod_{p\leq x} p$, (where p are primes), consider the interval $(k, N+k]$ and define $M^* := \prod_{x<p\leq x+y(x)} p$ where $y(x)$ is chosen such that M^* has the same number of prime divisors than the product $M'(k,N,t)$ $(t = N)$ defined in (12). Presently, if a prime number p verifies $p > N$ and $p|k+j$ with $j \leq N$ then $p > x$. Thus, $\frac{\varphi(M^*)}{M^*} \leq \frac{\varphi(M'(t))}{M'(t)}$ and to satisfy the assumption of Corollary 11 it suffices that the ratio $\frac{\varphi(M^*)}{M^*} = \prod_{x<p\leq x+y(x)}\left(1-\frac{1}{p}\right)$ converges to 1 as x tends to infinity. According to Mertens' formula, this is equivalent to having

$$\lim_{x\to\infty}\frac{\log\left(1+\frac{y(x)}{x}\right)}{\log x} = 0. \tag{16}$$

The inequalities

$$M^* \le M' = \prod_{k<n\le k+N} n'(t) \le (k+N)^N \le (2k)^N$$

lead to $\sum_{x<p\le x+y(x)} \log p \le 2N \sum_{p\le x} \log p$ and thus

$$\sum_{p\le x+y(x)} \log p \le (2N+1) \sum_{p\le x} \log p. \tag{17}$$

Using (13) in (17), we see that for any $\varepsilon > 0$, there exists $x_0(\varepsilon)$ such that $x \ge x_0(\varepsilon)$ implies

$$(\log 2 - \varepsilon)(x + y(x)) \le (2N+1)(2\log 2 + \varepsilon)x\,,$$

so that $\frac{y(x)}{x} \le cN$ for a positive constant c. Therefore, (16) holds and consequently (1) holds also, if we chose $x = x(N) \ge e^N$. Since $k(N) = \prod_{p\le x(N)} p \ge e^{c_1 x(N)}$, by taking $x(N) = e^{e^{e^N}}$ the limit

$$\lim_{N\to\infty} \frac{\log\log\log k}{N} = +\infty$$

holds as expected.

5. Davenport's Approach

Let $f : \mathbf{N} \to (0,1]$ be a multiplicative function. Assume that $0 < f(n) \le 1$ for all n; it is useful to introduce for any $x \in (0,1)$ the increasing sequence $a_k(x)$ of all integers a such that $f(a) \le x$ but $f(d) > x$ for every divisor d of a, $d \ne a$. In the case $f(n) = n/\sigma(n)$ (where $\sigma(n)$ is the sum of divisors of n) such an integer a is classically called primitive x-abundant number. In 1933, H. Davenport [1] using this notion proved that the sequence $n/\sigma(n)$ has a distribution function and found an explicit construction of it. In addition he gave sufficient conditions for f to have a distribution function. These conditions are easily verified for both sequences $n/\sigma(n)$ and $\varphi(n)/n$.

B.A. Venkov applied the same method in his paper [22] but for the sequence of ratios $\frac{\varphi(n)}{n}$. Following him, we introduce, for convenience, the definition of *x-numbers* (also called *primitive x-numbers* in [15]), that is to say integers $a > 0$ such that $\frac{\varphi(a)}{a} \le x$ and for every $d \,|\, a$ but $d \ne a$ one has $\frac{\varphi(d)}{d} > x$. We denote by $A(x)$ the set of all x-numbers ordered in increase magnitude *i.e.*,

$$a_1(x) < a_2(x) < a_3(x) < \cdots$$

From now on, the sequence $p_1, p_2, p_3, \ldots$ denotes the increasing sequence of all prime numbers.

Remark 16. From the above definitions we get the following properties.

(i) Every x-number is square-free.

(ii) Every square-free a is an x-number for some x. Concretely, if $a = q_1 q_2 \ldots q_m$ with $q_1 < q_2 < \cdots < q_m$, all prime numbers, then a is x-number for every x in the interval $\left[\prod_{i=1}^{m}\left(1 - \frac{1}{q_i}\right), \prod_{i=1}^{m-1}\left(1 - \frac{1}{q_i}\right)\right)$.

(iii) For every $i < j$ we have $a_i(x) \nmid a_j(x)$.

(iv) Let p_s be the s-th prime number and choose $x \in \left[1 - \frac{1}{p_s}, 1\right)$. Then $a_1(x) = p_1 = 2$, $a_2(x) = p_2 = 3$, $\ldots$, $a_s(x) = p_s$. Furthermore, if $x < 1 - \frac{1}{p_{s+1}}$ then for every $j > s$, the integer $a_j(x)$ cannot be a prime and $p_i \nmid a_j(x)$ for $i = 1, 2, \ldots, s$.

Proof. By (ii), prime numbers $p_1, p_2, \ldots, p_s$ are x-numbers for $x \geq 1 - \frac{1}{p_s}$. If for some j we have $p_1 \leq a_j(x) \leq p_s$ and $p \,|\, a_j(x)$, p prime, then $p \leq p_s$ and $a_j(x) = p$, since $pq \,|\, a_j(x)$ with $q > 1$ contradicts (iii).

Now, $x < 1 - \frac{1}{p_{s+1}}$ implies that p_{s+1} and any $p_k > p_s$ are not x-numbers, and by (iii) $p_i \nmid a_j(x)$ for $i = 1, \ldots, s$. □

(v) If $x \in \left[\prod_{i=1}^{s}\left(1 - \frac{1}{p_i}\right), \prod_{i=1}^{s-1}\left(1 - \frac{1}{p_i}\right)\right)$ then $a_1(x) = \prod_{i=1}^{s} p_i$.

Proof. By contradiction. The integer $a = \prod_{i=1}^{s} p_i$ is an x-number, hence $a_1(x) \leq a$. Assume that $a_1(x) < a$ and let $a_1(x) = p_{i_1} p_{i_2} \ldots p_{i_k}$ with $i_1 < i_2 < \cdots < i_k$, then $k < s$. By definition,

$$x \in \left[\prod_{j=1}^{k}\left(1 - \frac{1}{p_{i_j}}\right), \prod_{i=1}^{k-1}\left(1 - \frac{1}{p_{i_j}}\right)\right)$$

hence $\prod_{j=1}^{k}\left(1 - \frac{1}{p_{i_j}}\right) < \prod_{i=1}^{s-1}\left(1 - \frac{1}{p_i}\right)$ which implies $k > s - 1$, a contradiction. □

(vi) For every positive integer n and every $x \in (0, 1)$ we have

$$\frac{\varphi(n)}{n} \leq x \Longleftrightarrow \exists\, i \in \mathbb{N}\, (a_i(x) | n).$$

(vii) Assume that $0 < x < x' < 1$. Then for every x-number $a_i(x)$ there exists an x'-number $a_j(x')$ such that $a_j(x') | a_i(x)$. This property follows from (vi) and the fact that for $n = a_i(x)$ one has $\frac{\varphi(n)}{n} < x'$.

(viii) Let $[b_1,\dots,b_j]$ denote the least common multiple of the integers $b_1,\dots,b_j$, then the asymptotic density of the set

$$\{n\in\mathbb{N}; a_m(x)|n, a_1(x)\nmid n, a_2(x)\nmid n,\dots,a_{m-1}(x)\nmid n\}$$

is given by

$$A_m(x)=\frac{1}{a_m(x)}+\sum_{u=1}^{m-1}\sum_{1\le j_1<j_2<\cdots<j_u<m}\frac{(-1)^u}{[a_{j_1}(x),\dots,a_{j_u}(x),a_m(x)]}\,.$$

(ix) Define

$$B_n(x)=\{a\in\mathbb{N};\ a\,|\,n \text{ and } \exists\ i\in\mathbb{N}\ (a=a_i(x))\}. \tag{18}$$

In this paper we have defined $F_{(k,k+N]}(x)=\frac{1}{N}\sum_{k<n\le k+N}c_{[0,x)}\bigl(\frac{\varphi(n)}{n}\bigr)$ but in this part, due to the definition of x-number, we use $c_{[0,x]}$ in place of $c_{[0,x)}$. Applying (vi), we see that

$$F_{(k,k+N]}(x)=\frac{\#\{n\in(k,k+N]; B_n(x)\ne\emptyset\}}{N}. \tag{19}$$

(x) As suggested by (vi) and (ix) we have by B.A. Venkov [22] (see also H. Davenport [1]) the following theorem:

The asymptotic distribution function $g_0(x)$ of the sequence $\frac{\varphi(n)}{n}$, $n=1,2,3\dots$, can be expressed by

$$g_0(x)=\sum_{m=1}^{\infty}A_m(x)\,. \tag{20}$$

In fact, the right-hand side of (20) is the asymptotic density of all integers n divisible by some x-number.

Below we prove that the asymptotic distribution function $g(x)$ in (1) cannot be arbitrary. A similar result was known by Erdős for asymptotic averages (see [7], Theorem 8). The proof combines Lemma 4 and (20).

Theorem 17. *Assume that* $\lim_{m\to\infty}F_{(k_m,k_m+N_m]}(x)=g(x)$ *for all* $x\in[0,1]$. *Then* $g_0(x)\le g(x)$ *for all* $x\in[0,1]$.

Proof. Set

$$\begin{aligned} R^{(1)}_{(k,k+N]}(x) &:= \\ &\frac{\#\{n\in(k,k+N];B_n(x)\neq\emptyset,\exists\, a\in B_n(x)\big(\forall\, p\,(p\,\mathrm{prime\,and}\,p|a\Rightarrow p\le N)\big)\}}{N}, \\ R^{(2)}_{(k,k+N]}(x) &:= \\ &\frac{\#\{n\in(k,k+N];B_n(x)\neq\emptyset,\forall\, a\in B_n(x)\big(\exists\, p\,(p\,\mathrm{prime},p|a\,\mathrm{and}\,p>N)\big)\}}{N} \end{aligned}$$

where $B_n(x)$ is given in (18). By (19),

$$F_{(k,k+N]}(x)=R^{(1)}_{(k,k+N]}(x)+R^{(2)}_{(k,k+N]}(x). \tag{21}$$

The monotonicity of $R^{(1)}_{(k,k+N]}(x)$ ($x\in[0,1]$) follows from (vii) and then for the distribution functions $F_{(k,k+N]}(x)$ and $R^{(1)}_{(k,k+N]}(x)$ we can apply Helly selection principle to exhibit a subsequence of the intervals $(k_m,k_m+N_m]$, still denoted $(k_m,k_m+N_m]$, such that for all $x\in(0,1)$ we have both $\lim_{m\to\infty}F_{(k_m,k_m+N_m]}(x)=g(x)$ and $\lim_{m\to\infty}R^{(1)}_{(k_m,k_m+N_m]}(x)=g^{(1)}(x)$ for a suitable distribution function $g^{(1)}(x)$. Therefore, we also have the limit

$$\lim_{m\to\infty}R^{(2)}_{(k_m,k_m+N_m]}(x)=g^{(2)}(x)=g(x)-g^{(1)}(x).$$

Now we prove the equality

$$g^{(1)}(x)=g_0(x) \tag{22}$$

for all x, that is to say

$$g(x)=g_0(x)+g^{(2)}(x). \tag{23}$$

For the sequence $\frac{\varphi(n(t))}{n(t)}$, $n\in(k,k+N]$, $n(t)=\prod_{p|n p\le t}p$, where $t=N$, define

$$\tilde{F}_{(k,k+N]}(x):=\frac{\#\{n\in(k,k+N];\frac{\varphi(n(t))}{n(t)}\le x\}}{N}.$$

By property (vi), if $\frac{\varphi(n(t))}{n(t)}\le x$, then there exists x-number $a_i(x)$ such that $a_i(x)\,|\,n(t)$. Since $n(t)\,|\,n$ it follows that $a_i(x)|n$ and furthermore for all prime numbers p, $p\,|\,a_i(x)$ implies $p\le t$ $(=N)$. Reciprocally, if $a_i(x)|n$ and for all prime numbers p, $p\,|\,a_i(x)$ implies $p\le t$, then $a_i(x)\,|\,n(t)$ and $\frac{\varphi(n(t))}{n(t)}\le x$. Thus

$$\tilde{F}_{(k,k+N]}(x)=R^{(1)}_{(k,k+N]}(x)$$

and consequently, $\tilde{F}_{(k,k+N]}(x) \to g^{(1)}(x)$ too. By Erdős' Lemma 4

$$\int_0^1 x^s \mathrm{d}g^{(1)}(x) = \int_0^1 x^s \mathrm{d}g_0(x)$$

for $s = 1, 2, 3 \dots$ and thus $g^{(1)}(x) = g_0(x)$ for $x \in (0,1)$ a.e. □

Theorem 18. *For every distribution function $g(x)$ such that*

$$\lim_{m\to\infty} F_{(k_m, k_m+N_m]}(x) = g(x)$$

a.e. on $[0,1]$ (with $k_m, N_m \to \infty$), there exists a constant c_1 such that

$$\int_0^1 x^s\, dg(x) \le \int_0^1 x^s\, dg_0(x) \le \frac{c_1}{\log(s+1)}, \tag{24}$$

for every positive integer s.

Proof. The first inequality in (24) follows from Lemma 4, since $\left(\frac{\varphi(n)}{n}\right)^s \le \left(\frac{\varphi(n(t))}{n(t)}\right)^s$. It also follows from Theorem 17, because $\int_0^1 x^s \mathrm{d}g(x) \le \int_0^1 x^s \mathrm{d}g_0(x)$ is equivalent to $\int_0^1 x^{s-1} g(x)\mathrm{d}x \ge \int_0^1 x^{s-1} g_0(x)\mathrm{d}x$. The second inequality in (24) was proved by B. A. Venkov [22, Theorem 3] in the form

$$\lim_{s\to\infty} \left(\int_0^1 x^s \mathrm{d}g_0(x) \right) \log s = e^{-\gamma},$$

where γ is the Euler's constant. □

Theorem 19. *For every $\alpha \in (0,1)$ there exists a sequence of intervals $(k_m, k_m+N_m]$ ($k_m, N_m \to \infty$) such that $F_{(k_m,k_m+N_m]}(x)$ converges to a distribution function $g(x)$ with $g(x) = 1$ for $\alpha \le x \le 1$.*

Proof. Let $\alpha \in (0,1)$ be fixed and let p_s be the greatest prime number p_i verifying $\left(1 - \frac{1}{p_i}\right) \le \alpha$. The α-numbers being square free, we can select a subsequence of them $a_{s_1}(\alpha) < a_{s_2}(\alpha) < a_{s_3}(\alpha) < \dots$ pairwise co-prime. By the Chinese remainder theorem, there exists a positive integer k such that $k + i \equiv 0 \pmod{a_{s_i}(\alpha)}$ for $i = 1, \dots, N$. Therefore

$$\#\{n \in (k, k+N]; B_n(\alpha) \ne \emptyset\} = N$$

and thus, by (19),

$$F_{(k,k+N]}(\alpha) = 1.$$

□

Remark 20. If $1-\frac{1}{p_s} \le x$, then readily $1 \le g_0(x)+\prod_{i=1}^{s}(1-p_i^{-1})$ since the second term of this sum is the density of natural numbers coprime to $p_1 \cdots p_s$. So, inserting $g(x)$ from Theorem 19 and putting $\alpha = x$ gives

$$g_0(x) \ge 1 - \prod_{p \le \frac{1}{1-x}} \left(1-\frac{1}{p}\right) \ge 1 - \frac{c_2}{\log\left(\frac{1}{1-x}\right)}$$

for all $x \in (0,1)$. This inequality was first proved by B.A. Venkov [22]. He also proved

(i) $\lim_{\substack{x\to 1\\ x<1}} (1-g_0(x)) \log \frac{1}{1-x} = e^{-\gamma}$.

(ii) $\lim_{\substack{x\to 0\\ x>0}} \ x \log\log \frac{1}{g_0(x)} = e^{-\gamma}$.

(iii) Let p be a prime number. If $1-\frac{1}{p} \le x$, then

$$\frac{1}{p} = \sum_{n=0}^{\infty} (-1)^n (p-1)^n g_0\Big(x\Big(1-\frac{1}{p}\Big)^n\Big).$$

(iv) The function $g_0(x)$ at every value $x = \frac{\varphi(n)}{n}$, $n = 1,2,3,\ldots$, has an infinite left derivative.

In fact, (i), (ii) and (iv) are another way to express results proved or suggested by Erdős in [7] (Theorems 1 and 3).

The identity (21) can be rewritten as

$$F_{(k,k+N]}(x) = F_{(0,N]}(x) + \big(R^{(1)}_{(k,k+N]}(x) - F_{(0,N]}(x)\big) + R^{(2)}_{(k,k+N]}(x).$$

The equality (22) we have proved means

$$\lim_{k,N\to\infty} \big(R^{(1)}_{(k,k+N]}(x) - F_{(0,N]}(x)\big) = 0 \tag{25}$$

for every every $x \in (0,1)$. In the next theorem we give a quantitative form of (25). To this aim, we introduce

$$K_N(x) := \{a \in \mathbb{N};\ \exists\, m \,\big(a = a_m(x) \text{ and } \forall\, p \ (p \ \text{ prime and } p \,|\, a \Rightarrow p \le N)\big)\},$$

$$r_N(x) :=$$

$$\frac{1}{N}\sum_{m=1}^{\infty}\left(-\left\{\frac{N}{a_m(x)}\right\} - \sum_{u=1}^{m-1}(-1)^u \sum_{1\le j_1<\cdots<j_u<m}\left\{\frac{N}{[a_{j_1}(x),\ldots,a_{j_u}(x),a_m(x)]}\right\}\right).$$

and

$$\tilde{R}_{(k,k+N]}(x) = \frac{1}{N}\sum_{m=1}^{\infty}\left(\sum_{\left\{\frac{k}{a_m(x)}\right\}+\left\{\frac{N}{a_m(x)}\right\}\geq 1} 1 + \sum_{u=1}^{m-1}(-1)^u \sum_{\substack{1\leq j_1<\cdots<j_u<m \\ \left\{\frac{k}{[a_{j_1}(x),\ldots,a_{j_u},a_m(x)]}\right\}+\left\{\frac{N}{[a_{j_1}(x),\ldots,a_{j_u},a_m(x)]}\right\}\geq 1}} 1\right). \quad (26)$$

Theorem 21. *For every interval $(k, k+N]$ and every $x \in (0,1)$, we have*

$$R^{(1)}_{(k,k+N]}(x) - F_{(0,N]}(x) = \frac{1}{N}\sum_{\substack{m\geq 1 \\ a_m(x)\in K_N(x)}}\left(\sum_{\left\{\frac{k}{a_m(x)}\right\}+\left\{\frac{N}{a_m(x)}\right\}\geq 1} 1 + \sum_{u=1}^{m-1}(-1)^u \sum_{\substack{1\leq j_1<\cdots<j_u<m \\ \left\{\frac{k}{[a_{j_1}(x),\ldots,a_{j_u},a_m(x)]}\right\}+\left\{\frac{N}{[a_{j_1}(x),\ldots,a_{j_u},a_m(x)]}\right\}\geq 1}} 1\right). \quad (27)$$

Proof. With an obvious meaning one has

$$\#\{n \in (0,N]; a_m(x)|n, a_1(x)\nmid n, a_2(x)\nmid n, \ldots, a_{m-1}(x)\nmid n\}$$
$$= \left\lfloor\frac{N}{a_m(x)}\right\rfloor - \sum_{j<m}\left\lfloor\frac{N}{[a_j(x),a_m(x)]}\right\rfloor + \sum_{i<j<m}\left\lfloor\frac{N}{[a_i(x),a_j(x),a_m(x)]}\right\rfloor - \cdots.$$

Moreover,

$$\begin{aligned} F_{(0,N]}(x) &= \frac{1}{N}\left(\sum_{m,\,a_m(x)\leq N}\left(\left\lfloor\frac{N}{a_m(x)}\right\rfloor - \sum_{j<m}\left\lfloor\frac{N}{[a_j(x),a_m(x)]}\right\rfloor + \ldots\right)\right) \\ &= \sum_{m,\,a_m(x)\leq N}\left(\frac{1}{a_m(x)} - \sum_{j<m}\frac{1}{[a_j(x),a_m(x)]} + \ldots\right) \\ &\quad + \frac{1}{N}\left(\sum_{m,\,a_m(x)\leq N}\left(-\left\{\frac{N}{a_m(x)}\right\} + \sum_{j<m}\left\{\frac{N}{[a_j(x),a_m(x)]}\right\} - \cdots\right)\right). \end{aligned}$$

The restriction $a_m(x) \leq N$ can be omitted, so that

$$F_{(0,N]}(x) = g_0(x) + r_N(x). \quad (28)$$

Insert (28) in

$$NF_{(k,k+N]}(x) = (k+N)F_{(0,k+N]}(x) - kF_{(0,k]}(x)$$

to obtain

$$\begin{aligned} F_{(k,k+N]}(x) &= g_0(x) + \frac{1}{N}\big((k+N)r_{k+N}(x) - kr_k(x)\big) \\ &= g_0(x) + \frac{1}{N}\sum_{m=1}^{\infty}\Bigg(\Bigg(-\left\{\frac{k+N}{a_m(x)}\right\} + \left\{\frac{k}{a_m(x)}\right\}\Bigg) \\ &\qquad - \sum_{j<m}\left(-\left\{\frac{k+N}{[a_j(x),a_m(x)]}\right\} + \left\{\frac{k}{[a_j(x),a_m(x)]}\right\}\right) + \cdots\Bigg) \end{aligned}$$

and apply (3) to get

$$\begin{aligned} F_{(k,k+N]}(x) &= g_0(x) + r_N(x) + \tilde{R}_{(k,k+N]}(x) \\ &= F_{(0,N]}(x) + \tilde{R}_{(k,k+N]}(x), \end{aligned}$$

where $\tilde{R}_{(k,k+N]}(x)$ has the form (26). Now, divide the summation defining $\tilde{R}_{(k,k+N]}(x)$ into two parts, $\tilde{R}_{(k,k+N]}(x) = \tilde{R}^{(1)}_{(k,k+N]}(x) + \tilde{R}^{(2)}_{(k,k+N]}(x)$, where in $\tilde{R}^{(1)}_{(k,k+N]}(x)$ the summation runs over the integers m such that every prime divisor of $a_m(x)$ is less or equal to N and in $\tilde{R}^{(2)}_{(k,k+N]}(x)$, the summation runs over the rest of integers, *i.e.,* over integers m such that there exists a prime divisor of $a_m(x)$ strictly greater than N. Using the fact that if there exists a prime divisor p of $a_m(x)$, $p > N$ and $a_m(x)|k+u$ for an integer u, $1 \le u \le N$, then $a_m(x)$ cannot divide the other integers of the form $k+u'$, $1 \le u' \le N$, and the same property holds for $[a_i(x), a_m(x)]$, $[a_i(x), a_j(x), a_m(x)]$ and so on. By applying Lemma 2,

$$\tilde{R}^{(2)}_{(k,k+N]}(x) = \sum_{u=1}^{N} \sum_{\substack{m=1 \\ \exists\, p,\, p|a_m(x),\, p>N \\ a_m(x)|k+u}}^{\infty} \Bigg(1 - \sum_{\substack{i<m \\ [a_i(x),a_m(x)]|k+u}} 1 + \sum_{\substack{l<i<m \\ [a_l(x),a_i(x),a_m(x)]|k+u}} 1 - \cdots\Bigg).$$

If $a_m(x)|k+u$, the value of

$$\Bigg(1 - \sum_{\substack{i<m \\ [a_i(x),a_m(x)]|k+u}} 1 + \sum_{\substack{l<i<m \\ [a_l(x),a_i(x),a_m(x)]|k+u}} 1 - \cdots\Bigg) \tag{29}$$

is 0 or 1. More precisely, let $a_{i_1}(x), a_{i_2}(x), \ldots, a_{i_s}(x)$ be all x-numbers $a_i(x)$ such that $i < m$ and $a_i(x)$ divides $k+u$. Then the number of i, $i < m$, for which $[a_i(x), a_m(x)] \,|\, k+u$ is s; the number of tuples (j, i), with $1 \leq j < i < m$ and $[a_j(x), a_i(x), a_m(x)] \,|\, k+u$ is $s(s-1)/2$, etc. Thus, the expression (29) in this case has the form $(1-1)^s$ and hence equals 0. If such $a_{i_r}(x)$ do not exist, then the value of (29) is 1. Thus $\tilde{R}^{(2)}_{(k,k+N]}(x) = R^{(2)}_{(k,k+N]}(x)$ and $\tilde{R}^{(1)}_{(k,k+N]}(x) = R^{(1)}_{(k,k+N]}(x) - F_{(0,N]}(x)$. □

Note that for $\tilde{R}^{(1)}_{(k,k+N]}(x)$ we can also apply Lemma 2 in the form:

$$\left\{\frac{k}{d}\right\} + \left\{\frac{N}{d}\right\} \geq 1, \text{ if and only if there exists } j \text{ such that}$$

$$1 \leq j \leq N_d \ (= N - d\lfloor N/d \rfloor) \text{ and } d | k+j.$$

Remark 22. The following simple properties of $F_{(k,k+N]}(x)$, $r_N(x)$, $R^{(1)}_{(k,k+N]}(x)$ and $R^{(2)}_{(k,k+N}(x)$ hold.

(i) If for $x \in (0,1)$ one has $a_1(x) > k+N$ then, by (19), $F_{k,k+N}(x) = F_{(0,N]}(x) = 0$. Such $a_1(x)$ can be found by (v) in Remark 16.

(ii) A.S. Fajnlejb [10] (see also [15, p. 353 or English trans. p. 258]) proved that $F_{(0,N]}(x) = g_0(x) + \mathcal{O}\left(\frac{1}{\log\log N}\right)$ uniformly in $x \in [0,1]$. Hence, by applying (28) we see that $r_N(x) = \mathcal{O}\left(\frac{1}{\log\log N}\right)$.

(iii) The expression (23) implies, for any subsequences of integers k and N,

$$\lim_{k,N\to\infty} F_{(k,k+N]}(x) = g_0(x) \text{ if and only if } \lim_{k,N\to\infty} R^{(2)}_{(k,k+N]}(x) = 0$$

for every $x \in (0,1)$ and so, by Erdős' Theorem 9,

$$\lim_{k,N\to\infty} \frac{\log\log\log k}{N} = 0 \text{ implies } \lim_{k,N\to\infty} R^{(2)}_{(k,k+N]}(x) = 0.$$

(iv) If $\prod_{p\leq N} p \,|\, k$ (p are primes) then by Theorem 21, $R^{(1)}_{(k,k+N]}(x) = F_{(0,N]}(x)$.

(v) If $\prod_{p\leq N} p \,|\, k'$, then $R^{(1)}_{(k+k',k+k'+N}(x) - F_{(0,N]}(x) = R^{(1)}_{(k,k+N}(x) - F_{(0,N]}(x)$ because in Equation (27) the fractional parts verify, in the first sum under the main summation,

$$\left\{\frac{k+k'}{a_m(x)}\right\} + \left\{\frac{N}{a_m(x)}\right\} = \left\{\frac{k}{a_m(x)}\right\} + \left\{\frac{N}{a_m(x)}\right\}$$

and similarly for the other terms.

(vi) Define $K_N^*(x) := \#\{m \in \mathbb{N}; a_m(x) \le N\}$ and assume that k verifies $\prod_{p\le N} p \mid k$. Since for $a_m(x) \le N$ we have $a_m(x)|k + a_m(x)$, then the inequality

$$R^{(2)}_{(k,k+N]}(x) \le 1 - \frac{K_N^*(x)}{N}$$

implies $g^{(2)}(x) \le 1 - \underline{d}(x)$, where $\underline{d}(x)$ is the lower asymptotic density of x-numbers and we know that $R^{(2)}_{(k,k+N]}(x)$ converges to $g^{(2)}(x)$.

6. Using the Schinzel–Wang Theorem

A. Schinzel and Y. Wang [18] proved that for every fixed integer N the $(N-1)$-dimensional sequence

$$\left(\frac{\varphi(k+2)}{\varphi(k+1)}, \frac{\varphi(k+3)}{\varphi(k+2)}, \dots, \frac{\varphi(k+N)}{\varphi(k+N-1)}\right), \quad k = 1, 2, 3, \dots \tag{30}$$

is dense in $[0,\infty)^{N-1}$. Thus, for any given N-tuple $(\alpha_1, \alpha_2, \dots, \alpha_{N-1})$ in $[0,\infty)^{N-1}$ we can select an increasing sequence of integers k_m, such that the sequence of N-tuples

$$\left(\frac{\varphi(k_m+2)}{\varphi(k_m+1)}, \frac{\varphi(k_m+3)}{\varphi(k_m+2)}, \dots, \frac{\varphi(k_m+N)}{\varphi(k_m+N-1)}\right)$$

converges to $(\alpha_1, \alpha_2, \dots, \alpha_{N-1})$. Using the factorization

$$\frac{\varphi(k+n)}{k+n} = \frac{\varphi(k+n)}{\varphi(k+n-1)} \frac{\varphi(k+n-1)}{\varphi(k+n-2)} \cdots \frac{\varphi(k+2)}{\varphi(k+1)} \frac{\varphi(k+1)}{k+1} \frac{k+1}{k+n}$$

we can chose integers k_m such that the sequence of ratios $\frac{\varphi(k_m+1)}{k_m+1}$ converges, say to α, hence

$$\lim_{m\to\infty}\left(\frac{\varphi(k_m+1)}{k_m+1}, \frac{\varphi(k_m+2)}{k_m+2}, \dots, \frac{\varphi(k_m+N)}{k_m+N}\right)$$
$$= (\alpha, \alpha\alpha_1, \alpha\alpha_1\alpha_2, \dots, \alpha\alpha_1\alpha_2\dots\alpha_{N-1}).$$

In the following we apply the above fact but for an infinite sequence α_n, $n = 1, 2, 3 \dots$.

Theorem 23. *Let $\tilde{g}(x)$ be an arbitrary distribution function. There exists $\alpha \in (0,1]$ and a sequence of intervals $(k_m, k_m + N_m]$ such that the sequence of distribution functions $F_{(k_m,k_m+N_m]}(x)$ converges to a distribution function $g(x)$ such that for a.e. $x \in [0,1)$ one has*

$$g(x) = \begin{cases} \tilde{g}\left(\frac{x}{\alpha}\right) & \text{if } x \in [0,\alpha), \\ 1 & \text{if } x \in [\alpha,1]. \end{cases} \tag{31}$$

Proof. For an arbitrary distribution function $\tilde{g}(x)$ there exists a sequence α_n, $n = 1,2,\dots$ in $(0,\infty)$ such that for every $n = 1,2,3,\dots$ one has $\alpha_1\alpha_2\dots\alpha_n \in (0,1)$ and the sequence

$$\alpha_1\alpha_2\dots\alpha_n, \quad n = 1,2,\dots$$

has asymptotic distribution function $\tilde{g}(x)$. Now, using density of (30), for an arbitrary sequence $\varepsilon(N)$ with $\varepsilon(N) > 0$ and $\varepsilon(N)$ converging to 0, there exist integers $k = k(N)$ such that

$$\left|\frac{\varphi(k+2)}{\varphi(k+1)}\frac{\varphi(k+3)}{\varphi(k+2)}\cdots\frac{\varphi(k+n)}{\varphi(k+n-1)} - \alpha_1\alpha_2\dots\alpha_{n-1}\right| < \varepsilon(N) \tag{32}$$

for every $n = 2,\dots,N$ and

$$\left|\frac{k+1}{k+N} - 1\right| < \varepsilon(N). \tag{33}$$

From the sequence of pairs $(k(N),N)$, $N = 1,2,3,\dots$, select a subsequence (k',N'), $k' = k(N')$, such that

$$\frac{\varphi(k'+1)}{k'+1} \to \alpha \text{ as } N' \to \infty, \tag{34}$$

for some α in $(0,1]$. Then, from (32), (33) and (34) there exists a sequence of positive real numbers $\varepsilon'(N')$ that tends to 0 as N' go to infinity along a subsequence of integers such that

$$\left|\frac{\varphi(k'+n)}{k'+n} - \alpha\alpha_1\dots\alpha_{n-1}\right| < \varepsilon'(N') \tag{35}$$

for $n = 1,\dots,N'$.

Now we use the following fact: let x_n and y_n in $[0,1)$ for $n = 1,2,\dots,N$ and define on $[0,1]$ the step distribution functions

$$F_N^{(1)}(x) := \frac{1}{N}\sum_{n=1}^{N} c_{[0,x)}(x_n),\ F_N^{(2)}(x) := \frac{1}{N}\sum_{n=1}^{N} c_{[0,x)}(y_n).$$

By the triangular inequality,

$$\int_0^1 \left|F_N^{(1)}(x) - F_N^{(2)}(x)\right| \mathrm{d}x \leq \frac{1}{N}\sum_{n=1}^{N}\int_0^1 |c_{[0,x)}(x_n) - c_{[0,x)}(y_n)|\mathrm{d}x = \frac{1}{N}\sum_{n=1}^{N}|x_n - y_n|. \tag{36}$$

Choose

$$x_n = \frac{\varphi(k'+n)}{k'+n} \quad \text{and} \quad y_n = \alpha\alpha_1\ldots\alpha_{n-1}$$

for $n = 1, \ldots, N'$. By construction of y_n, the sequence of distribution functions $F_{N'}^{(2)}(x)$ converges to $\tilde{g}\left(\frac{x}{\alpha}\right)$ and from (35) and (36) the distribution function $F_{N'}^{(1)}(x)$, that is to say $F_{(k',k'+N']}(x)$, converges along a subsequence of integers N' to $g(x)$ almost everywhere and so, $g(x)$ satisfies (31). □

Remark 24. The value of α in (34) cannot be arbitrary. Applying (24) we see that

$$\int_0^\alpha x^s \mathrm{d}\tilde{g}\left(\frac{x}{\alpha}\right) = \alpha^s \int_0^1 x^s \mathrm{d}\tilde{g}(x) \leq \int_0^1 x^s \mathrm{d}g_0(x),$$

for every positive integer s. Recall that for $s = 1$ we have the classical result

$$\int_0^1 x \mathrm{d}g_0(x) = \frac{6}{\pi^2}$$

and more generally, we have (2). Consequently, with the distribution function $\tilde{g}(x) = x^2$ on $[0, 1]$ and $s = 1$ we obtain $\alpha \leq \frac{9}{\pi^2}$ and the case where $\tilde{g}(x)$ is the step function with jump 1 at $x = 1$ gives the inequality $\alpha \leq \frac{6}{\pi^2}$. It is worth comparing with Theorem 19 in which α is arbitrary but $g(x)$ is a special distribution function.

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References

[1] H. Davenport, Uber numeri abundantes, *Sitzungsber. Preuss. Akad. Wiss., Phys.-Math. Kl.* **26/29** (1933), 830–837.

[2] H. Delange, Sur les fonctions arithmétiques multiplicatives, *Ann. Sci. Éc. Norm. Sup., 3e série* **78** (1961), 273–304.

[3] M. Drmota and R. F. Tichy, *Sequences, Discrepancies and Applications*, Lecture Notes in Mathematics **1651**, Springer-Verlag, 1997.

[4] P.D.T.A. Elliot, *Probabilistic number theory I: Mean-Value theorems*, Grundlehren der Math. Wissen. **239**, Springer-Verlag, 1979.

[5] P. Erdős, On primitive abundant numbers, *J. London Math. Soc.* **9** (1935), 49–58.

[6] P. Erdős, On the smoothness of the asymptotic distribution of additive arithmetical functions, *Amer. J. Math.* **61** (1939), 722–725.

[7] P. Erdős, Some remarks about additive and multiplicative functions, *Bull. Amer. Mat. Soc.* **52** (1946), 527–537.

[8] P. Erdős, Remarks on number theory I. On primitive α-abundant numbers, *Acta Arith.* **5** (1958), 25–33.

[9] P. Erdős, On the distribution of numbers of the form $\sigma(n)/n$ and some related questions, *Pacific J. Math.* **52** (1974), 59–65.

[10] A.S. Fajnlejb, On the distribution of values of Euler's function (in russian), *Mat. zametki* **1**(6) (1967), 645–652.

[11] J. Galambos and I. Kátai, The gaps in a particular sequence of integers of positive density, *J. London Math. Soc. (2)* **36** (1987), 429–435.

[12] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, (Pure & Applied Math. Monograph) John Wiley & Sons Inc, 1974. Reprint: Dover Publications, Inc. Mineola, New York, 2006.

[13] E. Landau, *The Handbuch der Lehre von der Verteilung der Primzahlen*, Band 1. Verlag von B. G. Teubner, Leipzig, 1909.

[14] D.S. Mitrinovic, J. Sándor, and B. Crstici, *Handbook of Number Theory*, volume 351 of *Mathematics and its Applications*, Kluver Academic Publisher, Dordrecht, 1996.

[15] A.G. Postnikov, *Introduction to Analytic Number Theory* (in russian), Izd. Nauka, Moscow, 1971. English translation by G. A. Kandall, Translation of Mathematical Monographs Vol. 68, Amer. Math. Soc., Providence, RI, 1988.

[16] S. Ramanujan, *Collected Papers of Srinivasa Ramanujan*, Cambridge Univ. Press, 1927.

[17] J.B. Rosser and L. Schoenfeld, Appropriate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962), 64–94.

[18] A. Schinzel and Y. Wang, A note on some properties of the functions $\phi(n)$, $\sigma(n)$ and $\theta(n)$, *Ann. Polon. Math.* **4** (1958), 201–213.

[19] I.J. Schoenberg, Über die asymptotische Verteilung reeller Zahlen mod 1, *Math. Zeit.* **28** (1928), 171–199.

[20] I.J. Schoenberg, On asymptotic distribution of arithmetical functions, *Trans. Amer. Math. Soc.* **39** (1936), 315–330.

[21] O. Strauch and Š. Porubský, *Distribution of Sequences: A Sampler*, Peter Lang, Frankfurt am Main, 2005.

[22] B.A. Venkov, On one monotone function (in russian), *Učen. zap. Leningr. gos. un-ta. ser. matem.* **16** (1949), 3–19.