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ON RELATIVELY PRIME SETS COUNTING FUNCTIONS

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Abstract

This work is motivated by Nathanson's recent paper on relatively prime sets and a phi function for subsets of $\{1, 2, 3, \ldots, n\}$. We establish enumeration formulas for the number of relatively prime subsets and the number of relatively prime subsets of cardinality k of $\{1, 2, 3, \ldots, n\}$ under various constraints. Further, we show how this work links up with the study of multicompositions.

1. Background

Our paper is motivated by a recent paper of Nathanson [8] who defined a nonempty subset A of $\{1, 2, \ldots, n\}$ to be relatively prime if $\gcd(A) = 1$. He defined $f(n)$ to be the number of relatively prime subsets of $\{1, 2, \ldots, n\}$ and, for $k \geq 1$, $f_k(n)$ to be the number of relatively prime subsets of $\{1, 2, \ldots, n\}$ of cardinality k. Further, he defined $\Phi(n)$ to be the number of nonempty subsets A of the set $\{1, 2, \ldots, n\}$ such that $\gcd(A)$ is relatively prime to n and, for integer $k \geq 1$, $\Phi_k(n)$ to be the number of subsets A of the set $\{1, 2, \ldots, n\}$ such that $\gcd(A)$ is relatively prime to n and $\text{card}(A) = k$. He obtained explicit formulas for these functions and deduced asymptotic estimates. These functions were subsequently generalized by El Bachraoui [5] to subsets $A \in \{m+1, m+2, \ldots, n\}$ where m is any nonnegative integer, and then by Ayad and Kihel [3] to subsets of the set $\{a, a+b, \ldots, a+(n-1)b\}$ where a and b are any integers.

El Bachraoui [4] defined for any given positive integers $l \leq m \leq n$, $\Phi([l, m], n)$ to be the number of nonempty subsets of $\{l, l+1, \ldots, m\}$ which are relatively prime to n and $\Phi_k(l, m], n)$ to be the number of such subsets of cardinality k. He found formulas for these functions when $l = 1$ [4].

2. Introduction

It turns out that some of Nathanson's results are special cases of number theoretic functions investigated by Shonhiwa. In [10], Shonhiwa defined and investigated the following functions and established the following result.

Theorem 1 *Let*

$$S_k^m(n) = \sum_{\substack{1\le a_1,a_2,\ldots,a_k\le n\\(a_1,a_2,\ldots,a_k,m)=1}} 1;\ \forall\, n\ge k\ge 1,\ m\ge 1 \tag{1}$$

$$G_k(n) = \sum_{\substack{1\le a_1,a_2,\ldots,a_k\le n\\(a_1,a_2,\ldots,a_k)=1}} 1;\ \forall\, n\ge k\ge 1, \tag{2}$$

$$L_k^m(n) = \sum_{\substack{1\le a_1\le a_2\le\cdots\le a_k\le n\\(1_1,a_2,\ldots,a_k,m)=1}} 1;\ \forall\, n\ge k\ge 1,\ m\ge 1 \tag{3}$$

and

$$T_k^m(n) = \sum_{\substack{1\le a_1< a_2<\cdots< a_k\le n\\(a_1,a_2,\ldots,a_k,m)=1}} 1;\ \forall\, n\ge k\ge 1,\ m\ge 1. \tag{4}$$

Then

$$S_k^m(n) = \sum_{d|m}\mu(d)\left\lfloor\frac{n}{d}\right\rfloor^k,$$

$$L_k^m(n) = \sum_{d|m}\mu(d)L_k^1\left(\left\lfloor\frac{n}{d}\right\rfloor\right) = \sum_{d|m}\mu(d)\binom{\lfloor\frac{n}{d}\rfloor+k-1}{k},$$

and

$$T_k^m(n) = \sum_{d|m}\mu(d)T_k^1\left(\left\lfloor\frac{n}{d}\right\rfloor\right) = \sum_{d|m}\mu(d)\binom{\lfloor\frac{n}{d}\rfloor}{k}.$$

From above, it follows that

$$\Phi_k(n) = T_k^n = \sum_{d|m}\mu\left(\frac{n}{d}\right)\binom{d}{k} \tag{5}$$

and

$$\Phi(n) = \sum_{k=1}^{n} T_k^n(n) = \sum_{d|m} \mu(d) 2^{\frac{n}{d}}, \tag{6}$$

as shown therein and as proved in [8].

3. Main Results

The result obtained concerning the function $G_k(n)$ in [10] is incorrect and we provide the correction below. The corrected result makes use of the following theorem [1].

Theorem 2 (Generalized Möbius inversion formula) *If α is completely multiplicative we have*

$$G(x) = \sum_{n \leq x} \alpha(n) F\left(\frac{x}{n}\right) \Longleftrightarrow F(x) = \sum_{n \leq x} \mu(n)\alpha(n) G\left(\frac{x}{n}\right).$$

We may now prove our first result as follows.

Theorem 3 *We have*

$$G_k(n) = \sum_{j \leq n} \mu(j) \left\lfloor \frac{n}{j} \right\rfloor^k.$$

Proof. Since

$$\begin{aligned} G_k(n) = n^k - \sum_{j=2} \sum_{\substack{1 \leq a_1, a_2, \ldots, a_k \leq n \\ (a_1, a_2, \ldots, a_k) = j}} 1 &= n^k - \sum_{j=2}^{n} \sum_{\substack{1 \leq b_1, b_2, \ldots, b_k \leq \lfloor \frac{n}{j} \rfloor \\ (b_1, b_2, \ldots, b_k) = 1}} 1 \\ &= n^k - \sum_{j=2}^{n} G_k\left(\left\lfloor \frac{n}{j} \right\rfloor\right), \end{aligned}$$

we have

$$\sum_{j=1}^{n} G_k\left(\left\lfloor \frac{n}{d} \right\rfloor\right) = n^k.$$

Hence, by Theorem 2, it follows that

$$G(n) = \sum_{k=1}^{n} G_k(n) = \sum_{k=1}^{n}\sum_{j=1}^{n} \mu(j) \left\lfloor \frac{n}{j} \right\rfloor^k = \sum_{j=1}^{n} \mu(j) \sum_{k=1}^{j} \left\lfloor \frac{n}{j} \right\rfloor^k$$
$$= \sum_{j=1}^{n} \frac{\mu(j) \left\lfloor \frac{n}{j} \right\rfloor \left(1 - \left\lfloor \frac{n}{j} \right\rfloor^j\right)}{\left(1 - \left\lfloor \frac{n}{j} \right\rfloor\right)}. \qquad \square$$

Using our definition, results from [10], Nathanson's notation and arguing as above, it also follows that

$$f_k(n) = \sum_{\substack{1 \le a_1 < a_2 < \cdots < a_k \le n \\ (a_1, a_2, \ldots, a_k) = 1}} 1 = \binom{n}{k} - \sum_{j=2}^{n} f_k\left(\left\lfloor \frac{n}{j} \right\rfloor\right),$$

which gives

$$\sum_{j=1}^{n} f_k\left(\left\lfloor \frac{n}{j} \right\rfloor\right) = \binom{n}{k}.$$

Thus,

$$f_k(n) = \sum_{j=1}^{n} \mu(j) \binom{\left\lfloor \frac{n}{j} \right\rfloor}{k}.$$

From this it follows that

$$f(n) = \sum_{k=1}^{n} f_k(n) = \sum_{k=1}^{n}\sum_{j=1}^{n} \mu(j) \binom{\left\lfloor \frac{n}{j} \right\rfloor}{k}$$
$$= \sum_{j=1}^{n} \mu(j) \sum_{k=1}^{j} \binom{\left\lfloor \frac{n}{j} \right\rfloor}{k} = \sum_{j=1}^{n} \mu(j) \left(2^{\left\lfloor \frac{n}{j} \right\rfloor} - 1\right).$$

We now prove our next theorem.

Theorem 4 *Let*

$$H_k(n) = \sum_{\substack{1 \le a_1 \le a_2 \le \cdots \le a_k \le n \\ (a_1, a_2, \ldots, a_k) = 1}} 1.$$

Then

$$H_k(n) = \sum_{j \le n} \mu(j) \binom{\left\lfloor \frac{n}{j} \right\rfloor + k - 1}{k}.$$

Proof. Arguing as above it follows that

$$H_k(n) = \binom{n+k-1}{k} - \sum_{j=2}^{n} H_k\left(\left\lfloor \frac{n}{j} \right\rfloor\right)$$

which implies

$$\sum_{j=1}^{n} H_k\left(\left\lfloor \frac{n}{j} \right\rfloor\right) = \binom{n+k-1}{k},$$

and hence, by Theorem 2, we obtain the result

$$\begin{aligned} H(n) = \sum_{k=1}^{n} H_k(n) &= \sum_{j=1}^{n} \mu(j) \sum_{k=1}^{j} \binom{\left\lfloor \frac{n}{j} \right\rfloor + k - 1}{k} \\ &= \sum_{j=1}^{n} \mu(j) \binom{\left\lfloor \frac{n}{j} \right\rfloor + j}{j} - \sum_{j=1}^{n} \mu(j). \end{aligned}$$

□

We now define the corresponding totient function as

$$\Psi_k(n) = L_k^n(n) = \sum_{d|n} \mu\binom{n}{d}\binom{d+k-1}{k}.$$

Then

$$\Psi(n) = \sum_{k=1}^{n} \Psi_k(n) = \sum_{k=1}^{n} L_k^n(n) = \sum_{d|n} \binom{n}{d} \sum_{k=1}^{n} \binom{d+j-1}{j} = \sum_{d|n} \binom{n}{d}\binom{d+n}{n},$$

or equivalently,

$$\binom{2n}{n} = \sum_{d|n} \Psi(d) \Longleftrightarrow \sum_{n=1}^{\infty} \binom{2n}{n} x^n = \sum_{n=1}^{\infty} \Psi(n) \frac{x^n}{1-x^n} = \frac{1}{\sqrt{1-4x}} - 1.$$

It turns out the function $T_k^m(n)$ relates to other functions connected with the study of compositions of n into relatively prime summands as follows.

Gould [6] investigated the function

$$R_k(n) = \sum_{\substack{1\le a_1+a_2+\cdots+a_k=n\\(a_1,a_2,\ldots,a_k)=1}} 1 = \sum_{d|n} C_k(d)\mu\binom{n}{d} = \sum_{d|n}\mu\left(\frac{n}{d}\right)\binom{d-1}{k-1},$$

where $C_k(n)=\binom{n-1}{k-1}$ and obtained many other significant results concerning this function. Consequently,

$$\begin{aligned} T_k^n(n) &= \sum_{\substack{1\le a_1<a_2<\cdots<a_k\le n\\(a_1,a_2,\ldots,a_k,n)=1}} 1 \\ &= \sum_{d|n}\mu\left(\frac{n}{d}\right)\binom{d}{k} \\ &= \sum_{d|n}\mu\left(\frac{n}{d}\right)\left\{\binom{d-1}{k-1}+\binom{d-1}{k}\right\} \\ &= R_k(n)+R_{k+1}(n). \end{aligned}$$

Therefore, we may obtain results concerning either function by using known properties of the other. In particular, we may obtain the Lambert series for $T_k^n(n)$ as follows:

$$\begin{aligned} \sum_{n=1}^{\infty} T_k^n \frac{x^n}{1-x^n} &= \sum_{n=1}^{\infty} R_k^n \frac{x^n}{1-x^n} + \sum_{n=1}^{\infty} R_{k+1}^n \frac{x^n}{1-x^n} \\ &= \frac{x^k}{(1-x)^{k+1}} \\ &= \sum_{n=0}^{\infty} x^n \sum_{n=0}^{\infty} C_k(n)x^n \\ &= \sum_{n=1}^{\infty} x^n \sum_{j=0}^{n} C_k(n-j), \end{aligned}$$

which is equivalent to

$$\sum_{d|n} T_k^d(d) = \sum_{j=0}^{n} C_k(n-j) = \binom{n}{k},$$

as expected.

The inverse function of $R_k(n)$ is

$$A_k(n) = \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} \left\lfloor \frac{j}{k} \right\rfloor \tag{7}$$

and it is shown in [6] that these two satisfy the orthogonality relations.

Theorem 5 *We have*

$$\sum_{j=k}^{n} R_k(j) A_j(n) = \delta_k^n$$

and

$$\sum_{j=k}^{n} R_j(n) A_k(j) = \delta_k^n.$$

In [10], it is shown that the inverse of $T_k^n(n)$ is

$$K_k(n) = \sum_{j=k}^{n} (-1)^{n-j} \binom{n+1}{j+1} \left\lfloor \frac{j}{k} \right\rfloor,$$

and that:

Theorem 6 *We have*

$$\sum_{j=k}^{n} T_k^j(j) K_j(n) = \delta_k^n \text{ and } \sum_{j=k}^{n} K_k(j) T_j^j(n) = \delta_k^n.$$

It follows that

$$\begin{aligned} K_k(n) &= \sum_{j=k}^{n} (-1)^{n-j} \left\{ \binom{n}{j+1} + \binom{n}{j} \right\} \\ &= \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j+1} + A_k(n), \end{aligned}$$

so that

$$A_k(n) = K_k(n) + K_k(n-1). \tag{8}$$

Whilst we have a closed form expression for $A_k(n)$ it does not reveal enough regarding the structure of $A_k(n)$. Our next result responds to this concern for a special case of $A_k(n)$.

Theorem 7 *For $n \geq 1$, we have*

$$A_k(k+n) = (-1)^n \binom{n+k-1}{k-1}; \ \forall\, k \geq n+1.$$

Proof. From $T_k^n(n) = R_k(n) + R_{k+1}(n)$ and Theorem 3.4 above, it follows that

$$\sum_{j=k}^{n} A_j(n) T_k^j(j) = \delta_k^n + \delta_{k+1}^n. \tag{9}$$

So that for $n = k$,

$$A_k(k)T_k^k(k) = 1 \quad \text{implies} \quad A_k(k) = T_k^k = 1; \ \forall\ k \geq 1.$$

And for $n = k+1$,

$$A_k(k+1)T_k^k(k) + A_{k+1}(k+1)T_{k+1}^{k+1}(k+1) = 1 \Longrightarrow A_k(k+1) = -k; \ \forall\, k \geq 1.$$

For $n \geq k+2$ we may rewrite equation (9) above as

$$A_k(n) + T_k^n(n) = -\sum_{j=k+1}^{n-1} A_j(n) T_k^j(j),$$

then for $n = k+2$,

$$\begin{aligned} A_k(k+2) + T_k^{k+2}(k+2) &= -A_{k+1}(k+2)T_k^{k+1}(k+1) \\ &= (k+1)T_k^{k+1}(k+1) \\ &= (k+1)^2, \end{aligned}$$

since $A_k(k+1) = -k$. Hence

$$\begin{aligned} A_k(k+2) &= (k+1)^2 - \sum_{d|k+2} \mu(d) \binom{\frac{k+2}{d}}{k} \\ &= (k+1)^2 - \frac{(k+1)(k+2)}{2} \\ &= \frac{(-1)^2 k(k+1)}{2} \quad \text{provided } k \geq 3. \end{aligned}$$

Now assume the result holds for $k+1, k+2, \ldots, k+n-1$ and consider

$$\begin{aligned} A_k(k+n) + T_k^{k+n}(k+n) &= -\sum_{j=k+1}^{n+k-1} A_j(n+k)T_k^j(j) \\ &= \sum_{i=1}^{n-1} \frac{(-1)^{n-i}T_k^{k+i}(k+i)\prod_{j=i}^{n-1}(k+j)}{(n-i)!} \\ &= \prod_{j=1}^{n-1}(k+j)\left\{\sum_{i=1}^{n-1}\frac{(-1)^{n+1-i}(k+i)}{(n-i)!i!}\right\}. \end{aligned}$$

Then

$$\begin{aligned} A_k(k+n) &= \frac{\prod_{j=1}^{n-1}(k+j)}{n!}\left\{\sum_{j=1}^{n}(-1)^{n-j-1}\binom{n}{j}(k+j)\right\} \\ &= (-1)^n\binom{n+k-1}{k-1}, \end{aligned}$$

where we have used the inductive hypothesis as well as assumed that $k \geq n+1$. □

We note in passing that

$$R_k(k+n) = (-1)^n A_k(k+n) \text{ for all } k \geq n+1.$$

Following up on Gould's paper, Andrews [2] introduced the function $g_m(n)$, which gives the number of m-compositions of n with relatively prime positive summands so that

$$T(n) = \sum_{k=1}^{n} R_k(n) = g_1(n).$$

It is shown in [2] that the total number of m-compositions of n is $(m+1)^{n-1}$ and hence,

$$(m+1)^{n-1} = \sum_{d|n} g_m(d).$$

In a follow-up paper, Shonhiwa [11] provided an alternative investigation of the function $g_m(n)$.

From the equation

$$T_j^n = \sum_{d|n} \mu\left(\frac{n}{d}\right)\binom{d}{j}$$

it follows that

$$\sum_{j=1}^{n} T_j^n(n)x^j = \sum_{d|n} \left(\frac{n}{d}\right)(x+1)^d \text{ for all } \ n \geq 2.$$

Therefore

$$\begin{aligned} g_m(n) &= \sum_{d|n} \mu \binom{n}{d} (m+1)^{d-1} \\ &= (m+1)^{-1} \sum_{d|n} \mu \binom{n}{d} (m+1)^d \\ &= (m+1)^{-1} \sum_{j=1}^{n} T_j^n(n) M^j \text{ for all } n \geq 1. \end{aligned}$$

In particular, for $m = 1$, we obtain

$$\begin{aligned} g_1(n) = T(n) = \sum_{d|n} \mu \binom{n}{d} 2^{d-1} &= \sum_{d|n} \mu \binom{n}{d} (3-1)^{d-1} \\ &= \sum_{d|n} \mu \binom{n}{d} \sum_{j=0}^{d-1} \binom{d-1}{j} 3^j (-1)^{d-1-j} \\ &\equiv 0 \text{ (mod 3) for all } n \geq 3 \end{aligned}$$

(see [7]).

Further, from

$$g_m(i) = (m+1)^{-1} \sum_{j=1}^{i} T_j^i(i) m^j,$$

it follows that

$$K_i(n) g_m(i) = (m+1)^{-1} \sum_{j=1}^{i} T_j^i(i) K_i(n) m^j;$$

which implies that

$$\begin{aligned}\sum_{i=1}^{n} K_i(n) g_m(i) &= (m+1)^{-1} \sum_{i=1}^{n}\sum_{j=1}^{i} T_j^i(i) K_i(n) m^j \\ &= (m+1)^{-1} \sum_{j=1}^{n} m^j \delta_j^n \\ &= (m+1)^{-1} m^n,\end{aligned}$$

from above.

Hence

$$\begin{aligned}g_m(n) &= \frac{m^n}{m+1} - \sum_{i=1}^{n-1} K_i(n) g_m(i) \\ &= \frac{m^n}{m+1} + \sum_{i=1}^{n-1} K_i(n-1) g_m(i) - \sum_{i=1}^{n-1} A_i(n) g_m(i) \\ &= \frac{m^n}{m+1} + \frac{m^{n-1}}{m+1} - \sum_{i=1}^{n-1} A_i(n) g_m(i) \\ &= m^{n-1} - \sum_{i=1}^{n-1} A_i(n) g_m(i),\end{aligned}$$

as expected; see [11].

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> This publication is dedicated to the memory of Temba Shonhiwa's love of Mathematics and the development of a Mathematical Sciences Community in Africa. May his soul rest in peace. We would like to acknowledge and thank Dr. Augustine Munagi for finalizing and resubmitting this script.
>
> Many thanks,
>
> Fortune Shonhiwa

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