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ON THE KERNEL OF THE COPRIME GRAPH OF INTEGERS

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Abstract

Let (V, E) be the coprime graph with vertex set $V = \{1, 2, \ldots, n\}$ and edges $(i, j) \in E$ if $\gcd(i, j) = 1$. We determine the kernels of the coprime graph and its loopless counterpart as well as so-called simple bases for them (in case such bases exist), which means that basis vectors have entries only from $\{-1, 0, 1\}$. For the loopless version knowledge about the value distribution of Mertens' function is required.

1. Introduction

For each integer $n > 1$ the "traditional" coprime graph $TCG_n = (V, E)$ has the vertex set $V = \{1, 2, \ldots, n\}$ and edges $(i, j) \in E$ if and only if $\gcd(i, j) = 1$. Obviously, TCG_n has a loop at 1. Since one usually prefers loopless graphs, we also consider the slightly modified loopless coprime graph $LCG_n = (V', E')$ with $V' = V$ and $E' = E \setminus \{(1, 1)\}$. With regard to what we intend to prove LCG_n requires more involved techniques than TCG_n. For that reason we shall mainly deal with LCG_n and comment only in the final section on the corresponding results for TCG_n, which can be obtained by the same method with less effort.

The first problem concerning the coprime graph and its subgraphs was introduced by Erdős [8] in 1962. Meanwhile interesting features relating number theory and graph theory have been unearthed (for various "graphs on the integers" the reader is referred to [17], Chapter 20, 7.4):

- In 1984 Pomerance and Selfridge [18] proved Newman's coprime mapping conjecture: If $I_1 = \{1, 2, \ldots, n\}$ and I_2 is any interval of n consecutive integers, then there is a perfect coprime matching from I_1 to I_2. Note that the statement is not true if I_1 is also an arbitrary interval of n consecutive integers. Example: $I_1 = \{2, 3, 4\}$ and $I_2 = \{8, 9, 10\}$; any one-to-one correspondence between I_1 and I_2 must have at least one pair of even numbers in the correspondence.
- In a series of papers between 1994 and 1996 Ahlswede and Khachatrian (cf. [2], [3], [4]) and very recently Ahlswede and Blinovsky [1] proved results on extremal sets without coprime elements, extremal sets without $k+1$ pairwise coprime elements, and sets of integers with pairwise common divisors. Two edges in the coprime graph are not coprime if they are connected in the com-

plementary graph $\overline{TCG_n}$ of TCG_n. Therefore one has to search for maximal complete subgraphs in $\overline{TCG_n}$.

- In 1996 Erdős and G.N. Sárközy [9] gave lower bounds for the maximal length of cycles in the coprime graph. Three years later Sárközy [23] studied complete tripartite subgraphs in TCG_n.

Let $A_n = (a_{i,j})_{n\times n}$ be the adjacency matrix of LCG_n, i.e.,

$$a_{i,j} = \begin{cases} 0 & \text{if } \gcd(i,j) > 1 \text{ or } i = j = 1, \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

Apparently LCG_n is an undirected loopless graph, and A_n is symmetric.

For several decades spectra and eigenspaces of graphs (cf. [11]), that is, spectra and eigenspaces of their adjacency matrices, have been studied for quite a few different types of graphs (for references see [6], [7] or [10]). For reasons like characterization of graphs or computational advantages it is of particular interest to find so-called *simple* bases (all entries are $-1, 0, 1$) for eigenspaces, especially for the kernel of a graph. Such bases can be found for trees and forests (see [20], [5]), unicyclic graphs [21] and powers of circuit graphs [22].

Computational experiments provided evidence for the following observations:

- The dimensions of the kernels of the coprime graphs TCG_n and LCG_n, respectively, are growing with n.

- These kernels always have a simple basis in the above sense.

It is the purpose of this work to clarify the observations made. In fact, we shall prove precise formulae for $\dim \operatorname{Ker} TCG_n$ and $\dim \operatorname{Ker} LCG_n$ and construct an explicit simple basis for each of them – if one exists. In order to determine those kernels which have no simple basis, results about the Mertens function

$$M(n) := \sum_{k=1}^{n} \mu(k)$$

will be involved, where $\mu(n)$ denotes Möbius' function.

2. Basic Facts

We denote by $\kappa(m) = \prod_{p\in\mathbb{P},\ p|m} p$ the squarefree kernel of a positive integer m. For each squarefree integer $k > 1$ the vector $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ is called k-basic if for some m, $k < m \le n$, satisfying $\kappa(m) = k$,

$$b_j = \begin{cases} 1 & \text{for } j = k, \\ -1 & \text{for } j = m, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbf{b} \in \mathbb{R}^n$ is k-basic for some squarefree k, we call it a basic vector. The set of all basic vectors $\mathbf{b} \in \mathbb{R}^n$ will be denoted by $\mathcal{B}_n$.

Lemma 1 *The number* $\nu(n) := |\mathcal{B}_n|$ *of basic vectors satisfies*

$$\nu(n) = n - \sum_{k \le n} |\mu(k)|.$$

Proof. Associate with each non-squarefree positive integer $m \le n$ the basic vector $\mathbf{b} \in \mathbb{R}^n$ defined by

$$b_j = \begin{cases} 1 & \text{for } j = \kappa(m), \\ -1 & \text{for } j = m, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

This correspondence is apparently one-to-one. Now $\nu(n)$ precisely counts the non-squarefree positive integers $m \le n$. □

Proposition 2 *Let* $n > 1$ *be an arbitrary integer.*

(i) If $\mathbf{b} \in \mathcal{B}_n$ *then* $\mathbf{b} \in \mathrm{Ker} A_n$.

(ii) $\mathcal{B}_n$ *is linearly independent over* $\mathbb{R}$.

Proof. (i) Let $\mathbf{b} \in \mathcal{B}_n$, i.e., $\mathbf{b}$ is k-basic for some squarefree $k > 1$. Hence $\mathbf{b}$ has entries 0 apart from $b_k = 1$ and $b_m = -1$ for some m satisfying $k < m \le n$ and $\kappa(m) = k$. Now let $\mathbf{a}_i$ be the i-th row vector of A_n. Then we have for the scalar product

$$\mathbf{a}_i \cdot \mathbf{b} = a_{i,k} - a_{i,m} = 0,$$

because k and m have the same prime factors and therefore $\gcd(i, k)$ and $\gcd(i, m)$ are both 1 or both greater than 1. This means that $\mathbf{b}$ belongs to $\mathrm{Ker} A_n$.

(ii) Let $m \le n$ be a non-squarefree positive integer. Then there is precisely one basic vector $\mathbf{b} \in \mathcal{B}_n$ satisfying $b_m = -1$, namely the vector defined in (2). All other vectors $\mathbf{b}' \in \mathcal{B}_n$ have $b'_m = 0$. Thus, $\mathcal{B}_n$ is linearly independent. □

From Proposition 2 we obtain immediately

Corollary 3 *For any integer* $n > 1$ *we have* $\dim_{\mathbb{R}} \mathrm{Ker} A_n \ge \nu(n)$.

We shall prove in the sequel that in fact $\dim_{\mathbb{R}} \mathrm{Ker} A_n = \nu(n)$ for most n. This was suggested by numerical calculations. It turns out, however, that there are infinitely many exceptions.

3. Truncated Möbius Inversion and Mertens' Function

In the sequel we make use of the truncated version of the Möbius inversion formula (cf. [13, Chapter 6.4, Theorem 4.1]). Then an important role is played by Mertens' well-known function

$$M(n) := \sum_{k=1}^{n} \mu(k).$$

Trivially $|M(n)| \le n$ for all n. The relevance of this function becomes immediately clear from the facts that $M(n) = o(n)$ is equivalent to the prime number theorem and $M(n) = O(n^{\frac{1}{2}+\varepsilon})$ is equivalent to the Riemann hypothesis. The famous Mertens conjecture from 1897 saying that $|M(n)| < \sqrt{n}$ for all $x > 1$ was disproved by Odlyzko and te Riele [15] in 1985. For our purpose it is essential to know something about the value distribution of $M(n)$ (see Remark 6(ii)).

Proposition 4 *Let $n > 1$ be an arbitrary integer. A vector $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ lies in* $\mathrm{Ker} A_n$ *if and only if*

$$(M(n) - 1)\, b_1 = 0 \tag{3}$$

and for $2 \le k \le n,\ \mu(k) \ne 0$

$$\sum_{\substack{j=k \\ j \equiv 0 \bmod k}}^{n} b_j - b_1 = 0. \tag{4}$$

Proof. It is well-known that the summatory function $\varepsilon(n) = \sum_{d|n} \mu(d)$ of the Möbius function satisfies

$$\varepsilon(n) = \begin{cases} 1 & \text{for } n = 1, \\ 0 & \text{for } n > 1. \end{cases}$$

This implies

$$\varepsilon(\gcd(i,j)) = \sum_{\substack{d|i \\ d|j}} \mu(d) = \begin{cases} 1 & \text{for } \gcd(i,j) = 1, \\ 0 & \text{for } \gcd(i,j) > 1, \end{cases} \tag{5}$$

and therefore we have for the entries $a_{i,j}$ of the adjacency matrix of LCG_n (see (1))

$$a_{i,j} = \varepsilon(\gcd(i,j)) - \gamma_{ij} \tag{6}$$

for all $1 \le i, j \le n$, where γ_{ij} equals 1 for $i = j = 1$ and 0 otherwise.

For a given vector $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ let $f : \{1, 2, \ldots, n\} \to \mathbb{R}$ be defined by

$$f(k) := \mu(k) \sum_{\substack{j=1 \\ j \equiv 0 \bmod k}}^{n} b_j.$$

By (5) and (6) it follows that the summatory function g of f satisfies, for $1 \le i \le n$,

$$\begin{aligned} g(i) &:= \sum_{d|i} f(d) = \sum_{d|i} \mu(d) \sum_{\substack{j=1 \\ j \equiv 0 \bmod d}}^{n} b_j \\ &= \sum_{j=1}^{n} b_j \sum_{\substack{d|i \\ d|j}} \mu(d) = \sum_{j=1}^{n} b_j \varepsilon(\gcd(i,j)) \qquad (7) \\ &= \sum_{j=1}^{n} (a_{i,j} b_j + \gamma_{ij} b_j) = \sum_{j=1}^{n} a_{i,j} b_j + \gamma_{i1} b_1. \end{aligned}$$

A vector $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ lies in $\mathrm{Ker} A_n$ if and only if $\sum_{j=1}^{n} a_{i,j} b_j = 0$ for $1 \le i \le n$. By (7) this is equivalent to $g(i) = \gamma_{i1} b_1$ for $1 \le i \le n$. By the truncated version of the Möbius inversion formula (cf. [13], Chapt. 6.4, Theor. 4.1) this means that, for $1 \le k \le n$,

$$f(k) = \sum_{d|k} \mu(d) g\Big(\frac{k}{d}\Big) = \mu(k) g(1) = \mu(k) b_1,$$

and hence by the definition of f,

$$\mu(k) \sum_{\substack{j=1 \\ j \equiv 0 \bmod k}}^{n} b_j = \mu(k) b_1.$$

So far we have shown that $\mathbf{b} \in \mathrm{Ker} A_n$ if and only if

$$\sum_{\substack{j=1 \\ j \equiv 0 \bmod k}}^{n} b_j = b_1 \qquad (1 \le k \le n,\ \mu(k) \ne 0). \qquad (8)$$

We have

$$\begin{aligned} \sum_{\substack{k=2 \\ \mu(k) \ne 0}}^{n} \mu(k) \sum_{\substack{j=1 \\ j \equiv 0 \bmod k}}^{n} b_j &= \sum_{k=2}^{n} \mu(k) \sum_{\substack{j=2 \\ j \equiv 0 \bmod k}}^{n} b_j \\ &= \sum_{j=2}^{n} b_j \sum_{\substack{k=2 \\ k|j}}^{n} \mu(k) = \sum_{j=2}^{n} b_j \sum_{\substack{k=2 \\ k|j}}^{j} \mu(k) \qquad (9) \\ &= \sum_{j=2}^{n} b_j (\varepsilon(j) - 1) = -\sum_{j=2}^{n} b_j, \end{aligned}$$

and by adding the corresponding equations for $k = 2, \ldots, n$ with $\mu(k) \neq 0$ in (8) we obtain

$$-\sum_{j=2}^{n} b_j = \sum_{\substack{k=2 \\ \mu(k)\neq 0}}^{n} \mu(k) \sum_{\substack{j=1 \\ j\equiv 0 \bmod k}}^{n} b_j = \sum_{\substack{k=2 \\ \mu(k)\neq 0}}^{n} \mu(k) b_1 = b_1 \sum_{k=2}^{n} \mu(k) = b_1(M(n)-1).$$

The addition of this to the equation for $k = 1$ in (8) gives

$$b_1 = \sum_{j=1}^{n} b_j - \sum_{j=2}^{n} b_j = b_1 + b_1(M(n)-1),$$

and replacing the equation for $k = 1$ in (8) by this one does not change the set of solutions. This completes the proof of the proposition. □

4. Main Results

Theorem 5 *For any integer $n > 1$ we have*

$$\dim_{\mathbb{R}} \mathrm{Ker} LCG_n = \begin{cases} \nu(n) & \text{for } M(n) \neq 1, \\ \nu(n)+1 & \text{for } M(n) = 1, \end{cases} \tag{10}$$

where $\nu(n)$ is defined in Lemma 1. Consequently

$$\dim_{\mathbb{R}} \mathrm{Ker} LCG_n = \left(1 - \frac{6}{\pi^2}\right) n + O(\sqrt{n}). \tag{11}$$

Proof. By Proposition 4 a vector $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ lies in $\mathrm{Ker} LCG_n = \mathrm{Ker} A_n$ if and only if $\mathbf{b}$ satisfies the homogeneous system consisting of the linear equations (4) for $2 \leq k \leq n$, $\mu(k) \neq 0$, and, in addition, equation (3). Therefore we obtain

$$\left.\begin{array}{cccccccccc}
b_2 & +b_4 & +b_6 & +b_8 & & +b_{10} & \cdots & -b_1 & = 0 \\
& b_3 & +b_6 & & +b_9 & & \cdots & -b_1 & = 0 \\
& & b_5 & & & +b_{10} & \cdots & -b_1 & = 0 \\
& & & b_6 & & & \cdots & -b_1 & = 0 \\
& & & & b_7 & & \cdots & -b_1 & = 0 \\
& & & & & b_{10} & \cdots & -b_1 & = 0 \\
& & & & & & \ddots & \vdots & = \vdots \\
& & & & & & \ddots & \vdots & = \vdots \\
& & & & & & & (M(n)-1)b_1 & = 0
\end{array}\right\} . \tag{12}$$

Apparently (12) is a homogeneous system in row-echelon form with n variables. Hence the rank of the coefficient matrix B_n obviously satisfies

$$\text{rank } B_n = \begin{cases} \sum_{k=1}^{n} |\mu(n)| & \text{for } M(n) \neq 1, \\ \sum_{k=1}^{n} |\mu(n)| - 1 & \text{for } M(n) = 1. \end{cases}$$

Consequently $\dim_{\mathbb{R}} \mathrm{Ker} LCG_n = \dim_{\mathbb{R}} \mathrm{Ker} A_n = n - \text{rank } B_n$, and by Lemma 1 this proves (10).

It is well-known that

$$\sum_{k=1}^{n} |\mu(k)| = \frac{1}{\zeta(2)} n + O(\sqrt{n}) = \frac{6}{\pi^2} n + O(\sqrt{n})$$

(cf. [12], p. 270). Now Lemma 1 and (10) imply (11). □

Remarks 6

(i) The proof of Theorem 5 showed that

$$\mathbf{b} \in \mathrm{Ker} LCG_n \quad \Longleftrightarrow \quad B_n \tilde{\mathbf{b}} = \mathbf{0},$$

where B_n is the coefficient matrix of (12) and $\tilde{\mathbf{b}} := (b_2, b_3, \ldots, b_n, b_1)$.

(ii) Apparently $\dim_{\mathbb{R}} \mathrm{Ker} LCG_n$ depends on the value of $M(n)$, more precisely on whether $M(n) = 1$ or not. Results of Pintz and others (cf. [16]) show that $M(n)$ oscillates between $\pm\sqrt{n}$, and since $|M(n+1) - M(n)| \leq 1$, each value between these bounds is attained infinitely many times. In particular $|\{n : M(n) = 1\}| = \infty$. The smallest integers $n > 1$ with $M(n) = 1$ are $n = 94, 97, 98, 99, 100, 146, 147, 148, \ldots$.

Theorem 7 *If n is an integer satisfying $M(n) \neq 1$, we have the following:*

(i) *$\mathcal{B}_n$ is a basis of $\mathrm{Ker} LCG_n$.*

(ii) *$\mathrm{Ker} LCG_n$ has a simple basis, i.e., the components of all basis vectors are 0, 1 or −1.*

(iii) *Let $\iota : \mathbb{R}^n \to \mathbb{R}^{n+1}$ with $\iota(b_1, \ldots, b_n) := (b_1, \ldots, b_n, 0)$ be the canonical injection. Then we have $\iota(\mathrm{Ker} LCG_n) \subseteq \mathrm{Ker} LCG_{n+1}$.*

Proof. The assertion (i) follows from Theorem 5, Lemma 1 and Proposition 2. This immediately implies (ii).

It remains to show (iii). Note that putting a zero at the end of a basic vector of $\mathcal{B}_n$ turns it into a basic vector of $\mathcal{B}_{n+1}$, so that $\iota(\mathcal{B}_n) \subseteq \mathcal{B}_{n+1}$. The desired result now follows from part (i). □

Theorems 5 and 7 imply that in case $M(n) = 1$ the linearly independent set $\mathcal{B}_n$ of basic vectors needs a single additional vector $\tilde{\mathbf{b}} \in \mathbb{R}^n$, say, to obtain a basis of KerLCG_n. Such a vector $\tilde{\mathbf{b}} = (\tilde{b}_1, \ldots, \tilde{b}_n)$ can easily be defined recursively in the following fashion: First put $\tilde{b}_1 = 1$. Since all vectors in $\mathcal{B}_n$ have first entry 0, $\mathcal{B}_n \cup \{\tilde{\mathbf{b}}\}$ is linearly independent. Now the other coefficients of $\tilde{\mathbf{b}}$ are defined working down the subscripts according to (12) (where the ultimate equation disappears). Set $\tilde{b}_n = 1$ if n is squarefree, and 0 otherwise. If the coefficients $\tilde{b}_n, \tilde{b}_{n-1}, \ldots, \tilde{b}_{k+1}$ with $k \geq 2$ have been chosen, let

$$\tilde{b}_k := 1 - \sum_{2 \leq j \leq \frac{n}{k}} \tilde{b}_{j \cdot k}.$$

Obviously, $\tilde{\mathbf{b}}$ satisfies (12) and hence lies in KerLCG_n by Proposition 4.

Apparently, for sufficiently large n the vector $\tilde{\mathbf{b}}$ is not simple, hence $\mathcal{B}_n \cup \tilde{\mathbf{b}}_n$ is not a simple basis of KerLCG_n. In fact, we have

Theorem 8 *For any integer n satisfying $M(n) = 1$,* KerLCG_n *does not have a simple basis.*

Proof. In 1952, Nagura [14] gave a rather short proof for the fact that, given $x \geq 25$, there is always a prime p in the interval $x < p \leq \frac{6}{5}x$. Setting $x = \frac{n}{6}$ we obtain that for every integer $n \geq 150$ there is a prime p satisfying

$$\frac{n}{6} < p \leq \frac{n}{5}. \tag{13}$$

The primes 17 and 29, respectively, show that (13) is also valid if $94 \leq n \leq 100$ or $146 \leq n \leq 149$. By Remark 6(ii) we thus can find a prime p in the interval (13) for each integer $n > 1$ satisfying $M(n) = 1$. Alternatively, this follows from the more complicated estimates given later by Rosser and Schoenfeld [19].

By Proposition 4 the basis vectors of KerLCG_n are described by (12). Since $M(n) = 1$, the last equation of (12) disappears. Hence there is at least one basis vector $\mathbf{b}$, say, with $b_1 \neq 0$. We shall prove that $\mathbf{b}$ cannot be simple.

From the equations $b_{3p} - b_1 = 0$ and $b_{5p} - b_1 = 0$ of (12), we get $b_{3p} = b_{5p} = b_1$. The equation $b_{2p} + b_{4p} - b_1 = 0$ implies $b_{2p} + b_{4p} = b_1$. By inserting these into the equation $b_p + b_{2p} + b_{3p} + b_{4p} + b_{5p} - b_1 = 0$, we finally obtain $b_p = -2b_1$. So $\mathbf{b}$ has the entries $b_1 \neq 0$ and $b_p = -2b_1$, thus it is not simple. □

5. The Traditional Coprime Graph

Let us finally consider the traditional coprime graph TCG_n having a loop at the vertex 1, i.e., its adjacency matrix $\tilde{A}_n = (\tilde{a}_{i,j})_{n \times n}$ is defined as

$$\tilde{a}_{i,j} = \begin{cases} 0 & \text{if } \gcd(i,j) > 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then the analogue of Proposition 4 reads

Proposition 9 *Let $n > 1$ be an arbitrary integer. A vector $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ lies in $\mathrm{Ker}\tilde{A}_n$ if and only if*

$$b_1 = 0$$

and for $2 \le k \le n,\ \mu(k) \neq 0$

$$\sum_{\substack{j=k \\ j \equiv 0 \bmod k}}^{n} b_j = 0.$$

The proof of Proposition 9 as well as those of the subsequent main results are easily obtained by adjusting the proofs of Proposition 4 and Theorems 5 and 7 accordingly.

Theorem 10 *For any integer $n > 1$ we have $\dim_{\mathbb{R}} \mathrm{Ker}TCG_n = \nu(n)$, and*

$$\dim_{\mathbb{R}} \mathrm{Ker}TCG_n \ = \ (1 - \frac{6}{\pi^2})n + O(\sqrt{n}).$$

Theorem 11 *For each positive integer n, we have*

(i) $\mathcal{B}_n$ *is a basis of* $\mathrm{Ker}TCG_n$.

(ii) $\mathrm{Ker}TCG_n$ *has a simple basis, i.e., the components of all basis vectors are* 0, 1 *or* -1.

(iii) Let $\iota : \mathbb{R}^n \to \mathbb{R}^{n+1}$ with $\iota(b_1, \ldots, b_n) := (b_1, \ldots, b_n, 0)$ be the canonical injection. Then we have $\iota(\mathrm{Ker}TCG_n) \subseteq \mathrm{Ker}TCG_{n+1}$.

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