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A SHORT PROOF OF A SERIES EVALUATION IN TERMS OF HARMONIC NUMBERS[1]

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Abstract

We give another short and simple proof of

$$\sum_{n\ge1}\frac{1}{2^{2n-1}(2n-1)}\sum_k\binom{2n-1}{k}\frac{1}{k-j-n+\frac12}=-\frac{2}{j}\sum_{k=1}^{j}\frac{1}{2k-1}.$$

1. The Main Result

For positive integers j, consider

$$S(j)=\sum_{n\ge1}\frac{1}{2^{2n-1}(2n-1)}\sum_k\binom{2n-1}{k}\frac{1}{k-j-n+\frac12}.$$

This quantity arose in [4] and was subsequently evaluated in [3]. Further proofs of the final formula

$$S(j)=-\frac{2}{j}\sum_{k=1}^{j}\frac{1}{2k-1}$$

were given in [2, 1]. Here, we give another short and simple proof.

For our analysis, it is better to consider

$$T(j)=\sum_{n\ge1}\frac{1}{2^{2n-1}(2n-1)}\sum_k\binom{2n-1}{k}\frac{1}{k+j-n+\frac12},$$

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so that

$$S(j) = \sum_{n\geq 1} \frac{1}{2^{2n-1}(2n-1)} \sum_k \binom{2n-1}{2n-1-k} \frac{1}{k-j-n+\frac{1}{2}}$$

$$= \sum_{n\geq 1} \frac{1}{2^{2n-1}(2n-1)} \sum_k \binom{2n-1}{k} \frac{1}{(2n-1-k)-j-n+\frac{1}{2}}$$

$$= \sum_{n\geq 1} \frac{1}{2^{2n-1}(2n-1)} \sum_k \binom{2n-1}{k} \frac{1}{n-k-j-\frac{1}{2}}$$

$$= -T(j).$$

It will be advantageous to treat the sum

$$\widetilde{T}_j := \sum_{n\geq 1} \frac{1}{2^{2n-1}(2n-1)} \left[\frac{1}{2} \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{1}{k+j-n+\frac{1}{2}} - \frac{1}{2} \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{1}{k-j-n+\frac{1}{2}} \right]$$

$$= \frac{1}{2}\big(T(j) - S(j)\big) = T(j).$$

First we will give a representation of the sum $\sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{1}{k+m+\frac{1}{2}}$, with $m \in \mathbb{Z}$, as a curve integral in the complex plane.

Lemma 1 *We have*

$$\sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{1}{k+m+\frac{1}{2}} = \int_\Gamma u^{2m}(1+u^2)^{2n-1} du,$$

where the curve Γ is the upper half of the unit circle in the complex plane starting from -1 and ending at 1, i.e., $\Gamma = \{\cos(\pi - t) + i \sin(\pi - t) : t \in [0, \pi]\}$.

Proof. We have

$$\int_\Gamma u^{2m}(1+u^2)^{2n-1} du = \int_\Gamma \sum_{k=0}^{2n-1} \binom{2n-1}{k} u^{2k+2m} du$$

$$= \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{u^{2k+2m+1}}{2k+2m+1} \Bigg|_{-1}^{1}$$

$$= 2 \sum_{k=0}^{2n-1} \binom{2n-1}{k} \frac{1}{2k+2m+1}.$$

□

Thus we get

$$\frac{1}{2}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{1}{k+j-n+\frac{1}{2}}-\frac{1}{2}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{1}{k-j-n+\frac{1}{2}}$$
$$=\frac{1}{2}\int_\Gamma \big(u^{2j-2n}-u^{-2j-2n}\big)(1+u^2)^{2n-1}du$$
$$=\frac{1}{2}\int_\Gamma \big(u^{2j-1}-u^{-2j-1}\big)\Big(\frac{1+u^2}{u}\Big)^{2n-1}du,$$

and further

$$\widetilde{T}(j)=\sum_{n\geq 1}\frac{1}{2^{2n-1}(2n-1)}\frac{1}{2}\int_\Gamma \big(u^{2j-1}-u^{-2j-1}\big)\Big(\frac{1+u^2}{u}\Big)^{2n-1}du$$
$$=\sum_{n\geq 1}\int_\Gamma \frac{1}{2(2n-1)}\big(u^{2j-1}-u^{-2j-1}\big)\Big(\frac{1+u^2}{2u}\Big)^{2n-1}du. \tag{1}$$

Next we consider, for $j\in\mathbb{N}$ and $u\in\Gamma$, the series

$$Q_j(u):=\sum_{n\geq 1}\frac{1}{2(2n-1)}\big(u^{2j-1}-u^{-2j-1}\big)\Big(\frac{1+u^2}{2u}\Big)^{2n-1}.$$

Lemma 2 *The series $\widetilde{Q}_j(\varphi):=Q_j(e^{i\varphi})$ converges uniformly for $\varphi\in[0,\pi]$, i.e.,*

$$\widetilde{Q}_j(\varphi)=ie^{-i\varphi}\sin(2j\varphi)\frac{1}{2}\log\frac{1+\cos\varphi}{1-\cos\varphi}. \tag{2}$$

Proof. Substituting $u=e^{i\varphi}$, with $\varphi\in[0,\pi]$, we can write

$$\widetilde{Q}_j(\varphi)=Q_j(e^{i\varphi})=ie^{-i\varphi}\sum_{n\geq 1}\frac{1}{2n-1}\frac{e^{2ji\varphi}-e^{-2ji\varphi}}{2i}\Big(\frac{e^{i\varphi}+e^{-i\varphi}}{2}\Big)^{2n-1}$$
$$=ie^{-i\varphi}\sum_{n\geq 1}\frac{1}{2n-1}\sin(2j\varphi)(\cos\varphi)^{2n-1}.$$

Since we have

$$\sum_{n\geq 1}\frac{z^{2n-1}}{2n-1}=\frac{1}{2}\log\frac{1+z}{1-z},\quad \text{for } |z|<1, \tag{3}$$

we obtain the pointwise convergence of the series $\widetilde{Q}_j(\varphi)$, for $\varphi\in(0,\pi)$, to the function given in (2).

Obviously we also have $\widetilde{Q}_j(0)=\widetilde{Q}_j(\pi)=0$, which shows convergence of $\widetilde{Q}_j(\varphi)$, for all $\varphi\in[0,\pi]$. Since (3) converges uniformly for all z, with $|z|\leq q<1$, we obtain immediately that $\widetilde{Q}_j(\varphi)$ converges uniformly for all $\varphi \in [\delta, \pi - \delta]$, for arbitrary $0 < \delta < \frac{\pi}{2}$. But since for all $j \in \mathbb{N}$

$$\lim_{\varphi \to 0} \sin(2j\varphi) \log \frac{1 + \cos\varphi}{1 - \cos\varphi} = 0,$$

which can easily be shown, we obtain that for all $\epsilon > 0$ there exists a $\delta > 0$, such that

$$\Big| ie^{-i\varphi} \sum_{n \geq N} \frac{1}{2n-1} \sin(2j\varphi)(\cos\varphi)^{2n-1} \Big| = \sum_{n \geq N} \frac{1}{2n-1} \sin(2j\varphi)(\cos\varphi)^{2n-1}$$
$$\leq \sum_{n \geq 1} \frac{1}{2n-1} \sin(2j\varphi)(\cos\varphi)^{2n-1} < \epsilon,$$

for all $0 \leq \varphi < \delta$ and for all $N \in \mathbb{N}$. This, together with the obvious relation $\widetilde{Q}_j(\pi - \varphi) = -\widetilde{Q}_j(\varphi)$, shows that $\widetilde{Q}_j(\varphi)$ converges even uniformly for all $\varphi \in [0, \pi]$. □

After back-substitution, we obtain that the series $Q_j(u)$ converges uniformly for all $u \in \Gamma$ to the function

$$\big(u^{2j-1} - u^{-2j-1}\big) \frac{1}{4} \log\Big(\frac{1 + \frac{1+u^2}{2u}}{1 - \frac{1+u^2}{2u}} \Big).$$

Thus in equation (1) we can interchange summation and integration and obtain the integral representation

$$\widetilde{T}_j = \int_\Gamma \sum_{n \geq 1} \frac{1}{2(2n-1)} \big(u^{2j-1} - u^{-2j-1}\big) \Big(\frac{1 + u^2}{2u} \Big)^{2n-1} du$$
$$= \frac{1}{2} \int_\Gamma \big(u^{2j-1} - u^{-2j-1}\big) \frac{1}{2} \log\Big(\frac{1 + \frac{1+u^2}{2u}}{1 - \frac{1+u^2}{2u}} \Big) du$$
$$= \frac{1}{2} \int_\Gamma \big(u^{2j-1} - u^{-2j-1}\big) \frac{1}{2} \log\Big(-\frac{(1+u)^2}{(1-u)^2} \Big) du. \quad (4)$$

Remark Using the substitution $u = e^{i\varphi}$ one obtains the following representation of the sum $\widetilde{T}_j$ as a real integral:

$$\widetilde{T}_j = \frac{1}{2} \int_0^\pi \sin(2j\varphi) \log \frac{1 + \cos\varphi}{1 - \cos\varphi} d\varphi,$$

but it seems more involved to evaluate this integral.

We use now that, for $u \in \Gamma$:

$$\frac{1}{2} \log\Big(-\frac{(1+u)^2}{(1-u)^2} \Big) = \log\Big((-i) \frac{1+u}{1-u} \Big),$$

and the correct determination of the (multi-valued) logarithm function is obtained when considering the real analogue of this equation:

$$\frac{1}{2}\log\frac{1+\cos\varphi}{1-\cos\varphi} = \log\frac{\cos\frac{\varphi}{2}}{\sin\frac{\varphi}{2}}, \quad \text{for } \varphi \in (0,\pi).$$

Then equation (4) gives

$$\begin{aligned}\widetilde{T}_j &= \frac{1}{2}\int_\Gamma \left(u^{2j-1} - u^{-2j-1}\right)\log\Big((-i)\frac{1+u}{1-u}\Big)du \\ &= \frac{\log(-i)}{2}\int_\Gamma \left(u^{2j-1}-u^{-2j-1}\right)du + \int_\Gamma \left(u^{2j-1}-u^{-2j-1}\right)\frac{1}{2}\log\Big(\frac{1+u}{1-u}\Big)du \\ &= \int_\Gamma \left(u^{2j-1}-u^{-2j-1}\right)\frac{1}{2}\log\Big(\frac{1+u}{1-u}\Big)du, \qquad (5)\end{aligned}$$

since obviously the first integral vanishes.

In order to proceed we consider, for $j \in \mathbb{N}$ and $u \in \Gamma$, the series

$$R_j(u) := \sum_{n\geq 1}\left(u^{2j-1}-u^{-2j-1}\right)\frac{u^{2n-1}}{2n-1}.$$

Lemma 3 *There is uniform convergence of the series $R_j(u)$, for $u \in \Gamma$, to the function*

$$f_j(u) = \left(u^{2j-1}-u^{-2j-1}\right)\frac{1}{2}\log\Big(\frac{1+u}{1-u}\Big). \qquad (6)$$

Proof. It is well-known that equation (3) even holds, with the exception of $z = 1$ and $z = -1$, for all complex z with $|z| = 1$, which proves pointwise convergence of $R_j(u)$ to $f_j(u)$ for $u \in \Gamma \setminus \{-1, 1\}$.

Obviously we also have $R_j(-1) = R_j(1) = 0$, which shows convergence of $R_j(u)$, for all $u \in \Gamma$. Furthermore, since

$$R_j(u) = -\sum_{m=-j+1}^{j}\frac{u^{2m-2}}{2(m+j)-1} + \sum_{m\geq j+1}\frac{4j}{(2(m-j)-1)(2(m+j)-1)}u^{2m-2},$$

as can be shown easily, we obtain by simple majorization arguments that $R_j(u)$ converges even uniformly for all $u \in \Gamma$ to the function $f_j(u)$. □

Thus in equation (5) we can replace $f_j(u)$ by the series $R_j(u)$ and interchange summation and integration and get

$$\widetilde{T}_j = \sum_{n\geq 1}\int_\Gamma \left(u^{2j-1}-u^{-2j-1}\right)\frac{u^{2n-1}}{2n-1}du,$$

which can be evaluated easily:

$$\begin{aligned}
\widetilde{T}_j &= \sum_{n\geq 1}\int_\Gamma\Big(\frac{u^{2n+2j-2}}{2n-1}-\frac{u^{2n-2j-2}}{2n-1}\Big)du \\
&= \sum_{n\geq 1}\Big(\frac{u^{2n+2j-1}}{(2n-1)(2n+2j-1)}-\frac{u^{2n-2j-1}}{(2n-1)(2n-2j-1)}\Big)\Big|_{-1}^{1} \\
&= 2\sum_{n\geq 1}\Big(\frac{1}{(2n-1)(2n+2j-1)}-\frac{1}{(2n-1)(2n-2j-1)}\Big) \\
&= \frac{1}{j}\sum_{n\geq 1}\Big(\frac{1}{2n-1}-\frac{1}{2n+2j-1}\Big)-\frac{1}{j}\sum_{n\geq 1}\Big(\frac{1}{2n-2j-1}-\frac{1}{2n-1}\Big) \\
&= \frac{1}{j}\sum_{k=1}^{j}\frac{1}{2k-1}-\frac{1}{j}\sum_{k=1}^{j}\frac{-1}{2k-1}=\frac{2}{j}\sum_{k=1}^{j}\frac{1}{2k-1}.
\end{aligned}$$

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