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A PLANAR INTEGRAL SELF-AFFINE TILE WITH CANTOR SET INTERSECTIONS WITH ITS NEIGHBORS

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Abstract

We give an example of a planar self-affine tile that intersects some of its neighbors in the tiling it generates in Cantor sets. The reasoning also shows that it has nontrivial fundamental group. The technique used is to obtain information from a good approximation of the tile, namely, an estimation of its convex hull.

1. Introduction

A planar integral self-affine tile is a compact set T in $\mathbb{R}^2$ of positive Lebesgue measure for which there is an expanding integral matrix B and a digit set $\mathcal{D} = \{d_0, \ldots, d_{m-1}\} \subset \mathbb{Z}^2$ such that

$$B(T) = \bigcup_{i=0}^{m-1} (T + d_i),$$

with $(T + d_i) \cap (T + d_j)$ of measure zero. In this case, we must have $|\det B| = |\mathcal{D}| = m$. See [8].

We consider the tile $T = T(B, \mathcal{D})$, where

$$B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad \mathcal{D} = \left\{ d_i = \begin{pmatrix} i \\ 0 \end{pmatrix}, i = 0, \ldots, 7 \right\}. \tag{1}$$

We know that $T + \mathbb{Z}^2$ is a tiling of $\mathbb{R}^2$ (see Lagarias–Wang [8, Theorem 1.2(ii)]). We will prove the following two theorems.

Theorem 1 *Let $T = T(B, \mathcal{D})$ be the self-affine tile with B, $\mathcal{D}$ given in Equation (1). In the planar tiling generated by T, there are neighbors intersecting in Cantor sets, that is, totally disconnected perfect sets. More precisely, T intersects with at least four of its neighbors in Cantor sets. (See Figure 1.)*

Theorem 2 *T has nontrivial fundamental group. Hence its fundamental group is uncountably generated* [10].

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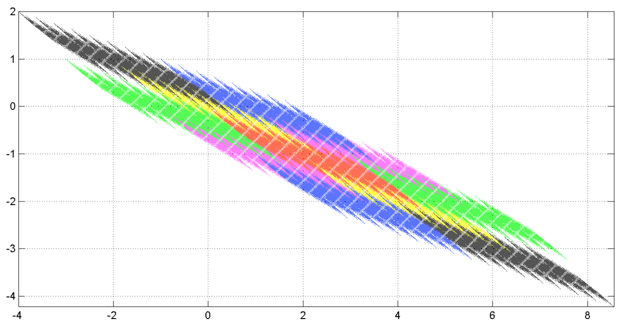

Figure 1: The central tile T, painted red, intersects four of its neighbors in Cantor sets, the blue and the black ones.

To put these into perspectives, we recall two theorems on tilings. Leung and Lau [9] give a necessary and sufficient condition for certain planar self-affine tiles to be disklike.

Theorem A [9, Theorem 1.4] Let $C \in M_2(\mathbb{Z})$ be an expanding matrix with characteristic polynomial $f(x) = x^2 + px + q$. Let $v \in \mathbb{Z}^2 - \{(0,0)\}$ with v, Cv independent. Let $\mathcal{D} = \{0, v, 2v, \ldots, (|q|-1)v\}$. Then $T := T(C, \mathcal{D})$ is disklike if and only if $2|p| \leq |q+2|$.

Akiyama and Thuswaldner [1, 2] gave a proof for the special case of tilings generated by canonical number systems, hence requiring $f(x)$ to be irreducible, $-1 \leq p \leq q$ and $q \geq 2$, that T is disklike if and only if $2p \leq q+2$.

We next state a theorem of Bandt and Gelbrich. Recall that two tiles T', T'' are tiling neighbors if $T' \cap T'' \neq \emptyset$. They are vertex neighbors if their intersection is a point. They are edge neighbors if their intersection contains a point inside $(T' \cup T'')^\circ$.

Theorem B [3, Lemma 5.1] *Let T be a topological disk which tiles $\mathbb{R}^2$ by the lattice $\mathcal{L}$. Then in the tiling $T + \mathcal{L}$, one of the followings is true:*

(i) *T has no vertex neighbors and six edge neighbors $T \pm \alpha$, $T \pm \beta$, $T \pm (\alpha + \beta)$ for some $\alpha, \beta \in \mathcal{L}$, with $\mathbb{Z}\alpha + \mathbb{Z}\beta = \mathcal{L}$.*

(ii) *T has four edge neighbors $T \pm \alpha$, $T \pm \beta$, and four vertex neighbors $T \pm \alpha \pm \beta$ for some $\alpha, \beta \in \mathcal{L}$, with $\mathbb{Z}\alpha + \mathbb{Z}\beta = \mathcal{L}$.*

From Theorem A, the tile $T = T(B, D)$ is not disklike, with $B, \mathcal{D}$ given in Equation (1). In contrast to Theorem B for disklike tiles,Theorem 1 shows that Cantor set intersections among neighbors can occur for non disklike self-affine $\mathbb{Z}^2$ tiles. While one can construct non self-affine tiles that have this property, the authors are not aware of any example that is self-affine. There are certainly self-affine tiles that intersect in finite but more than one point with some of its neighbors. For example, the fundamental domain of the canonical number system of base $-2+i$.

There are different ways to be non disklike, even for fractal tiles. For example, the Heighway dragon is pinched, but doesn't have holes. Theorem 2 says that the tile we study has holes, and that is not clear from its picture.

The method used here is to get conclusions from a good enough approximation of the tile T, namely, an estimation of the convex hull of T from the outside. This estimation is given by Duda [4]. He gives an example where the estimation gives exactly the convex hull. For our tile, the method also works very well. Indeed, a little more work gives the exact convex hull in our case.

The paper is organized as follows. In Section 2, we give an estimation of the convex hull of the tile we study. We prove Theorem 1 in Section 3 and Theorem 2 in Section 4.

2. Convex Hull Estimation

We recall the theory of convex hull estimation [4] for the tiles we are dealing with. Let T be self-affine tiles satisfying the equation

$$T = \bigcup_{i=0,\dots,m-1} A(T+d_i), \quad d_i = (i,0)^t,$$

where $A = B^{-1}$ for an integral expanding matrix B. Let $\mathbf{S} \subset \mathbb{R}^2$ be the unit circle and $d \in \mathbf{S}$ be a direction. The width function centered at zero is the function $h : \mathbf{S} \to \mathbb{R}$ given by

$$h(d) := \inf\{t : T \subset H(d,t)\},$$

where $H(d,t)$ is the half space $\{x \in \mathbb{R}^2 : x \cdot d \leq t\}$. We have the following theorem.

Theorem C [4, Observation 4] *Let T be as above. Then for $d \in \mathbf{S}$,*

$$h(d) = \sum_{j=0}^{\infty} \|A^j d\| h^*(\widehat{A^j d}).$$

Here, $\widehat{A^j d} = A^j d / \|A^j d\|$, and for any $v \in \mathbf{S}$, $h^*(v) = \max_{i=0,\dots,m-1} v \cdot Ad_i$.

Theorem D [4, Observation 5] *Let x_0 be the center of symmetry of T: $T - x_0 = x_0 - T$. Then*

$$x_0 = \frac{1}{2}(B-I)^{-1} d_{m-1}.$$

We now estimate the convex hull of $T(B, \mathcal{D})$, with $B, \mathcal{D}$ as given in Equation (1).

Proposition 2.1 *Let $B, \mathcal{D}$ be given as in equation 1. Then $T(B, \mathcal{D})$ lies in the parallelogram P with vertices $(0,0), (7/2, -7/6), (14/3, -7/3), (7/6, -7/6)$. Moreover T contains $(0,0), (14/3, -7/3)$.*

Proof. Let $A = B^{-1}$. The eigenvalues of A are $\lambda_1 = 1/2$ and $\lambda_2 = 1/4$, with eigenvectors $v_1 = (-1,1)^t, v_2 = (1,1)^t$ respectively. Notice that the vectors $Ad_i = (3i/8, -i/8)^t$ are nonnegative multiples of $(3,-1)^t$.

Let $d = \widehat{(1,3)^t}$. As the eigenvector $v_1 = (-1,1)$ correspond to the larger eigenvalue and $v_2 = (1,1)$ corresponds to the smaller one, $\widehat{A^j d} \to (-1/\sqrt{2}, 1/\sqrt{2})^t$ 'monotonically' as $j \to \infty$. Hence for $i = 0, \ldots, 7$ and nonnegative integer j, $\widehat{A^j d} \cdot Ad_i \leq 0$. By definition, $h^*(\widehat{A^j d}) = \max_{i=0,\ldots,7} \widehat{A^j d} \cdot Ad_i = 0$ for $j = 0, 1, 2, \ldots$, implying that $h(d) = 0$ from Theorem C. That is, T is below the line $x + 3y = 0$. See Fig. 2. The same reasoning applied to $d = \widehat{(-1,-1)^t}$ shows that T is to the right of the line $x + y = 0$.

From Theorem D, the center of symmetry of T is $(7/3, -7/6)$. Obviously, $(0,0) \in T$ from the definition of T. Hence $(14/3, -7/3) \in T$. Therefore T and its convex hull lies in the region bounded by the lines

$$x + 3y = 0, \quad x + y = 0, \quad x + y = 7/3, \quad x + 3y = -7/3. \qquad \square$$

See Figure 2 for a picture for the convex hull approximation. Indeed, a little further work can give us the exact convex hull of the tile we are working with.

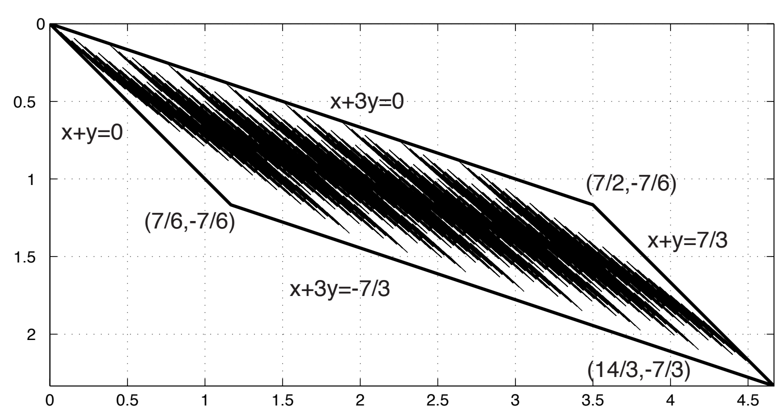

Figure 2: An approximation of the convex hull of the tile T.

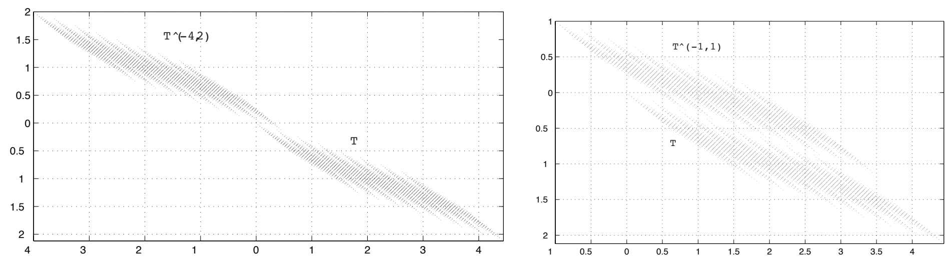

Figure 3: (left) $T^{(-4,2)} \cap T$ is a Cantor set. The same is true for $T^{(-1,1)} \cap T$ (right)

3. Cantor Set Intersections: Proof of Theorem 1

In this section, we show that there are neighbors in the tiling generated by $T(B, \mathcal{D})$ that intersects in Cantor sets. See Fig. 3. We first introduce some notations. They are general though we only state them for our tile.

For $i = 0, \ldots, 7$, let $f_i(x) := A(x + d_i)$ for $x \in \mathbb{R}^2$. Let $f_{i_1 \cdots i_k} := f_{i_1} \circ \cdots \circ f_{i_k}$ be their compositions, where k is a positive integer and for $j = 1, \ldots, k$, $i_j \in \{0, \ldots, 7\}$. For $E \subset \mathbb{R}^2$, we also write $E_{i_1 \cdots i_k}$ for $f_{i_1 \cdots i_k}(E)$. Let $\mathcal{F}^k(E) = \bigcup_{i_1, \ldots, i_k = 0}^{7} f_{i_1 \cdots i_k}(E)$. For $(i, j) \in \mathbb{Z}^2$, define $E^{(i,j)} := E + (i, j)^t$ to be the translate of E by $(i, j)^t$. We will write E for $E^{(0,0)}$. We call $T^{(i,j)}_{i_1 \cdots i_k}$ a k-th level piece of $T^{(i,j)}$, with the tile $T^{(i,j)}$ itself being the 0-th level piece. For the polygon P given in Proposition 2.1, we will call (for convenience) $P^{(i,j)}_{i_1 \cdots i_k}$ a k-th level polygon of $P^{(i,j)}$. Notice that $P^{(i,j)}_{i_1 \cdots i_k}$ contains the k-th level piece $T^{(i,j)}_{i_1 \cdots i_k}$ of $T^{(i,j)}$.

We now show that $T \cap T^{(-4,2)}$ is a Cantor set. Then we can conclude that $T \cap T^{(-1,1)}$, $T \cap T^{(1,-1)}$ and $T \cap T^{(4,-2)}$ are also Cantor sets as they are either a translate of $T \cap T^{(-4,2)}$ or unions of translates of $T \cap T^{(-4,2)}$ (see Fig. 1). The following lemma shows that it suffices to figure out $\mathcal{F}^k(P) \cap \mathcal{F}^k(P)^{(-4,2)}$ for each positive integer k.

Lemma 4 *For all positive integers k,*

$$T \cap T^{(-4,2)} = \bigcap_{k=1}^{\infty} [\mathcal{F}^k(P) \cap \mathcal{F}^k(P)^{(-4,2)}].$$

Proof. As $\mathcal{F}^1(P) \subset P$, we have $\mathcal{F}^k(P)^{(i,j)}$ decreasing in k, and

$$T = \bigcap_{k=1}^{\infty} \mathcal{F}^k(P), \quad T^{(-4,2)} = \bigcap_{k=1}^{\infty} \mathcal{F}^k(P)^{(-4,2)};$$

see, for example, [5]. The lemma follows. $\square$

Lemma 5 *Let P be the parallelogram in Proposition 2.1. Let k be a positive integer and for $j = 1, \ldots, k$, $i_j \in \{0, \ldots, 7\}$. Then*

(a) $AP^{(-4,2)}_{i_1\cdots i_k} = P^{(-4,2)}_{6i_1\cdots i_k}$.

(b) $AP_{i_1\cdots i_k} = P_{0i_1\cdots i_k}$.

(c) $P^{(-4,2)}_{6i_1\cdots i_k} + (3/8, -1/8)^t = P^{(-4,2)}_{7i_1\cdots i_k}$. *Notice that* $(3/8, -1/8)^t = Ad_1$.

As usual, A stands for the map 'multiplication by A'.

Proof. By the definitions given at the beginning of this section,

$$P^{(-4,2)}_{i_1\cdots i_k} = f_{i_1\cdots i_k}(P) + (-4,2)^t = A^k P + \sum_{j=1}^{k} A^j d_{i_j} + (-4,2)^t.$$

Hence

$$AP^{(-4,2)}_{i_1\cdots i_k} = A^{k+1}P + \sum_{j=1}^{k} A^{j+1} d_{i_j} + A(-4,2)^t.$$

On the other hand,

$$P^{(-4,2)}_{6i_1\cdots i_k} = A^{k+1}P + \sum_{j=1}^{k} A^{j+1} d_{i_j} + Ad_6 + (-4,2)^t.$$

As $(A - I)(-4,2)^t = Ad_6$, (a) is proved.

(b) can be proved similarly. (c) follows from $Ad_1 = (3/8, -1/8)^t$. □

Lemma 6 *For $k = 1, 2, 3, \ldots,$*

$$\mathcal{F}^k(P)^{(-4,2)} \cap \mathcal{F}^k(P) = \bigcup_{i_1,\ldots,i_k=6,7} [P^{(-4,2)}_{i_1\cdots i_k} \cap P_{(i_1-6)\cdots(i_k-6)}].$$

Proof. We use induction. For $k = 1$, it can be verified directly using the equations of the boundary of P, or can be seen clearly from Figure 4 that

$$\mathcal{F}^1(P)^{(-4,2)} \cap \mathcal{F}^1(P) = (P^{(-4,2)}_6 \cap P_0) \cup (P^{(-4,2)}_7 \cap P_1).$$

Suppose the statement is true for k. For $k + 1$, as

$$\begin{aligned}\mathcal{F}^{k+1}(P)^{(-4,2)} \cap \mathcal{F}^{k+1}(P) &\subset \mathcal{F}^1(P)^{(-4,2)} \cap \mathcal{F}^1(P) \\ &= (P^{(-4,2)}_6 \cap P_0) \cup (P^{(-4,2)}_7 \cap P_1),\end{aligned}$$

we just need to find out the portions of $\mathcal{F}^{k+1}(P)^{(-4,2)} \cap \mathcal{F}^{k+1}(P)$ in $P^{(-4,2)}_6 \cap$

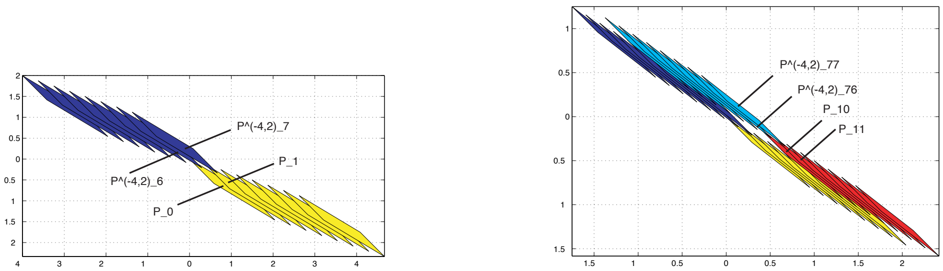

Figure 4: (left) The intersection $\mathcal{F}^1(P)^{(-4,2)} \cap \mathcal{F}^1(P)$, consisting of intersections of first level polygons $P_6^{(-4,2)} \cap P_0$ and $P_7^{(-4,2)} \cap P_1$. (right) The polygons drawn are second level polygons of $P^{(-4,2)}$ and P inside $P_6^{(-4,2)}, P_7^{(-4,2)}$ and P_0, P_1. The intersection shown is $\mathcal{F}^2(P)^{(-4,2)} \cap \mathcal{F}^2(P)$. It consists of four pieces, two in $P_6^{(-4,2)} \cap P_0$ and two in $P_7^{(-4,2)} \cap P_1$

P_0 and in $P_7^{(-4,2)} \cap P_1$. Notice that for $i = 6, 7$, the only 1-st level piece of P intersecting $P_i^{(-4,2)}$ is P_{i-6}. Similarly, for $i = 0, 1$, the only 1-st level piece of $P^{(-4,2)}$ intersecting P_i is $P_{i+6}^{(-4,2)}$. Hence the portion of $\mathcal{F}^{k+1}(P)^{(-4,2)} \cap \mathcal{F}^{k+1}(P)$ in $P_6^{(-4,2)} \cap P_0$ is

$$\begin{aligned}
&\left(\bigcup_{i_2,\ldots,i_{k+1}=0}^{7} P_{6i_2\cdots i_{k+1}}^{(-4,2)} \right) \cap \left(\bigcup_{i_2,\ldots,i_{k+1}=0}^{7} P_{0i_2\cdots i_{k+1}} \right) \\
&\qquad = \left(\bigcup_{i_2,\ldots,i_{k+1}=0}^{7} AP_{i_2\cdots i_{k+1}}^{(-4,2)} \right) \cap \left(\bigcup_{i_2,\ldots,i_{k+1}=0}^{7} AP_{i_2\cdots i_{k+1}} \right) \\
&\qquad = A \left[\left(\bigcup_{i_2,\ldots,i_{k+1}=0}^{7} P_{i_2\cdots i_{k+1}}^{(-4,2)} \right) \cap \left(\bigcup_{i_2,\ldots,i_{k+1}=0}^{7} P_{i_2\cdots i_{k+1}} \right) \right] \\
&\qquad = A[\mathcal{F}^k(P)^{(-4,2)} \cap \mathcal{F}^k(P)],
\end{aligned}$$

where we have used Lemma 3.2(a)(b) in the first equality. By the induction hypothesis, this is equal to

$$\begin{aligned}
&A \bigcup_{i_2,\ldots,i_{k+1}=6,7} [P_{i_2\cdots i_{k+1}}^{(-4,2)} \cap P_{(i_2-6)\cdots(i_{k+1}-6)}] \\
&\qquad = \bigcup_{i_2,\ldots,i_{k+1}=6,7} [P_{6i_2\cdots i_{k+1}}^{(-4,2)} \cap P_{0(i_2-6)\cdots(i_{k+1}-6)}].
\end{aligned}$$

Next, the part of $\mathcal{F}^{k+1}(P)^{(-4,2)} \cap \mathcal{F}^{k+1}(P)$ in $P_7^{(-4,2)} \cap P_1$ is, using Lemma 3.2 and the induction hypothesis,

$$\begin{aligned}
&\left(\bigcup_{i_2,\ldots,i_{k+1}=0}^{7} P^{(-4,2)}_{7i_2\cdots i_{k+1}}\right) \cap \left(\bigcup_{i_2,\ldots,i_{k+1}=0}^{7} P_{1i_2\cdots i_{k+1}}\right) \\
&\quad = \left(\bigcup_{i_2,\ldots,i_{k+1}=0}^{7} P^{(-4,2)}_{6i_2\cdots i_{k+1}}\right) \cap \left(\bigcup_{i_2,\ldots,i_{k+1}=0}^{7} P_{0i_2\cdots i_{k+1}}\right) + (3/8,-1/8)^t \\
&\quad = A\left[\left(\bigcup_{i_2,\ldots,i_{k+1}=0}^{7} P^{(-4,2)}_{i_2\cdots i_{k+1}}\right) \cap \left(\bigcup_{i_2,\ldots,i_{k+1}=0}^{7} P_{i_2\cdots i_{k+1}}\right)\right] + (3/8,-1/8)^t \\
&\quad = A \bigcup_{i_2,\ldots,i_{k+1}=6,7} [P^{(-4,2)}_{i_2\cdots i_{k+1}} \cap P_{(i_2-6)\cdots(i_{k+1}-6)}] + (3/8,-1/8)^t \\
&\quad = \bigcup_{i_2,\ldots,i_{k+1}=6,7} [P^{(-4,2)}_{6i_2\cdots i_{k+1}} \cap P_{0(i_2-6)\cdots(i_{k+1}-6)}] + (3/8,-1/8)^t \\
&\quad = \bigcup_{i_2,\ldots,i_{k+1}=6,7} [P^{(-4,2)}_{7i_2\cdots i_{k+1}} \cap P_{1(i_2-6)\cdots(i_{k+1}-6)}].
\end{aligned}$$

Hence

$$\mathcal{F}^{k+1}(P)^{(-4,2)} \cap \mathcal{F}^{k+1}(P) = \bigcup_{i_1,\ldots,i_{k+1}=6,7} [P^{(-4,2)}_{i_1\cdots i_{k+1}} \cap P_{(i_1-6)\cdots(i_{k+1}-6)}]. \qquad \square$$

Lemma 7 *The union on the right-hand side of Lemma 3.3 is a disjoint union.*

Proof. Use induction on k. For $k=1$, $P_6^{(-4,2)} \cap P_0$ and $P_7^{(-4,2)} \cap P_1$ are disjoint, as can be verified directly, or seen clearly from Figure 4.

Suppose the statement is true for k. Consider the pieces

$$R := P^{(-4,2)}_{i_1\cdots i_{k+1}} \cap P_{(i_1-6)\cdots(i_{k+1}-6)}, \quad S := P^{(-4,2)}_{l_1\cdots l_{k+1}} \cap P_{(l_1-6)\cdots(l_{k+1}-6)}$$

in $\mathcal{F}^{k+1}(P)^{(-4,2)} \cap \mathcal{F}^{k+1}(P)$. That is, $i_j, l_j \in \{6,7\}$ for all $j = 1,\ldots,k+1$. If $i_j \neq l_j$ for some $j \in \{1,\ldots,k\}$, then R and S are disjoint as they are contained in different pieces of $\mathcal{F}^k(P)^{(-4,2)} \cap \mathcal{F}^k(P)$, which are disjoint by the induction hypothesis. Suppose that $i_j = l_j$ for $j \in \{1,\ldots,k\}$, and $i_{k+1} \neq l_{k+1}$, say, $i_{k+1} = 6$ and $l_{k+1} = 7$. Then R and S are respectively the images of the disjoint sets $P_6^{(-4,2)} \cap P_0$ and $P_7^{(-4,2)} \cap P_1$ under the same sequence of affine maps consisting of $A(\cdot)$ and $A(\cdot) + (3/8,-1/8)^t$, and hence are disjoint. $\square$

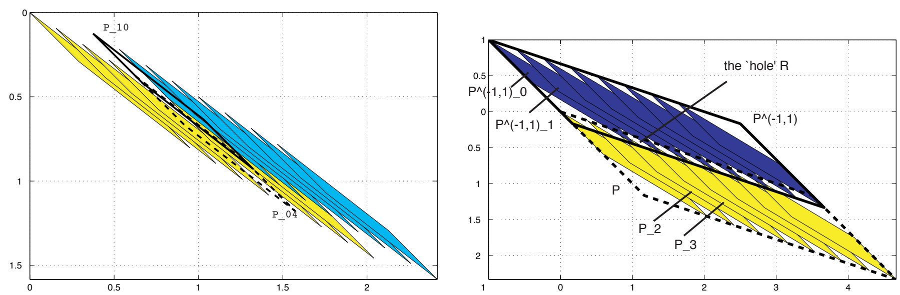

Figure 5: (left) Second level polygons of P in P_0, P_1 are drawn. P_{10} and P_{04} are drawn non-filled, bounded by solid and dotted lines respectively. Part of the intersection $P_{10} \cap P_{04}$ is not covered by other second level polygons of P. (right) The affine map F, defined in step 2 in section 4, maps $P^{(-1,1)} \cup P$ onto $P_{10} \cup P_{04}$.

Proof of Theorem 1. The maps $A(\cdot)$ and $A(\cdot) + (3/8, -1/8)^t$ are contractions. As the eigenvalues of A are not bigger than $1/2$, for any bounded set $E \subset \mathbb{R}^2$ with diameter diam E, diam $A(E) \leq 1/2(\text{diam } E)$. The same is true for the map $A(\cdot) + (3/8, -1/8)^t$. Hence from Lemmas 5, 6, and 7, each piece $P^{(-4,2)}_{i_1 \cdots i_k} \cap P_{(i_1-6)\cdots(i_k-6)}$ of $\mathcal{F}^k(P)^{(-4,2)} \cap \mathcal{F}^k(P)$ contains two disjoint pieces of $\mathcal{F}^{k+1}(P)^{(-4,2)} \cap \mathcal{F}^{k+1}(P)$, each of diameter less than $1/2$ of its own. This is true for every $k = 1, 2, \ldots$. It is standard to prove that $\bigcap_{k=1}^{\infty}[\mathcal{F}^k(P) \cap \mathcal{F}^k(P)^{(-4,2)}]$ is a totally disconnected perfect set. Theorem 1 follows from Lemma 4. □

4. Nontrivial Fundamental Group: Proof of Theorem 2

In this section, we prove that $\pi_1(T)$ is nontrivial. In short, the reason is that there are affine images of $T_6^{(-4,2)} \cup T_7^{(-4,2)} \cup T_0 \cup T_1$ (an area studied in section 3) in T, with the hole it bounds mapped to points not in T.

Proof of Theorem 2. Step 1. Look at the second level polygons P_{04} and P_{10} of P (see Figure 5). They have nonempty intersection. Part of this intersection is not covered by other second level polygons of P. Call this set

$$W := (P_{04} \cap P_{10}) - \bigcup \{P_{i_1 i_2} : (i_1, i_2) \neq (0,4), (1,0)\}.$$

We will show that there is a hole of T in W.

Step 2. The affine map $F(x) = A^2(x + d_4)$ for $x \in \mathbb{R}^2$ maps P to P_{04}, and $P^{(-1,1)}$ to P_{10}. For example, the reason for the latter is

$$F(P^{(-1,1)}) = A^2(P + (-1,1)^t + d_4) = A^2 P + A d_1 = P_{10}.$$

So F maps $P_0^{(-1,1)} \cup P_1^{(-1,1)} \cup P_2 \cup P_3$ homeomorphically onto $P_{100} \cup P_{101} \cup P_{042} \cup P_{043}$, also mapping the hole R bounded by the former (see Fig. 5) to a hole called S bounded by the latter.

Step 3. The set S (the hole) does not intersect the second level polygons in P other than P_{04} and P_{10}. To see this, we can check that the images under $F^{-1}(\cdot) = B^2(\cdot) - d_4$ of these other second level polygons of P do not intersect R. For example, we check whether P_{11} intersects S as follows:

$$\begin{aligned} F^{-1}(P_{11}) &= F^{-1}(A^2P + A^2d_1 + Ad_1) \\ &= B^2(A^2P + A^2d_1 + Ad_1) - d_4 \\ &= P + (0,1)^t, \end{aligned}$$

and it doesn't intersect the hole R. Others are checked similarly. Therefore, S is not contained in $\mathcal{F}^2(P)$ and hence not in T.

Step 4. Notice that the intersection $(P_0^{(-1,1)} \cup P_1^{(-1,1)}) \cap (P_2 \cup P_3)$ is congruent to $(P_6^{(-4,2)} \cup P_7^{(-4,2)}) \cap (P_0 \cup P_1)$, the set we have studied in detail in Section 3. They differ by a translation of $2 \times (3/8, -1/8)^t$. Similarly, the intersection $T^{(-1,1)} \cap T$ is congruent to three copies of $T^{(-4,2)} \cap T$, differing by translations by multiples of $(3/8, -1/8)^t$ (see Figure 3). Now pick any point $p \in T \cap T^{(-1,1)}$ in $P_0^{(-1,1)} \cap P_2$, and any point $q \in T \cap T^{(-1,1)}$ in $P_1^{(-1,1)} \cap P_3$.

Step 5. Notice that T is connected [7] and hence path connected [6]. The path connectedness of $T^{(-1,1)}$ guarantees that there is a path $\gamma_1 \subset T^{(-1,1)} \subset \bigcup_{i=0}^{7} P_i^{(-1,1)}$ joining p, q. The same reason gives a path $\gamma_2 \subset T \subset \bigcup_{i=0}^{7} P_i$ joining p, q. Then $\gamma_1 \cup \gamma_2$ is a nontrivial loop in $T \cup T^{(-1,1)}$, surrounding the hole R. Hence $F(\gamma_1 \cup \gamma_2)$ is a nontrivial loop in T because of the hole S.

Step 6. By Luo and Thuswaldner [10], $\pi_1(T)$ nontrivial implies that it is uncountably generated. $\square$

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