NONEXISTENCE OF A GEOMETRIC PROGRESSION THAT CONTAINS FOUR TRIANGULAR NUMBERS

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Abstract

We prove that there is no geometric progression that contains four (distinct) triangular numbers.

1. Introduction

The integers of the form $T_n = n(n+1)/2$, $n \in \mathbf{N}$, are called triangular numbers. Sierpinski asked whether there exist four(distinct) triangular numbers in geometric progression (see [1, D23]). Bennett [2] claimed that there are no four(distinct) triangular numbers in geometric progression. In fact, Bennett's proof is under the assumption that the geometric progression has an integral common ratio. Chen and Fang [6] removed the assumption "integral common ratio" and solved Sierpinski's problem.

By employing the theory of Pell's equations and a result of Y. Bilu, G. Hanrot and P. M. Voutier on primitive divisors of Lucas and Lehmer numbers [3], Yang and He [4] claimed that there is no geometric progression which contains four (distinct) triangular numbers. In their paper, they misunderstood the phrase "in geometric progression," and claimed that Bennett's proof is not complete and that they solved Sierpinski's problem completely. In fact, their proof is also under the assumption that the geometric progression has an integral common ratio.

In this paper, "integral common ratio" is removed. We use only the Störmer theorem on Pell's equation to prove the following result.

Theorem. *No geometric progression contains four (distinct) triangular numbers.*

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2. Proof of Theorem

In this paper, we use the following lemma.

Lemma (Störmer theorem[5]) *If Pell's equation $x^2 - Dy^2 = \pm 1 (D > 0)$ has a positive integral solution (x_1, y_1), and every prime divisor of y_1 divides D, then (x_1, y_1) is the fundamental solution.*

Proof of the theorem. Suppose that there is a geometric progression which contains four (distinct) triangular numbers T_x, T_y, T_u, T_v. Let the common ratio of the geometric progression be $q = b/a$ with $a \geq 1$ and $(a, b) = 1$. Obviously, we can consider the question in a finite geometric progression, thus we may assume $0 < q < 1$, so $a > b$. We can arrange T_x, T_y, T_u, T_v so that there exist positive integers $A, r_1, r_2, r_3 (0 < r_1 < r_2 < r_3)$ satisfying

$$8T_x = A, \quad 8T_y = Aq^{r_1}, \quad 8T_u = Aq^{r_2}, \quad 8T_v = Aq^{r_3}.$$

By the form of triangular numbers, we have

$$A + 1 = m_1^2, \quad Aq^{r_1} + 1 = m_2^2, \quad Aq^{r_2} + 1 = m_3^2, \quad Aq^{r_3} + 1 = m_4^2,$$

where m_1, m_2, m_3, m_4 are all positive integers, and $m_1 > m_2 > m_3 > m_4$. Since $Aq^{r_3} \in \mathbb{N}$, we have $a^{r_3} \mid Ab^{r_3}$. Since $(a, b) = 1$, we have $a^{r_3} \mid A$. Let $A = a^{r_3} a_0$. By the above equations, we have

$$m_1^2 - a^{r_3} a_0 = 1, \quad m_2^2 - a^{r_3 - r_1} b^{r_1} a_0 = 1, \tag{1}$$

$$m_3^2 - a^{r_3 - r_2} b^{r_2} a_0 = 1, \quad m_4^2 - b^{r_3} a_0 = 1. \tag{2}$$

Case 1. $2 \mid r_3$. By (1) and the lemma, $(m_1, a^{(r_3-2)/2})$ is the basic solution of the Pell's equation, $x^2 - a_0 a^2 y^2 = 1$. If $2 \mid r_1$, then $r_3 \geq r_1 + 2$. By (1) we have $m_2^2 - a^{r_3 - r_1 - 2} b^{r_1} a_0 a^2 = 1$. Thus, since $a > b$, we have

$$\begin{aligned} m_2 + a^{\frac{r_3 - r_1 - 2}{2}} b^{\frac{r_1}{2}} \sqrt{a_0 a^2} &= (m_1 + a^{\frac{r_3 - 2}{2}} \sqrt{a_0 a^2})^k \\ &\geq m_1 + a^{\frac{r_3 - 2}{2}} \sqrt{a_0 a^2} \\ &> m_2 + a^{\frac{r_3 - r_1 - 2}{2}} b^{\frac{r_1}{2}} \sqrt{a_0 a^2}, \end{aligned}$$

a contradiction. If $2 \mid r_2$, then $r_3 \geq r_2 + 2$. By (2) we have $m_3^2 - a^{r_3 - r_2 - 2} b^{r_2} a_0 a^2 = 1$. Thus, since $a > b$, we have

$$\begin{aligned} m_3 + a^{\frac{r_3 - r_2 - 2}{2}} b^{\frac{r_2}{2}} \sqrt{a_0 a^2} &= (m_1 + a^{\frac{r_3 - 2}{2}} \sqrt{a_0 a^2})^k \\ &\geq m_1 + a^{\frac{r_3 - 2}{2}} \sqrt{a_0 a^2} \\ &> m_3 + a^{\frac{r_3 - r_2 - 2}{2}} b^{\frac{r_2}{2}} \sqrt{a_0 a^2}, \end{aligned}$$

a contradiction.

If $2 \nmid r_1$ and $2 \nmid r_2$, then by (1) and (2) we have

$$m_2^2 - a^{r_3 - r_1 - 1} b^{r_1 - 1} a_0 ab = 1, m_3^2 - a^{r_3 - r_2 - 1} b^{r_2 - 1} a_0 ab = 1.$$

By the lemma, both $(m_2, a^{(r_3-r_1-1)/2}b^{(r_1-1)/2})$ and $(m_3, a^{(r_3-r_2-1)/2}b^{(r_2-1)/2})$ are the basic solutions of Pell's equation $x^2 - a_0aby^2 = 1$. This is impossible.

Case 2. $2 \nmid r_3$. By (1) and the lemma, the basic solution of Pell's equation $x^2 - a_0ay^2 = 1$ is $(m_1, a^{(r_3-1)/2})$. If $2 \mid r_1$, then $2 \mid r_3 - r_1 - 1$. By (1) we have $m_2^2 - a^{r_3-r_1-1}b^{r_1}a_0a = 1$. So, since $a > b$, we have

$$\begin{aligned} m_2 + a^{\frac{r_3-r_1-1}{2}}b^{\frac{r_1}{2}}\sqrt{a_0a} &= (m_1 + a^{\frac{r_3-1}{2}}\sqrt{a_0a})^k \\ &\geq m_1 + a^{\frac{r_3-1}{2}}\sqrt{a_0a} \\ &> m_2 + a^{\frac{r_3-r_1-1}{2}}b^{\frac{r_1}{2}}\sqrt{a_0a}, \end{aligned}$$

a contradiction. If $2 \mid r_2$, then $2 \mid r_3 - r_2 - 1$. By (2) we have $m_3^2 - a^{r_3-r_2-1}b^{r_2}a_0a = 1$. So, since $a > b$, we have

$$\begin{aligned} m_3 + a^{\frac{r_3-r_2-1}{2}}b^{\frac{r_2}{2}}\sqrt{a_0a} &= (m_1 + a^{\frac{r_3-1}{2}}\sqrt{a_0a})^k \\ &\geq m_1 + a^{\frac{r_3-1}{2}}\sqrt{a_0a} \\ &> m_3 + a^{\frac{r_3-r_2-1}{2}}b^{\frac{r_2}{2}}\sqrt{a_0a}, \end{aligned}$$

a contradiction.

If $2 \nmid r_1$ and $2 \nmid r_2$, then since $2 \nmid r_3$ and $0 < r_1 < r_2 < r_3$, we have: $r_3 \geq r_1 + 2, r_3 \geq r_2 + 2, 2 \mid r_3 - r_1 - 2, \quad 2 \mid (r_3 - r_2 - 2), 2 \mid (r_1 - 1)$, and $2 \mid (r_2 - 1)$. By (1) and (2), we have $m_2^2 - a^{r_3-r_1-2}b^{r_1-1}a_0a^2b = 1$ and $\mathrm{m}_3^2 - a^{r_3-r_2-2}b^{r_2-1}a_0a^2b = 1$. By the lemma, both $(m_2, a^{(r_3-r_1-2)/2}b^{(r_1-1)/2})$ and $(m_3, a^{(r_3-r_2-2)/2}b^{(r_2-1)/2})$ are the basic solutions of the Pell equation $x^2 - a_0a^2by^2 = 1$, which is impossible.

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