INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 7 (2007), #A12

A GRAPHIC GENERALIZATION OF ARITHMETIC

Bilal Khan
Department of Mathematics and Computer Science,
John Jay College of Criminal Justice, City University of New York,
New York, NY 10019, USA.
bkhan@jjay.cuny.edu

Kiran R. Bhutani
Department of Mathematics,
The Catholic University of America, Washington DC 20064, USA.
bhutani@cua.edu

Delaram Kahrobaei
New York City College of Technology, City University of New York,
New York, NY 11201, USA.
DKahrobaei@CityTech.Cuny.edu

Received: 1/21/05, Revised: 1/17/07, Accepted: 1/31/07, Published: 2/28/07

Abstract

In this paper, we introduce a natural arithmetic on the set of all *flow graphs*, that is, the set of all finite directed connected multigraphs having a pair of distinguished vertices. The proposed model exhibits the property that the natural numbers appear as a submodel, with the directed path of length n playing the role of the standard integer n. We investigate the basic features of this model, including associativity, distributivity, and various identities relating the order relation to addition and multiplication.

1. Introduction

The *language of arithmetic* $\mathcal{L}$ consists of two constants $\mathbf{0}$ and $\mathbf{1}$, one binary relation $\leqslant$, and two binary operations $+$ and $\times$. In this paper, we generalize classical arithmetic defined over the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, to the set F consisting of all *flow graphs*: finite directed connected multigraphs [1] in which a pair of distinguished vertices is designated as the *source* and *target* vertex. We give natural interpretation for $\mathcal{L}$ on the set F. To avoid confusion with the standard model of arithmetic, the corresponding operations in

[1] By multigraph we mean graphs in which parallel and loop edges are permitted.

F are denoted with a circumscribed circle. The new model $\mathcal{F} = \langle F, 0, 1, \leqslant, +, \times\rangle$ is a natural extension of the standard model $\mathcal{N} = \langle \mathbb{N}, 0, 1, \leqslant +, \times\rangle$.

Specifically, we exhibit an embedding $i : \mathcal{N} \overset{i}{\hookrightarrow} \mathcal{F}$ satisfying:

$$\begin{array}{rcl} i(0) &=& 0, \\ i(1) &=& 1, \\ \forall x, y \in \mathbb{N},\ \ x \leqslant y &\Leftrightarrow& i(x) \textcircled{\leqslant} i(y), \\ \forall x, y \in \mathbb{N},\ \ i(x+y) &=& i(x) \textcircled{+} i(y), \\ \forall x, y \in \mathbb{N},\ \ i(x \times y) &=& i(x) i(y). \end{array}$$

There have been other attempts to define algebraic and metric structures on the set of all graphs. In [6, 2, 1], the authors used graph embeddings to define a metric on the set of all simple connected graphs of a given order. This work differs from those investigations in that it considers an infinite collection of graphs in order to extend the standard model of arithmetic, and in doing so does not seek to establish a metric structure. The classical operations on graphs [9] (including extensive literature on graph products [5]) have yielded many results and a deep mathematical theory. There has also been considerable prior work on addition and multiplication of ordinals and partially ordered sets [3, 4, 7, 8]. To date, these prior investigations have not yielded an interpretation of the language of arithmetic on graphs. This paper presents results and open questions in this direction.

2. Flow Graphs

Definition 2.1 (Flow graph)**.** *We define a* flow graph *A to be a triple (G_A, s_A, t_A), where $G_A = (V_A, E_A)$ is a finite*[2] *directed connected multigraph and E_A is a multisubset of $V_A \times V_A$. Note that this definition permits parallel and loop edges*[3]*. Given vertices u and v, we denote $\sharp(u, v)$ to be the number of edges from u to v. Individual parallel edges from u to v will be referred to as $(u, v)_1, (u, v)_2, \ldots, (u, v)_i, \ldots, (u, v)_{\sharp(u,v)}$. However, if the argument does not depend on a specific edge from u to v, the subscript will be dropped–the expression (u, v) will be used to mean any one of (possibly many) parallel edges from u to v. The vertices $s_A, t_A \in V_A$ are called the* source *and the* target *vertex of A, respectively. The set of all flow graphs is denoted F.*

Definition 2.2 (Flow graph morphism)**.** *Let $A = (G_A, s_A, t_A)$ and $B = (G_B, s_B, t_B)$ be two flow graphs with $G_A = (V_A, E_A)$ and $G_B = (V_B, E_B)$. A map $\phi : A \to B$ is*

[2] In this paper, we focus on finite flow graphs, although many of our results continue to hold in the formulation which considers infinite flow graphs as well.

[3] We say that two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ are *parallel* if $u_1 = u_2$ and $v_1 = v_2$. An edge $e = (u, v)$ is called a *loop edge* if $u = v$.

called a flow graph morphism *if (1) As a map of vertex sets, $\phi : V_A \to V_B$ respects edge structure: $e = (u, v) \in E_A \Rightarrow \phi(e) = (\phi u, \phi v) \in E_B$ (2) Source and target are preserved, i.e. $\phi(s_A) = s_B$, $\phi(t_A) = t_B$. A flow graph morphism is said to be a flow graph* embedding *of A into B if additionally ϕ is injective on both V_A and E_A. Flow graphs A and B are considered* isomorphic *if there is a flow graph embedding $\phi : A \to B$ for which $\phi(E_A) = E_B$.*

Clearly, flow graph isomorphism defines an equivalence relation on flow graphs. In this paper, we shall only consider properties of flow graphs which are invariant with respect to this equivalence relation. Consequently, when discussing an equivalence class of flow graphs, we will conduct our analysis by restricting ourselves to an arbitrary representative from the class. Whenever we refer to "A flow graph F", we shall intend "Any flow graph from the equivalence class of F", but we will use the former phrase for succinctness. Likewise, we write $A = B$ for flowgraphs to indicate only that A and B are isomorphic as flow graphs.

Definition 2.3 (Trivial flow graph)**.** *A* flow graph *$A = (G_A, s_A, t_A)$ is called* the trivial flow graph *if $|V[G_A]| = 1$ and $|E[G_A]| = 0$. All other flow graphs are considered* non-trivial.

Definition 2.4. *Given any flow graph A, let A' be the flow graph obtained by swapping the source and the target of A.*

Definition 2.5 (Reflective flow graphs)**.** *A* flow graph *$A = (G_A, s_A, t_A)$ is called an* reflective *flow graph if $A = A'$. The set of all reflective flow graphs is denoted $\mathcal{H}$.*

Definition 2.6 (Infinitesimal flow graphs)**.** *A* flow graph *$A = (G_A, s_A, t_A)$ is called an* infinitesimal *flow graph if $s_A = t_A$. The set of all infinitesimal flow graphs is denoted $\mathcal{I}$. Note that an infinitesimal flow graph is necessarily reflective. The converse is false as the reflective example in Figure 1 shows.*

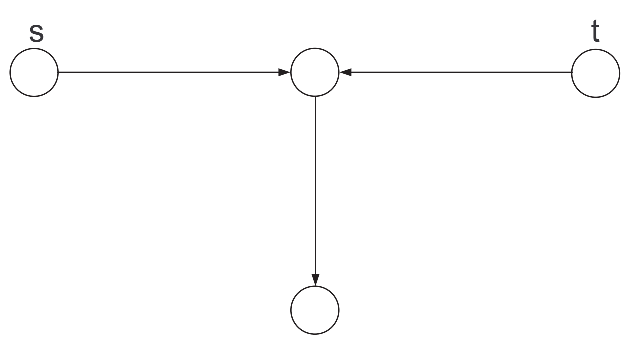

Figure 1: A non-infinitesimal flow graph in $\mathcal{H}$.

Definition 2.7. *Given any flow graph A, let A^* be the flow graph obtained by reversing all the arrows of A.*

Definition 2.8 (Reversible flow graphs)**.** *A flow graph $A = (G_A, s_A, t_A)$ is called a* reversible *flow graph if $A = A^*$. The set of all reversible flow graphs is denoted $\mathcal{J}$. Note that if for all vertices u, v in V_A we have $\sharp(u, v) = \sharp(v, u)$, then A is necessarily reversible. The converse is false as the reversible example in Figure 2 shows.*

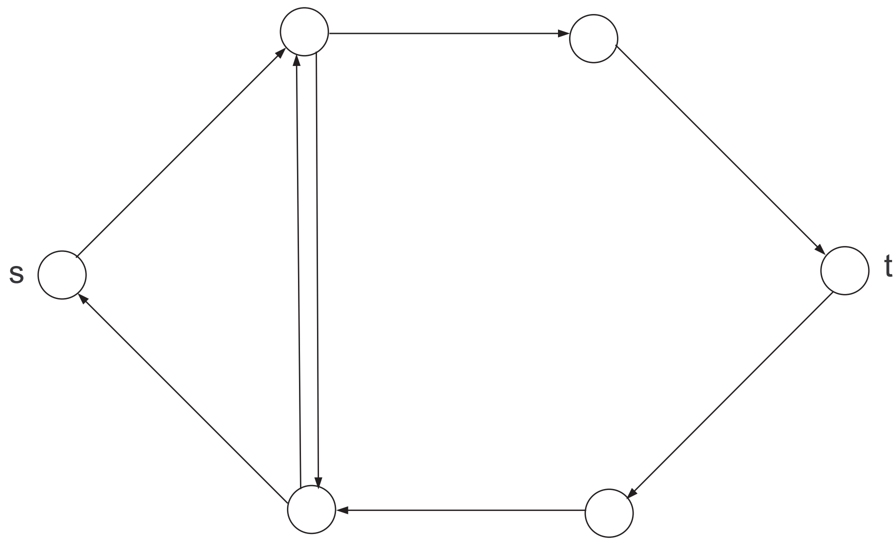

Figure 2: A flow graph in $\mathcal{J}$ having a non-symmetric adjacency matrix.

Definition 2.9 (Self-conjugate flow graphs)**.** *A flow graph $A = (G_A, s_A, t_A)$ is called an* self-conjugate[4] *flow graph if $A = A'^* = A^{*\prime}$.*

The set of all self-conjugate flow graphs is denoted $\mathcal{K}$. Note that if a flow graph is both reflective and reversible, it is necessarily self-conjugate. The converse is false as the self-conjugate example in Figure 3 shows.

Indeed, no two of the sets $\mathcal{H}, \mathcal{J}$, and $\mathcal{K}$ are contained in each other. The flow graph in Figure 1 belongs to $\mathcal{H} \backslash (\mathcal{J} \cup \mathcal{K})$. The flow graph in Figure 2 belongs to $\mathcal{J} \backslash (\mathcal{H} \cup \mathcal{K})$. The flow graph in Figure 3 belongs to $\mathcal{K} \backslash (\mathcal{H} \cup \mathcal{J})$.

Definition 2.10. *The* **rose** with n petals *is defined to be the infinitesimal flow graph R_n having one vertex and n loop edges. Roses R_1, R_2, R_3 are shown in the bottom left panel of Figure 4.*

Definition 2.11. *The* **star (antistar)** with n edges *is defined to be the infinitesimal flow graph S_n (S_n^*) having $n+1$ vertices $v_1, v_2, \ldots, v_n$ and $u = s = t$, with n edges from u to v_i (v_i to u) for each $i = 1, \ldots, n$. Stars S_1, S_2 and S_3 are shown in the bottom center panel of Figure 4, while anti-stars S_1^*, S_2^* and S_3^* are shown on the bottom right panel.*

[4]The motivation for the term *self-conjugate* will be clarified later, in item 4 of Section 3.4.

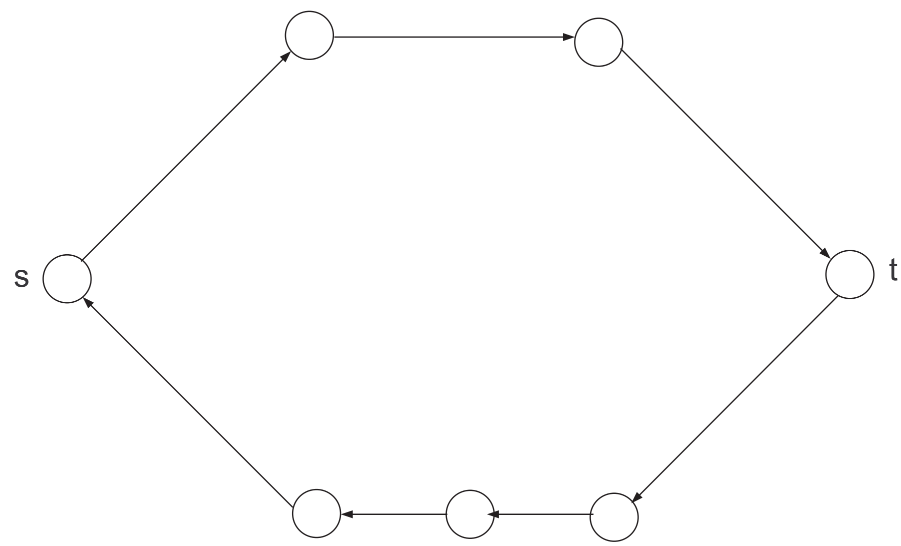

Figure 3: A flow graph in $\mathcal{K}$ that is neither reflective nor reversible.

Definition 2.12 (Graphical natural number)**.** *We represent the natural number n as a directed chain of length n, having $n+1$ vertices. More formally, let P_n be a directed chain of length n (having $n+1$ vertices) where each vertex has in-degree $\leqslant 1$ and out-degree $\leqslant 1$. Denote by s_n, the unique vertex in P_n having in-degree 0, and let t_n be the unique vertex in P_n having out-degree 0. The flow graph $F_n = (P_n, s_n, t_n)$ is referred* the graphic natural number n. *Define the map $i : \mathcal{N} \to \mathcal{F}$ as*

$$i : n \quad \mapsto \quad F_n.$$

We denote F_0 as 0 and F_1 as 1. Graphical natural numbers F_1, F_2 and F_3 are shown in the top left panel of Figure 4, while the corresponding reverse flow graphs are shown in the top right panel.

3. Arithmetic on Flow Graphs

3.1. Addition

In Definition 2.1, we represented the natural number n by the flow graph F_n. It follows that we interpret the addition of two numbers n_1 and n_2 inside $\mathcal{F}$ as "concatenating" F_{n_1} with F_{n_2}. Consider, for example, the addition of 3 and 2 depicted in Figure 5.

To extend this definition of $+$ to all of F, we define general addition of flow graphs as follows: Given two flow graphs A and B, define $A+B$ to be the flow graph obtained by identifying t_A with s_B and defining $s_{A+B} = s_A$ and $t_{A+B} = t_B$. An example of such an addition is shown in Figure 6.

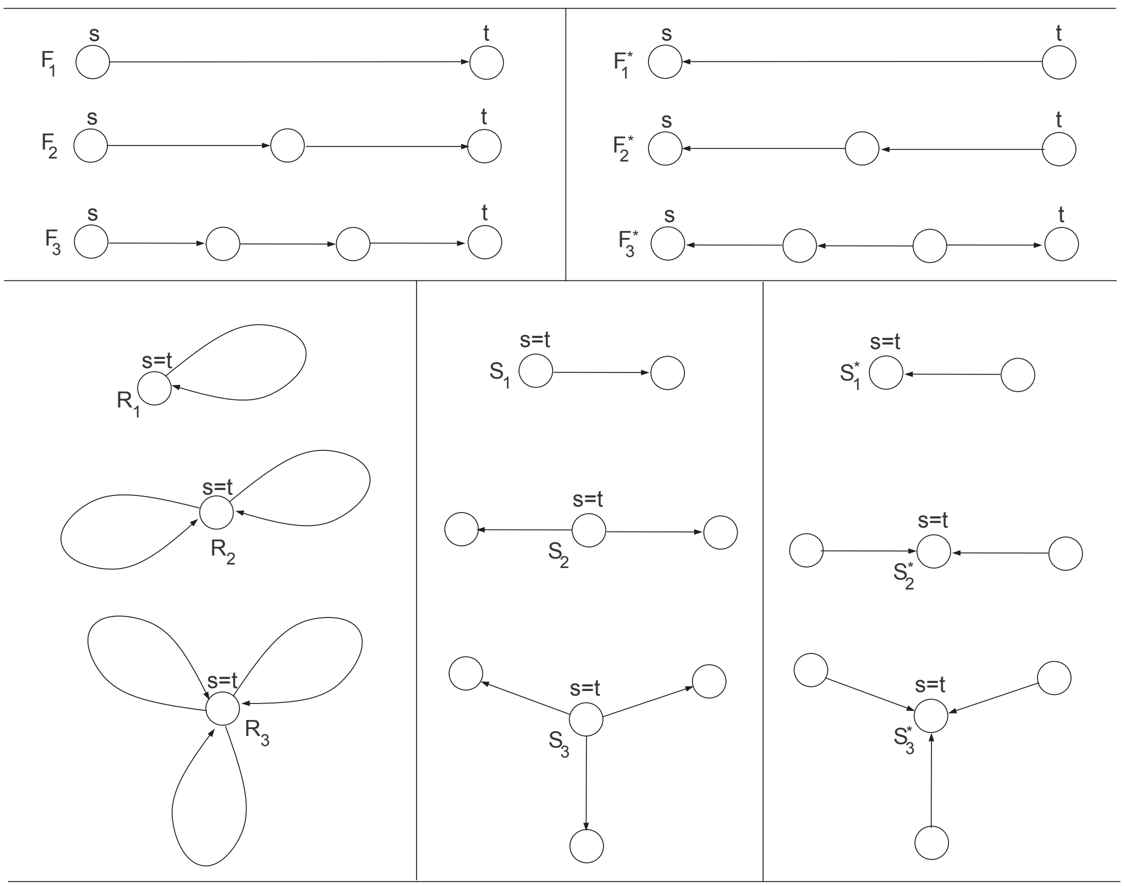

Figure 4: Some examples of special flow graphs: the graphical natural numbers F_1, F_2, F_3, the anti-paths F_1^*, F_2^*, F_3^*, the roses R_1, R_2, R_3, the stars S_1, S_2, S_3, and the anti-stars S_1^*, S_2^*, S_3^*.

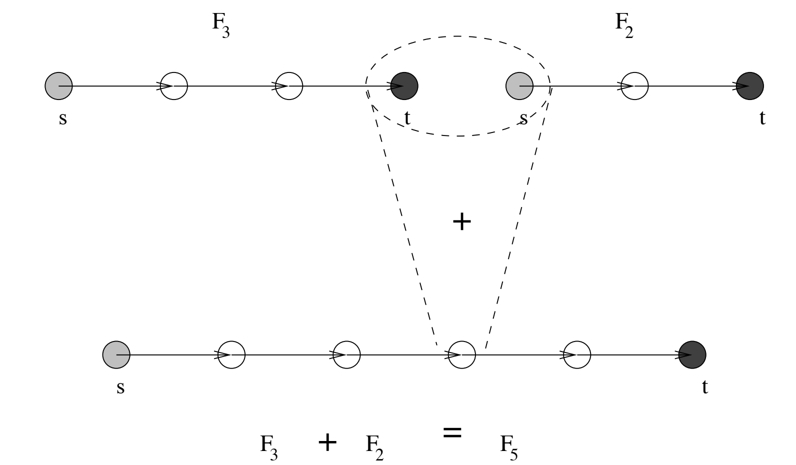

Figure 5: Interpreting addition of natural numbers inside $\mathcal{F}$.

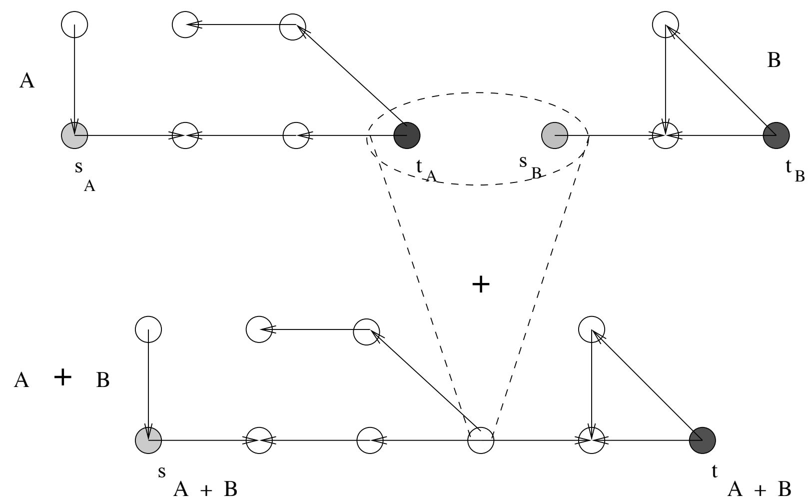

Figure 6: General addition of flow graphs.

We begin by defining the following "Vertex gluing" operation on directed multigraphs:

Definition 3.1 (Vertex gluing of directed graphs)**.** *Given directed graphs G_1 and G_2, and vertices $u_1 \in V[G_1]$, $u_2 \in V[G_2]$, we define*

$$G_1 \underset{u_1 \approx u_2}{+} G_2 \stackrel{def}{=} (G_1 \sqcup G_2)/(u_1 \approx u_2)$$

to be the graph obtained by taking disjoint copies of G_1 and G_2 and identifying vertex u_1 in G_1 with vertex u_2 in G_2. Note the obvious and natural graph embeddings

$$\begin{aligned} \sigma^+_{u_1 \approx u_2} : G_1 &\hookrightarrow G_1 \underset{u_1 \approx u_2}{+} G_2 \\ \tau^+_{u_1 \approx u_2} : G_2 &\hookrightarrow G_1 \underset{u_1 \approx u_2}{+} G_2.. \end{aligned} \tag{1}$$

Now we can define addition of flow graphs:

Definition 3.2. *Given two flow graphs $A = (G_A, s_A, t_A)$ and $B = (G_B, s_B, t_B)$, we define*

$$A+B \stackrel{\text{def}}{=} (G_A \underset{t_A \approx s_B}{+} G_B, s_A, t_B).$$

Since A and B are connected, it follows that $A+B$ is connected.

The next lemma follows immediately from Definitions 2.12 and 3.2.

Lemma 3.3. *Let m, n be natural numbers. Then $i(n + m) = i(n)+i(m)$.*

We present some properties of $+$.

Lemma 3.4. *The operation $+$ is associative.*

Proof. Given flow graphs A, B, C,

$$\begin{aligned}
(A+B)+C &= (G_A \underset{t_A \approx s_B}{+} G_B, s_A, t_B)+C \\
&= ((G_A \underset{t_A \approx s_B}{+} G_B) \underset{t_B \approx s_C}{+} G_C, s_A, t_C) \\
&= (G_A \underset{t_A \approx s_B}{+} (G_B \underset{t_B \approx s_C}{+} G_C), s_A, t_C) \\
&= A+(G_B \underset{t_B \approx s_C}{+} G_C, s_B, t_C) \\
&= A+(B+C).
\end{aligned}$$

□

One can check that $1+R_1 \neq R_1+1$. Thus we obtain

Lemma 3.5. *The operation $+$ is not commutative.*

Definition 3.6. *A flow graph A is called* $+$-reducible *if there exist non-trivial flow graphs B, C, such that $A = B+C$. Otherwise, A is called $+$-irreducible.*

Definition 3.7 (Scalar multiplication of flow graphs)**.** *Given a flow graph A, and a positive natural number k in $\mathbb{N}$, we define left scalar multiplication inductively as follows:*

$$\begin{aligned}
1A &= A \\
kA &= (k-1)A+A.
\end{aligned}$$

Right scalar multiplication is defined analogously. As we have seen, $+$ is associative, and so the two notions coincide. We shall subsequently consider only left scalar multiplication by integer scalars.

Remark 3.8. *Note that if A is a flow graph with p_A vertices and q_A edges, and B is a flow graph with p_B vertices and q_B edges, then $A+B$ is a flow graph having $p_A + p_B - 1$ vertices and $q_A + q_B$ edges.*

3.2. Multiplication

In the previous section, we presented an interpretation of addition in $\mathcal{F}$ that is a natural extension of addition on the natural numbers. In this section, we present an interpretation of multiplication in $\mathcal{F}$ which generalizes the multiplication of natural numbers. In doing this, we must respect the fact that for each pair of natural numbers n_1, n_2, the following identity holds in $\mathcal{N}$:

$$\underbrace{n_2 + n_2 + \cdots + n_2}_{n_1 \text{ times}} = n_1 n_2 = \underbrace{n_1 + n_1 + \cdots + n_1}_{n_2 \text{ times}}.$$

So, in particular, the definition of multiplication in $\mathcal{F}$ must satisfy

$$n_1 F_{n_2} = F_{n_1} F_{n_2} = F_{n_1} n_2. \tag{2}$$

Given that we represent the natural number n by the flow graph F_n, the product of two graphical numbers F_{n_1} and F_{n_2} (denoted $F_{n_1} F_{n_2}$) can be made to satisfy relation (2) if we take multiplication to be the act of replacing each edge of F_{n_2} with a copy of F_{n_1}. For example, the multiplication of graphical natural numbers F_3 and F_2 is illustrated in Figure 7.

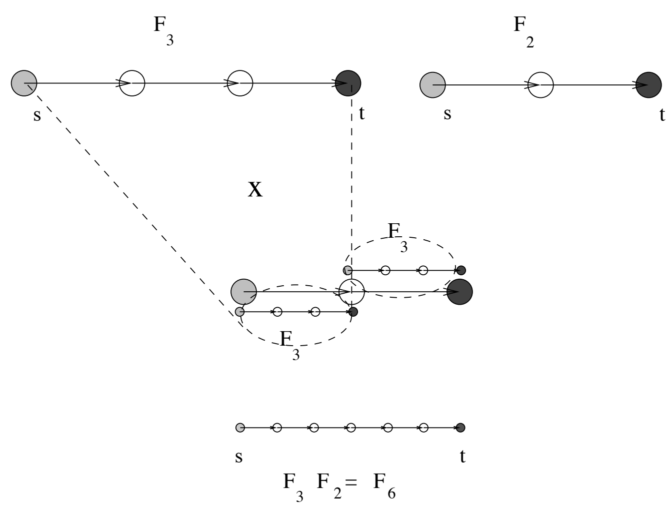

Figure 7: Standard multiplication of natural numbers in $\mathcal{F}$ (represented as flow graphs).

To extend this definition of multiplication to all of F, we define general multiplication of flow graphs as follows: Given two flow graphs A and B, define AB to be the flow graph obtained by replacing every edge e (from $E[G_B]$) with a copy of A as follows: For each edge $e = (u, v)$ in B, we remove e and replace it with a graph A_e isomorphic to A, by identifying u with s_{A_e}, and v with t_{A_e}. An example of such a multiplication operation is shown in Figure 8. We now formally define multiplication of flow graphs:

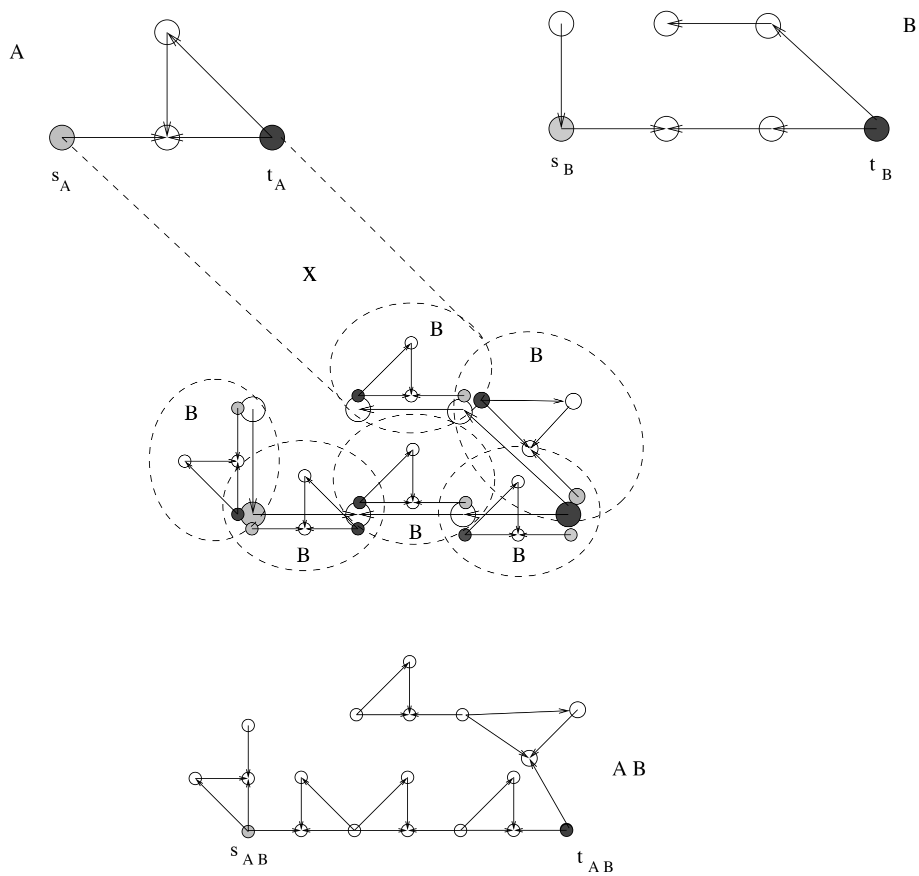

Figure 8: General multiplication of flow graphs.

Definition 3.9. *Let $A = (G_A, s_A, t_A)$ and $B = (G_B, s_B, t_B)$ be any two flow graphs. We define an equivalence relation $\sim_R$ on $V_A \times E_B$, as follows: Given vertices u_1, u_2 in V_A, and edges $e_1 = (v_1, w_1)$ and $e_2 = (v_2, w_2)$ in E_B, let $(u_1, (v_1, w_1)) \sim_R (u_2, (v_2, w_2))$ iff the following holds: whenever u_1 is the source (target) and u_2 is the source (target) then (respectively) the tail (head) of e_1 coincides with the tail (head) of e_2 in B. Then $\sim_R$ is an equivalence relation.*

We define the flow graph $AB = (G_{AB}, s_{AB}, t_{AB})$ as follows. Let $G_{AB} = (V_{AB}, E_{AB})$, where $V_{AB} = (V_A \times E_B)/\sim_R$ and $((u_1, e_1), (u_2, e_2)) \in E_{AB}$ if $(u_1, u_2) \in E_A$ and $e_1 = e_2$ in E_B. Define $s_{AB} = (s_A \times e)/\sim_R$ where $e = (s_B, w)$ for any $w \in V_B$ and $t_{AB} = (t_A \times e)/\sim_R$ where $e = (v, t_B)$ for any $v \in V_B$.

Since A and B are connected, it follows that AB is connected.

We remark that there is an obvious symmetric definition for multiplication in which the roles of two flow graphs being multiplied is exchanged. To remain in agreement with conventions of ordinal and poset multiplication established by Cantor [3] and others subsequently [4, 7, 8], we chose the definition above.

We now present some properties of multiplication.

Lemma 3.10. *Let A be a flow graph with p_A vertices and q_A edges, and B be a flow graph having p_B vertices and q_B edges. Then AB has $q_A q_B$ edges. If A is either trivial or infinitesimal then AB has $1+q_B(p_A-1)$ vertices. If A is non-trivial and non-infinitesimal then AB has $p_B + q_B(p_A-2)$ vertices.*

Proof. The flow graph AB is obtained by replacing each edge $e = (u, v)$ in G_B with a copy of A as follows: remove e from G_B and replace it with a flow graph A_e isomorphic to A, identifying u with s_{A_e} and v with t_{A_e}. Thus each edge of G_B produces q_A edges in AB and so the by doing the same operation with every edge of G_B, we see that the number of edges in AB will be $q_A q_B$. If A is non-trivial and non-infinitesimal, then each edge $e = (u, v)$ of G_B produces in addition to its end vertices u and v, an additional $(p_A - 2)$ vertices. Thus the number of vertices in AB is $p_B + q_B(p_A - 2)$. If A is trivial, then each edge of G_B under this operation of multiplication by A collapses into one vertex and sequentially applying this operation to all edges results in the graph AB which consists of a single vertex with no edges, that is, results in a trivial graph. If A is infinitesimal and non-trivial, then each edge $e = (u, v)$ is replaced by a copy of A with $s_{A_e} = t_{A_e}$ identified with u collapsed with v . Thus the number of vertices produced by an edge $e = (u, v)$ besides the collapsed vertex $u = v$ is $(p_A - 1)$. Hence the total number of vertices in AB is $1 + q_B(p_A - 1)$. □

Lemma 3.11. *Flow graph multiplication is associative.*

Proof. Given flow graphs $A = (G_A, s_A, t_A)$, $B = (G_B, s_B, t_B)$, $C = (G_C, s_C, t_C)$, we want to show:

$$(AB)C \quad = \quad A(BC).$$

We define a bijection Λ between the vertices $(AB)C$ and the vertices of $A(BC)$, and then show that Λ respects the edge relation. Let

$$\Lambda : ((v_1, (v_2, w_2)), e_3) \mapsto (v_1, ((v_2, e_3), (w_2, e_3))),$$

where v_1 is any vertex in A, v_2 and w_2 are two vertices in B, and e_3 is any edge in C. An edge in $(AB)C$ is of the form

$$(\ ((v_1, (v_2, w_2)), e_3),\ ((v_1', (v_2, w_2)), e_3)\ ),$$

where $(v_1, v_1') \in E[G_A]$. The image of this edge under Λ is

$$(\ v_1, ((v_2, e_3), (w_2, e_3))\ )$$

which is an edge in $A(BC)$. Hence $(AB)C$ is a subgraph of $A(BC)$. Proceeding in the same way using Λ^{-1}, one can show that $A(BC)$ is a subgraph of $(AB)C$. The Lemma is proved. □

One can check that $R_1F_2 \neq F_2R_1$. Thus we obtain

Lemma 3.12. *Flow graph multiplication is not commutative.*

The next lemma follows immediately from Definitions 2.12 and 3.9.

Lemma 3.13. *Let m, n be natural numbers. Then $i(n \times m) = i(n)i(m)$.*

Definition 3.14 (Scalar exponentiation of flow graphs)**.** *Given a flow graph A, and a positive natural number k in $\mathbb{N}$, we define right-exponentiation inductively as follows:*

$$\begin{aligned} A^1 &= A \\ A^k &= A^{k-1}A. \end{aligned}$$

Left-exponentiation is defined analogously. As we have seen, $\times$ is associative, and so the two notions coincide. We shall subsequently consider only right-exponentiation by integer scalars.

3.3. Zero Divisors and Units

The next lemma shows that $\mathcal{F}$ has no members which behave like zero divisors.

Lemma 3.15. *Given flow graphs G and H:*

$$GH = 0 \quad \Leftrightarrow \quad H = 0 \quad \text{or} \quad G = 0. \tag{3}$$

Proof. If $G = 0$ then $GH = HG = 0$. For the reverse implication, we appeal to Lemma 3.10, noting that $GH = 0$ implies $q_G q_H = 0$, so either $q_G = 0$ or $q_H = 0$. It follows that either $H = 0$ or $G = 0$. □

The next Lemma shows that the only units are 1 and $1'$.

Lemma 3.16. *Given flow graphs G and H:*

$$GH = 1 \quad \Leftrightarrow \quad G = 1 = H \quad \text{or} \quad G = 1' = H. \tag{4}$$

Proof. By Lemma 3.10, we know that $q_G q_H = 1$, hence $q_G = 1$ and $q_H = 1$. It follows that $G, H \in \{F_1, F_1', S_1, S_1^*, R_1\}$. Since 1 is not infinitesimal, it follows that $G, H \in \{F_1, F_1'\}$ Then since

$$\begin{aligned} F_1F_1' = F_1'F_1 = F_1' &\neq 1 \\ F_1F_1 = F_1'F_1' &= 1 \end{aligned}$$

the result follows. □

3.4. Structural Unitary Operators

The unary operations of $'$ (Definition 2.4) and $*$ (Definition 2.7) interact nicely with the binary operations of addition and multiplication. The following identities are easy to verify:

1. Nilpotency of * and $'$ operations: $A^{**} = A = (A')'$

2. Distributivity of * and $'$ over addition and multiplication:

$$\begin{aligned} (A{+}B)^* &= A^*{+}B^* \\ (A{+}B)' &= B'{+}A' \\ (AB)^* &= A^*B \\ (AB)' &= AB' \end{aligned}$$

3. Multiplicative definitions of $'$ and *:

$$\begin{aligned} A^* &= (1^*)A \\ A' &= A(1') \end{aligned}$$

4. Commutativity of $'$ and * operations: $(A^*)' = (A')^* = 1^*A1^*$.

   Note that this identity is the justification for the term *self-conjugate* in Definition 2.9, since if $A = A'^*$, then $A = 1^*A1^*$, and thus a self-conjugate graph A is isomorphic to itself conjugated by the only non-identity unit 1^*.

3.5. Identity

Lemma 3.17. *The flow graph $0 \stackrel{\text{def}}{=} F_0$ is the unique one-sided identity on each side with respect to $+$. That is, for all flow graphs $A, G \in \mathcal{F}$,*

$$A{+}G = A \quad\Leftrightarrow\quad G = 0 \quad\Leftrightarrow\quad G{+}A = A.$$

Proof. If $G = 0$ then $A{+}G = G{+}A = A$. For the reverse implication, we appeal to Remark 3.8, noting that $A{+}G = A$ implies $p_A + p_G - 1 = p_A$ and $q_A + q_G = q_A$. Hence $p_G = 1$ and $q_G = 0$, so $G = 0$. An analogous argument shows that $G{+}A = A$ implies $G = 0$. □

We note $F_n^* = F_n{}'$ for all n. Considering addition,

$$F_n{+}F_m = (F_n^*{+}F_m^*)^* = (F_m{}'{+}F_n{}')' = F_{m+n}.$$

Considering multiplication,

$$\begin{aligned} F_n F_m = F_n^* F_m^* = {F_n}' {F_m}' &= F_{mn} \\ F_n F_m^* = F_n^* F_m &= F_{mn}^* \\ F_n F_m' = F_n' F_m &= F_{mn}' \end{aligned}$$

These observations are mirrored in the natural numbers, where for any n, m, we have that

$$\begin{aligned} n + m &= -(-n + -m) \\ nm &= (-n)(-m) \\ n(-m) &= (-n)m = -(nm). \end{aligned}$$

Thus, we suggest viewing the $'$ and * operations as two different kinds of "negation" on flow graphs, considering ${F_n}' = F_n^*$ to be different interpretations of the number[5] $-n$. Following this metaphor, the reversible flow graphs $\mathcal{J}$ and reflective flow graphs $\mathcal{H}$ have the property of being isomorphic to their own negations. We shall see that the behavior of multiplication will satisfy certain identities as long as the parameters lie outside of these two pathological sets, in much the same way that certain multiplicative identities hold for the natural numbers as long as certain parameters are assumed to be nonzero.

We now consider the right multiplicative identity. Note that for any flow graph H, $H = H1$ and $H' = H1'$. So if H is a reflective flow graph then $H = H1'$, hence 1 and $1'$ are both right identities on $\mathcal{H}$. The next lemma shows that on $F \backslash \mathcal{H}$, there is a unique right identity, 1.

Lemma 3.18. *Let G, H be non-trivial flow graphs with $G \notin \mathcal{I}$ and $H \notin \mathcal{H}$. Then*

$$HG = H \quad \Leftrightarrow \quad G = 1. \tag{5}$$

Proof. If $G = 1$ then $HG = GH = H$. For the reverse implication, we appeal to Lemma 3.10, noting that $HG = H$ implies $p_G + q_G(p_H - 2) = p_H$ and $q_H q_G = q_H$. Hence $q_G = 1$, $p_G = 2$ and so $G \in \{1, 1'\, S_1, S_1^*\}$. But G cannot be S_1 or S_1^* since G is not infinitesimal. Likewise, G cannot be $1'$ since $H1' = H'$ and $H' \neq H$ since $H \notin \mathcal{H}$. It follows that $G = 1$. □

We now turn to the existence of left identity. Note that if $H \in \mathcal{J}$, then $1H = H = H^* = 1^* H$ so both 1 and 1^* are left identities on $\mathcal{J}$. If $H \in \{S_n \mid n \in \mathbb{N}\}$ then $S_1 H = H$, so both 1 and S_1 are left identities on $\{S_n \mid n \in \mathbb{N}\}$. If $H \in \{S_n^* \mid n \in \mathbb{N}\}$ then $S_1^* H = H$, so both 1 and S_1^* are left identities on $\{S_n^* \mid n \in \mathbb{N}\}$. The next lemma shows that on $F \backslash (\mathcal{J} \cup \{S_n \mid n \in \mathbb{N}\} \cup \{S_n^* \mid n \in \mathbb{N}\})$ there is a unique left identity, 1.

[5] The metaphor holds only up to a point, however, since for $n > m > 0$, we have $F_n + F_m^* \neq F_{n-m}$. Cancellation is not witnessed between positive and "negative" flow graphs.

Lemma 3.19. *Let G, H be non-trivial flow graphs with $H \notin \mathcal{J}$ and $S_1H \neq H$ and $H \neq S_1^*H$. Then*

$$GH = H \quad \Leftrightarrow \quad G = 1. \tag{6}$$

Proof. If $G = 1$ then $GH = H$. For the reverse implication, we appeal to Lemma 3.10.

a. Suppose $G \in \mathcal{I}$. Then $GH = H$ implies $1+q_H(p_G-1) = p_H$ and $q_Hq_G = q_H$. Hence $q_G = 1$, so $G \in \{1, 1^*, R_1, S_1, S_1^*\}$. Since $G \in \mathcal{I}$, it cannot be 1 or 1^*. Likewise, G cannot be R_1 since $GH = H$ implies $R_1H = R_{q_H}$ and so $H = R_{q_H}$, contradicting that $H \notin \mathcal{J}$. Suppose $G = S_1$ (or S_1^*) then $GH = H$ implies that $H = S_{q_H}$ (or $H = S^*_{q_H}$) which contradicts the hypothesis $S_1H \neq H$ (or $H \neq S_1^*H$). Hence G cannot be in $\mathcal{I}$.

b. So now we consider $G \notin \mathcal{I}$. Then $GH = H$ implies $p_H + q_H(p_G - 2) = p_H$ and $q_Hq_G = q_H$. Hence $q_G = 1$ which implies $G \in \{1, 1^*, R_1, S_1, S_1^*\}$. Since $G \in \mathcal{I}$, it must be 1 or 1^*. But G cannot be 1^* since $1^*H = H^*$ and $H^* \neq H$ since $H \notin \mathcal{J}$. It follows that $G = 1$.

□

3.6. Infinitesimals

The following observations motivate our choice of the term *infinitesimal* for flow graphs whose source and target vertices coincide.

Proposition 3.20. *Let B and C be non-trivial flow graphs. Then $B+C$ is infinitesimal, if and only if both B and C are infinitesimal.*

Proof. If B (resp. C) is not infinitesimal then $s_B \neq t_B$ (resp. $s_C \neq t_C$), hence $s_{B+C} \neq t_{B+C}$. So $B+C$ is not infinitesimal.

If B and C are infinitesimal then $s_B = t_B$ and $s_C = t_C$ hence $s_{B+C} = t_{B+C}$. So $B+C$ is infinitesimal. □

The next Proposition shows that with respect to multiplication, the set of infinitesimals behaves, in some sense, like a prime ideal inside F.

Proposition 3.21. *Let G and H be non-trivial flow graphs, then GH is infinitesimal if and only if at least one of the two factors is infinitesimal.*

Proof. If H or G is infinitesimal, then $s_{GH} = t_{GH}$ in GH and $s_{HG} = t_{HG}$ in HG. Hence GH and HG are both infinitesimal.

On the other hand, suppose G and H are non-trivial flow graphs that are both non-infinitesimal. Then $s_{GH} \neq t_{GH}$ in GH and $s_{HG} \neq t_{HG}$ in HG. Hence GH and HG are both non-infinitesimal. □

The reader may wish to compare the above Proposition with assertion (3) of Lemma 3.15 which showed that $\{0\}$ also behaves, in some sense, like a prime ideal inside F. The next two propositions show that $\mathcal{H}$ and $\mathcal{J}$ behave like one-sided ideals in $\mathcal{F}$.

Proposition 3.22. *Let G be any flow graph and H be a reflective flow graph. Then GH is a reflective flow graph.*

Proof. $(GH)' = GH' = GH$, since $H = H'$. □

Proposition 3.23. *Let G be a reversible flow graph and H be any flow graph. Then GH is a reversible flow graph.*

Proof. $(GH)^* = G^*H = GH$, since $G = G^*$. □

3.7. Infinitesimalizing Unary Operators

It is also possible to define natural infinitesimalizing unary operations on flow graphs. We introduce the following:

Definition 3.24. *Given a flow graph A, define*

- *A^+ as the graph A with the target moved down to coincide with the source.*
- *A^- as the graph A with the source moved up to coincide with the target.*
- *A° as the graph A with the source and target nodes identified.*

The following identities are easy to verify:

5. Idempotency: Given two operations $x, y \in \{+, -, \circ\}$:

$$(A^x)^y = A^x.$$

   More generally: applying $^+, ^-$ or $^\circ$ to an infinitesimal has no effect, and $A = A^\circ = A^+ = A^-$ if and only if A is infinitesimal.

The infinitesimalizing operators interact nicely with the unary operations of $'$ (Definition 2.4) and * (Definition 2.7). The following identities are easy to verify:

6. Commutativity of * with any operation $x \in \{+, -, \circ\}$: $(A^x)^* = (A^*)^x$

7. Interaction of $'$ with infinitesimalizing operations:

$$\begin{aligned} (A')^+ = \quad & A^- \quad = (A^-)' \\ (A')^- = \quad & A^+ \quad = (A^+)' \\ (A')^\circ = \quad & A^\circ \quad = (A^\circ)' \end{aligned}$$

8. Multiplicative definitions of $^+$, $^-$ and $^\circ$:

$$\begin{aligned} A^+ &= A1^+ = AS_1 \\ A^- &= A1^- = AS_1^* \\ A^\circ &= A1^\circ = AR_1 \end{aligned}$$

9. Non-distributivity of $^+$, $^-$ and $^\circ$ over addition and multiplication. Taking $A = B = F_2$, one sees:

$$\begin{aligned} (A{+}B)^+ &\neq A^+{+}B^+ \\ (A{+}B)^- &\neq A^-{+}B^- \\ (A{+}B)^\circ &\neq A^\circ{+}B^\circ \\ (AB)^+ &\neq A^+B^+ \\ (AB)^- &\neq A^-B^- \\ (AB)^\circ &\neq A^\circ B^\circ \end{aligned}$$

10. Left-identities other than 1 on stars, anti-stars, and roses:

$$\begin{aligned} S_kA = \quad & k1^+A \quad = S_{k|E[G_A]|} \\ S_k^*A = \quad & k1^-A \quad = S^*_{k|E[G_A]|} \\ R_kA = \quad & k1^\circ A \quad = R_{k|E[G_A]|}. \end{aligned}$$

Taking $k = 1$ and A to be a star, anti-star, or rose (respectively), the above identities show that S_1, S_1^* and R_1 act as left identities on the set of stars, anti-stars, and roses (respectively).

4. Order

Given our representation of the natural number n by the flow graph F_n in Definition 2.12, comparing the order of two numbers n_1 and n_2 (as flow graphs) requires simply comparing the lengths of the corresponding chain graphs F_{n_1} and F_{n_2}. To generalize this to all of $\mathcal{F}$, however, we cannot refer to "length". In what follows, we present two possible interpretations of $\leqslant$ in $\mathcal{F}$. To avoid confusion, we denote these distinct interpretations by the symbols $\preccurlyeq$, and $\ll$—these are referred to as the strong and induced orders respectively.

4.1. Strong Order $\preccurlyeq$

We now define an ordering on $\mathcal{F}$. Given two flow graphs A and B, informally, we say that $A \preccurlyeq B$ iff two copies of G_A appear in G_B; one as a neighborhood of s_B and one as a neighborhood of t_B. The next definition makes this statement precise.

Definition 4.1 (Strong order)**.** *Given two flow graphs* $A = (G_A, s_A, t_A)$ *and* $B = (G_B, s_B, t_B)$*, we say* $A \preccurlyeq B$ *iff there are graph embeddings*[6] *where* $\phi_s : G_A \to G_B$ *and* $\phi_t : G_A \to G_B$ *which satisfy* $\phi_s(s_A) = s_B$ *and* $\phi_t(t_A) = t_B$*.*

Consider the comparison of F_3 and F_5 depicted in Figure 9; clearly $F_3 \preccurlyeq F_5$.

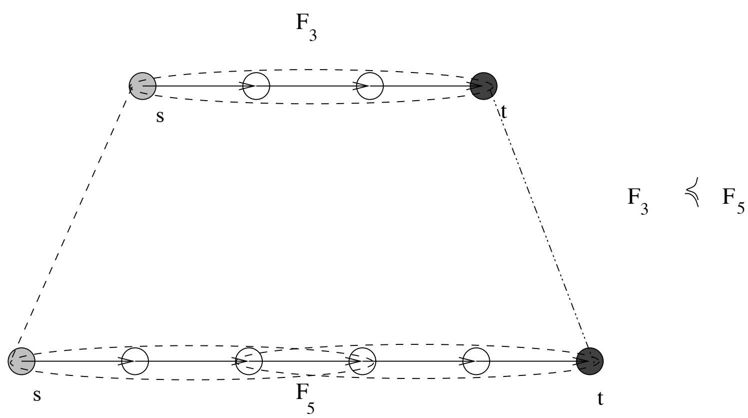

Figure 9: Standard strong ordering of natural numbers (represented as flow graphs).

The proof of the following lemma is immediate.

Lemma 4.2. *Let* m, n *be natural numbers. Then* $n \leqslant m \Leftrightarrow i(n) \preccurlyeq i(m)$*.*

Figure 10 illustrates a more general example in which strong order is used to compare two elements of $\mathcal{F}$ which are *not* graphical natural numbers.

[6] These embeddings are merely directed graph embeddings whose image need not be an induced subgraph of G_B).

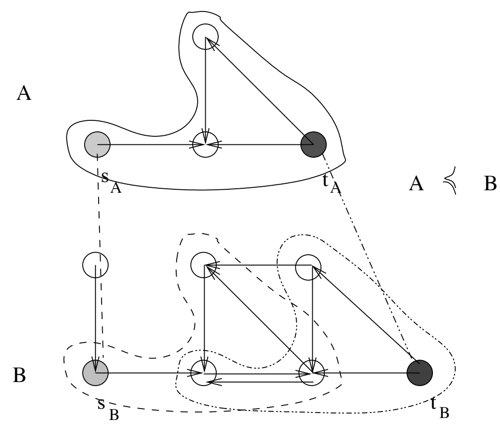

Figure 10: General strong ordering of flow graphs.

The next Proposition follows immediately from Lemmas 3.3, 3.17, 3.13, 3.18, and 4.2.

Proposition 4.3. *Under the embedding $i : n \mapsto F_n$, the standard model $\mathcal{N} = \langle \mathbb{N}, 0, 1, \leqslant, +, \times \rangle$ is a submodel of $\mathcal{F} = \langle F, 0, 1, \preccurlyeq, +, \times \rangle$, where $0 = F_0$, $1 = F_1$, and the relations $+$, $\times$ and $\preccurlyeq$ reinterpret $+, \times$ and $\leqslant$ inside $\mathcal{F}$.*

4.2. Induced Order $\ll$

We now give an alternate ordering on $\mathcal{F}$. Given two flow graphs A and B, informally, we say that $A \ll B$ iff B can be transformed into A by a series of edge contractions[7]. The next two definitions make this statement precise.

Definition 4.4 (Edge contraction)**.** *Given a flow graphs $A = (G_A, s_A, t_A)$ and an edge $e = (u, v) \in E[G_A]$, the flow graph A/e is obtained from A by deleting e in G_A and identifying vertex u with v. If u or v was the source (resp. target) of A, then the identified vertex $u \approx v$ will be taken as the source (resp. target) of A/e.*

The next two observations consider the effect of contracting an edge $e = (u, v)$ in a flow graph $A = (G_A, s_A, t_A)$.

Observation 4.5. $|E[G_{A/e}]| = |E[G_A]| - 1$. *If e is a non-loop edge then $|V[G_{A/e}]| = |V[G_A]| - 1$; if e is a loop edge then $|V[G_{A/e}]| = |V[G_A]|$.*

Definition 4.6. *Let $A = (G_A, s_A, t_A)$ be a flow graph, where $G_A = (V, E)$. Fix $X \subset E$ and define an equivalence relation R_X on the vertices of A by taking $(v_1, v_2) \in R_X$ iff v_1 and v_2 are in the same connected component of (V, X). We define G/R_X to be the graph obtained by considering the quotient of the edge relation E by the equivalence relation R_X. Note that the vertex set of G/R_X is $\{[v] \mid v \in V\}$.*

[7]The induced order was the outcome of discussions held when these results were presented at the City University of New York Logic Workshop, September 2004.

Definition 4.7 (Induced order)**.** *Given two flow graphs $A = (G_A, s_A, t_A)$ and $B = (G_B, s_B, t_B)$, we say $A \ll B$ iff there is set of edges $X \subset E[G_B]$ such that B/R_X is isomorphic to A.*

The proof of the following lemma is immediate.

Lemma 4.8. *Let m, n be natural numbers. Then $n \leqslant m \Leftrightarrow i(n) \ll i(m)$.*

Observation 4.9. *Given two vertices u and v in $V[G_A]$, the distance between u and v in $G \backslash e$ does not exceed the distance between u and v in G.*

There is no obvious relationship between induced order and the afforementioned strong orders.

Figure 13 shows flow graphs A and B for which the strong order relationship $A \preccurlyeq B$ holds. However, since $d_B(s_B, t_B) = 1$ and $d_A(s_A, t_A) = 2$, by Observation 4.9 no sequence of edge contractions can transform B into A, and hence $A \not\ll B$.

In the reverse direction, let $A = F_1 + R_1 + F_1$ and $B = F_2 + R_1 + F_2$. Then $A \ll B$ since each F_2 summand in B can be edge contracted to become F_1. Note that A contains a unique vertex with a loop edge attached, and this vertex is at distance 1 from s_A and t_A. In contrast, in B there is a unique vertex with a loop edge attached, and this vertex is at distance 2 from s_B and t_B. It follows that $A \not\preccurlyeq B$.

The next Proposition follows from Lemmas 3.3, 3.17, 3.13, 3.18, and 4.8.

Proposition 4.10. *Under the embedding $i : n \mapsto F_n$, the standard model $\mathcal{N} = \langle \mathbb{N}, 0, 1, \leqslant, +, \times \rangle$ is a submodel of $\mathcal{F} = \langle F, 0, 1, \ll, +, \times \rangle$, where $0 = F_0$, $1 = F_1$, and the relations $+$, $\times$ and $\ll$ reinterpret $+, \times$ and $\leqslant$ inside $\mathcal{F}$.*

The unary operations of $'$ (Definition 2.4) and $*$ (Definition 2.7) interact nicely with the two orders $\preccurlyeq$ and $\ll$. The following assertions are easily verified.

$$
\begin{array}{ccc|ccc}
A \preccurlyeq B & \Leftrightarrow & A' \preccurlyeq B' & A \ll B & \Leftrightarrow & A' \ll B' \\
A \preccurlyeq B & \Leftrightarrow & A^* \preccurlyeq B^* & A \ll B & \Leftrightarrow & A^* \ll B^* \\
A \preccurlyeq B & \Rightarrow & A^+ \preccurlyeq B^+ & A \ll B & \Rightarrow & A^+ \ll B^+ \\
A \preccurlyeq B & \Rightarrow & A^- \preccurlyeq B^- & A \ll B & \Rightarrow & A^- \ll B^- \\
A \preccurlyeq B & \Rightarrow & A^\circ \preccurlyeq B^\circ & A \ll B & \Rightarrow & A^\circ \ll B^\circ
\end{array}
$$

The last three implications are not reversible, since:

- If $A = F_1$, $B = F_1^+$, then $A^+ \preccurlyeq B^+$ and $A^+ \ll B^+$ but $A \not\preccurlyeq B$ and $A \not\ll B$.
- If $A = F_1$, $B = F_1^-$, then $A^- \preccurlyeq B^-$ and $A^- \ll B^-$ but $A \not\preccurlyeq B$ and $A \not\ll B$.
- If $A = F_1$, $B = F_1^\circ$, then $A^\circ \preccurlyeq B^\circ$ and $A^\circ \ll B^\circ$ but $A \not\preccurlyeq B$ and $A \not\ll B$.

4.3. Other embeddings of $\mathcal{N}$ into $\mathcal{F}$

Propositions 4.3 and 4.10 show that the set of all graphical natural numbers $\{F_n \mid n \in \mathbb{N}\}$ induces a submodel of $\mathcal{F}$ that is isomorphic to $\mathcal{N}$. There are other embeddings of $\mathcal{N}$ into $\mathcal{F}$. For example, consider the set of roses R_n (Definition 2.10). As a substructure of the flowgraphs, these are isomorphic to $\mathcal{N}$, since $R_{n+m} = R_n + R_m = R_m + R_n$, and $R_{mn} = R_m R_n = R_n R_m$ for all $n, m, \in \mathbb{N}$. Note that R_1 is not a multiplicative identity on all of $\mathbb{F}$, but it is on the subset of roses. Alternatively, we can embed $\mathcal{N}$ into the infinitesimals using either the stars S_n or the anti-stars S_n^* (Definition 2.11). Let us define:

$$\begin{aligned} i_{F^*} : n &\mapsto F_n^* \\ i_R : n &\mapsto R_n \\ i_S : n &\mapsto S_n \\ i_{S^*} : n &\mapsto S_n^*. \end{aligned}$$

By carrying out a similar analysis for these functions, one can show that

$$i_{F^*}, i_R, i_S, i_{S^*} : \mathcal{N} \hookrightarrow \mathcal{F}$$

are embeddings of structures, and thus the submodels induced by their images in $\mathcal{F}$ (i.e. set of all anti-paths (F_n^*), roses, stars, and anti-stars) are each isomorphic to the natural numbers. As we shall see, however, there are aesthetic advantages to the mapping which represents the natural number n by the flow graph F_n (e.g. Proposition 5.12, pp. 26).

5. Properties of Flow Graphs

In this section we show that $\times$ left-distributes over $+$ but does not right-distribute. We define left and right divisibility of flow graphs, and show that right divisibility distributes over $+$, but left divisibility does not. We introduce the notion of a prime flow graph, and show that the concepts of left-prime and right-prime coincide. Finally, we explore the properties and relationships of the different orders, and describe the interaction between the orders introduced in Section 4 and the operations of $+$ and $\times$.

5.1. Multiplicative Properties

Lemma 5.1 (Left-distributivity of $\times$ over $+$)**.** *For any flow graphs A, B, C,*

$$C(A+B) = CA + CB$$

Proof. Fix $e \in E[G_{C(A+B)}]$. Then define $\beta_0(e) = \Lambda_{C,A+B}(e)$. Note that $\beta_0(e) = (f, e')$, where f is an edge in $E[G_C]$ and e' is an edge in $E[G_{A+B}]$. Define $\beta_1 : E[G_{A+B}] \to E[G_A] \cup E[G_B]$ so that

$$\beta_1(e) \quad = \quad \begin{cases} \sigma^{+-1}_{t_A \approx s_B}(e) & \text{if} \quad e \in Im(\sigma^{+-1}_{t_A \approx s_B}) \\ \tau^{+-1}_{t_A \approx s_B}(e) & \text{if} \quad e \in Im(\tau^{+-1}_{t_A \approx s_B}). \end{cases}$$

Then $\beta_1 \circ \beta_0$ maps $E[G_{C(A+B)}]$ injectively into $(E[G_C] \times E[G_A]) \cup (E[G_C] \times E[G_B])$. Define β_2 by taking

$$\beta_2(e) \quad = \quad \begin{cases} \Lambda^{-1}_{C,A}(e) & \text{if} \quad e \in E[G_C] \times E[G_A] \\ \Lambda^{-1}_{C,B}(e) & \text{if} \quad e \in E[G_C] \times E[G_B]. \end{cases}$$

Then β_2 maps $(E[G_C] \times E[G_A]) \cup (E[G_C] \times E[G_B])$ into $E[G_{CA}] \cup E[G_{CB}]$ injectively. Finally, define β_3 by taking

$$\beta_3(e) \quad = \quad \begin{cases} \sigma^{+}_{t_{CA} \approx\, s_{CB}}(e) & \text{if} \quad e \in E[G_{CA}] \\ \tau^{+}_{t_{CA} \approx\, s_{CB}}(e) & \text{if} \quad e \in E[G_{CB}]. \end{cases}$$

Then β_3 maps $E[G_{CA}] \cup E[G_{CB}]$ injectively into $E[G_{(CA)+(CB)}]$. The composite map $\beta_3 \circ \beta_2 \circ \beta_1 \circ \beta_0$ maps the edges of $C(A+B)$ injectively into the edges of $CA+CB$, and is the desired flow graph isomorphism demonstrating the claimed equality. $\square$

Let A be the flow graph consisting of a directed cycle of length 3 taking source and target vertices to be any two distinct vertices on this cycle. Observe that $(F_1+F_1)A = F_2A$, while $(F_1A)+(F_1A) = A+A = 2A = AF_2$. Referring to Figure 11, we see that $AF_2 \neq F_2A$.

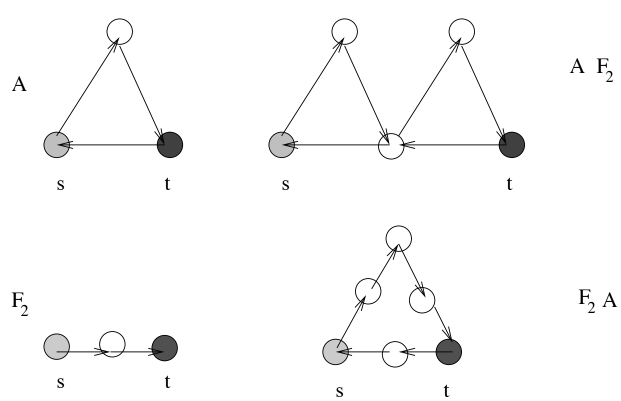

Figure 11: $(F_1+F_1)A \neq AF_1+AF_1$.

Lemma 5.2 (Non Right-distributivity of multiplication over addition)**.** *There exist flow graphs $A, B,$ and C,*

$$(B+C)A \quad \neq \quad BA+CA$$

Remark 5.3 (Violation of left and right cancellation). *Let B be non-reflective and A be reversible. Since A is reversible, $A = A^*$, so then $BA = BA^* = B1^*A = B'A$. Since B is not reflective, $B \neq B'$ violating right cancellation. For example, taking $B = 1$, we get that $1A = A = A^* = 1^*A$, but $1 \neq 1^*$.*

If A be reflective, and B be non-reversible. Since A is reflective, $A = A'$, so then $AB = A'B = A1'B = AB^*$. Since B is not reversible, $B \neq B^*$ violating left cancellation. For example, taking $B = 1$, we get that $A1 = A = A' = A1'$, but $1 \neq 1'$.

Definition 5.4. *Given flow graphs A, B at least one of which is non-trivial, we define A/B as the set of flow graphs C for which $A = BC$. Analogously, we define $A\backslash B$ as the set of flow graphs C for which $A = CB$. If $|A/B| = 0$ (resp. $|A\backslash B| = 0$) then we say that A is* not right-divisible *(resp.* not left-divisible*) by B. Note that the sets A/B and $A\backslash B$ may have size bigger than one. For example, if $A = A^*$, then $A = 1^*A = 1A$, so $A\backslash A$ contains both 1 and 1^*. If $A = A'$, then $A = A1' = A1$, so A/A contains both 1 and $1'$. By convention, we say that $0/0$ and $0\backslash 0$ are undefined.*

Clearly if m and n are standard integers then F_m is right-divisible by F_n iff F_m is left-divisible by F_n iff $\exists k \in \mathbb{N}$ for which $F_m = F_nF_k = F_kF_n$ iff m is divisible by n.

We extend multiplication of flow graphs to multiplication of sets of flow graphs in the obvious way:

Definition 5.5. *Given two nonempty sets of flow graphs $\mathcal{A}$ and $\mathcal{B}$, we define*

$$\mathcal{A}\widetilde{\times}\mathcal{B} = \{AB \mid A \in \mathcal{A}, B \in \mathcal{B}\},$$

$$\mathcal{A}\widetilde{+}\mathcal{B} = \{A+B \mid A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Note that for all flow graphs A, C, if $A/C \neq \emptyset$ then there exists a flow graph B such that $A/C \supseteq B/C\widetilde{\times}A/B$ as sets; simply take $B = C$. In contrast, the next lemma concerns cancellation in products:

Lemma 5.6. *For all flow graphs A, B, C, if $B/C \neq \emptyset$ and $A/B \neq \emptyset$ then*

$$B/C\widetilde{\times}A/B \quad \subseteq \quad A/C.$$

Proof. Suppose $K_2 \in B/C$ and $K_1 \in A/B$. Then by Definition 5.4, $B = CK_2$ and $A = BK_1$, which implies $A = (CK_2)K_1$ which by by Lemma 3.11 implies $A = C(K_2K_1)$, so $K_2K_1 \in A/C$. Thus, $B/C\widetilde{\times}A/C \subseteq A/C$. □

Note that Lemma 5.6 is not an equality, that is $B/C\widetilde{\times}A/B$ need not equal A/C even if $B/C \neq \emptyset$ and $A/B \neq \emptyset$. To see this, fix $n \geqslant 3$ odd, and take $i > 2$, $j = (n-1)/2 \geqslant 1$, and $k = n$. Put $A = R_{ijk}$, $B = R_{ij}$ and $C = R_i$. Then A/C is the set of all flow graphs having jk edges, while B/C (resp. A/B) is the set of all flow graphs having j (resp. k)

edges. Thus, to show that the inclusion in Lemma 5.6 is sometimes a proper inclusion, it suffices to show that not every flow graph having jk edges can be expressed as the product of two flow graphs which have j and k edges, respectively. Take, for example, the flow graph G obtained by considering any tournament[8] on n vertices, with distinct source and target. Since the number of edges in G is $n(n-1)/2 = jk$, we know that G is in A/C. We claim that G cannot be expressed as a product HK where H has j edges and K has k edges, i.e. that G is not in $B/C\widetilde{\times}A/B$. Suppose towards contradiction that a factorization $G = HK$ was possible. Since G was constructed to be non-infinitesimal, by Proposition 3.21, both H and K must be non-infinitesimal. Then by Lemma 3.10, we have that the number of vertices p and the number of edges q in H, K and G are related by the expression $p_K + q_K(p_H - 2) = p_G$, which in the specific setting becomes

$$p_K + n(p_H - 2) = n. \tag{7}$$

Examining equation (7) we see that if $p_H \leqslant 2$ then $p_K > n - 1 = q_K - 1$, violating that K is connected, and if $p_H \geqslant 3$ then $p_K < 0$, violating that K is a flow graph. It follows that no such factorization of G exists, and thus the inclusion in Lemma 5.6 is sometimes proper.

There is an analogous result to Lemma 5.6 concerning left-divisibility, namely for all flow graphs A, B, C, if $B\backslash C \neq \emptyset$ and $A\backslash B \neq \emptyset$ then

$$A\backslash B\widetilde{\times}B\backslash C \subseteq A\backslash C \tag{8}$$

It is unclear whether analogous examples can be constructed to demonstrate that the inclusion in the left divisibility analogue (8) is proper. The authors conjecture that expression (8) is actually an equality.

Lemma 5.7 (Restricted distributivity of right-divisibility over +)**.** *For all flow graphs A, B, C,*

$$A/B\widetilde{+}C/B \;\subseteq\; (A{+}C)/B$$

Proof. Suppose $K_1 \in A/B$ and $K_2 \in C/B$. Then by Definition 5.4, $A = BK_1$ and $C = BK_2$. Thus $A{+}C = (BK_1){+}(BK_2)$ which by Lemma 5.1, is $B(K_1{+}K_2)$. This implies that $K_1{+}K_2$ belongs to $(A{+}C)/B$ contains . Thus $A/B\widetilde{+}C/B$ is contained in $(A{+}C)/B$. ☐

To see that Lemma 5.7 is not necessarily an equality, that is $A/B\widetilde{+}C/B$ is not equal to $(A{+}C)/B$, consider the following example. Let $A = C = F_1$ and $B = F_2$. Since $F_1{+}F_1 = F_2F_1$ it means $F_1 \in (A + C)/B$. Now F_1 cannot be in $A/B\widetilde{+}C/B$ since only possibility is that $F_1 = F_0{+}F_1$ or $F_1 = F_1{+}F_0$ and F_0 is not in A/B or A/C.

[8]By tournament we mean a complete graph in which every two vertices u and v are connected by *either* the edge (u, v) or the edge (v, u).

Observation 5.8 (Non-distributivity of left-divisibility over $+$)**.** *Note that Lemma 5.2 can be used to construct examples that demonstrate non-distributivity of left-divisibility over $+$. For example, let B be a directed cycle of length 3 with any two distinct vertices as s_B and t_B. Take $A = F_2B$. Then $F_2 \in A\backslash B$. Now take $C = B$. Then $F_1 \in C\backslash B$ and so $F_2{+}F_1 = F_3 \in A\backslash B\widetilde{+}C\backslash B$. Since $A{+}C \neq F_3B$ this means that $F_2{+}F_1 = F_3 \notin (A{+}C)\backslash B$. Thus, the example shows that for some A, B, and C, the set $(A\backslash B)\widetilde{+}(C\backslash B)$ is not contained in the set $(A{+}C)\backslash B$.*

In Lemma 3.16, we determined that 1 and $1'$ are the only units in the set of flow graphs. This motivates the following definition of a prime flow graph:

Definition 5.9. *A flow graph A is called* prime *if A is neither trivial nor a unit, and $A = BC$ implies that either B or C is a unit.*

If we consider Definition 5.9 in the case when A is assumed to be non-infinitesimal, we see that B and C must both be non-infinitesimal and hence, $|E[G_B]| = 1$ (resp. $|E[G_C]| = 1$) implies that $B = 1$ or $1'$ (resp. $C = 1$ or $1'$). Accordingly, let us say a flow graph A is *right-prime* if for all flow graphs B, $A/B \neq \emptyset$ implies one of the following hold:

$RP1)$ $B = 1^*$ and $A^* \in A/B$.

$RP2)$ $B = A'$ and $1' \in A/B$.

$RP3)$ $B = A$ and $1 \in A/B$.

$RP4)$ $B = 1$ and $A \in A/B$.

Likewise, let us say that a flow graph A is *left-prime* if for all flow graphs C, $A\backslash C \neq \emptyset$ implies one of the following hold:

$LP1)$ $C = A^*$ and $1^* \in A\backslash C$.

$LP2)$ $C = 1'$ and $A' \in A\backslash C$.

$LP3)$ $C = A$ and $1 \in A\backslash C$.

$LP4)$ $C = 1$ and $A \in A\backslash C$.

Note that a natural number n is prime iff the non-infinitesimal flow graph F_n is prime iff F_n is right-prime iff F_n is left-prime. More generally:

Lemma 5.10. *Let A be a non-infinitesimal flow graph. Then A is right-prime iff A is left-prime.*

Proof. Suppose A is right-prime. To show that A is left-prime we must show that for all flow graphs C, $A \backslash C \neq \emptyset$ implies that at least one of LP1-LP4 holds. Suppose $B \in A \backslash C$; then $A = BC$, and hence $C \in A/B$. Since A is assumed to be right prime, we know that at least one of RP1-RP4 holds for B. Suppose that RP1 holds. Then $B = 1^*$ and $A^* = C \in A/B$, which implies LP1 holds. Similarly one can show that if RPi holds, then LPi holds (for $i = 2, 3, 4$). Thus A is left-prime.

A similar argument shows that left-prime implies right-prime. □

The previous lemma shows that the notions of prime, left-prime and right-prime coincide on non-infinitesimals. However, an infinitesimal flow graph can be prime while being neither left-prime nor right-prime. To see this, let A be any infinitesimal flow graph for which $|V[G_A]| > 1$ and $|E[G_A]|$ is prime. Fix a vertex $t^\dagger \in V[G_A]$, for which $t^\dagger \neq s_A, t_A$, and take $A^\dagger$ to be the flow graph $(G_A, s_A, t^\dagger)$. Then $A^\dagger S_1 = A$, so A is neither left-prime, nor right prime. But A is prime, since by Lemma 3.10, any factorization of A into a product BC must satisfy $|E[G_A]| = |E[G_A]| \cdot |E[G_A]|$. Indeed, any flow graph with a prime number of edges is necessarily a prime flow graph.

Definition 5.11. *Given a set of flow graphs $S \subset F$, we say that A is* central *in S if $A \in S$ and for all flow graphs $B \in S$, we have that $AB = BA$. The set of all flow graphs that are central in S is denoted as $Z(S)$.*

Proposition 5.12. *$Z(F) = \{0, 1\}$.*

Proof. Suppose A is central. Then $AS_1 = S_1A$. Since $AS_1 = A^+$ and $S_1A = S_{|E[G_A]|}$, it follows that A^+ is a star, and thus A (viewed as a directed graph) is also a star. This implies that A^* (viewed as a directed graph) is an antistar. Since A is central, $A1^* = 1^*A$; but $A1^* = A1' = A'$ and $1^*A = A^*$. Thus $A' = A^*$. So A' (viewed as a directed graph) is also an antistar. This means A as directed graph is a star with in-degree(s) in A equal to out-degree(t) in A and in-degree(t) in A equal to out-degree(s) in A. It follows that A has at most one edge. But this is possible if and only if A is 0 or 1. □

5.2. Order Properties

In this section we explore the relationship between strong order (denoted $\preccurlyeq$), and induced order (denoted $\ll$). While these orders coincide on the graphical natural numbers, only induced order is anti-symmetric on all of $\mathcal{F}$, and only the strong order and induced order are transitive. We consider several standard laws that govern the relationship between $\leqslant$, $+$ and $\times$ in $\mathcal{N}$, and show that these laws continue to hold for induced order $\ll$ but several are violated under the strong and induced orders.

5.2.1. Strong Order $\preccurlyeq$

We begin by describing the properties of $\mathcal{F}$ under the strong order.

Lemma 5.13 (Strong Order Preservation)**.** *For flow graphs A, B, C, if $A \preccurlyeq B$ then $CA \preccurlyeq CB$.*

Proof. Let $A = (G_A, s_A, t_A)$, $B = (G_B, s_B, t_B)$. Since $A \preccurlyeq B$ there are graph embeddings $\phi_s : G_A \to G_B$ and $\phi_t : G_A \to G_B$ which satisfy $\phi_s(s_A) = s_B$ and $\phi_t(t_A) = t_B$. Define $\gamma_s : E[G_C] \times E[G_A] \to E[G_C] \times E[G_B]$ by

$$(f, e) \mapsto (f, \phi_s(e)).$$

Then the composite map

$$\Phi_s^C : E[G_{CA}] \xrightarrow{\Lambda_{C,A}} E[G_C] \times E[G_A] \xrightarrow{\gamma_s} E[G_C] \times E[G_B] \xrightarrow{\Lambda_{C,B}^{-1}} E[G_{BC}]$$

defines an embedding of $G_{CA} \to G_{CB}$ which takes s_{CA} to s_{CB}. An analogous construction can be carried out to produce a map Φ_t^C which embeds $G_{CA} \to G_{CB}$ and sends t_{CA} to t_{CB}. □

Lemma 5.14 (Strong Order Violations)**.** *There exist flow graphs A, B and C for which $A \preccurlyeq B$ but:*

$$\begin{array}{rccc} (i) & A{+}C & \not\preccurlyeq & B{+}C \\ (ii) & C{+}A & \not\preccurlyeq & C{+}B \\ (iii) & AC & \not\preccurlyeq & BC. \end{array}$$

Proof. See Figure 12. □

We consider possible anti-symmetry of $\preccurlyeq$. Suppose $A \preccurlyeq B$ and $B \preccurlyeq A$. There is a graph embedding $\phi_s : G_A \to G_B$ which satisfies $\phi_s(s_A) = s_B$. Hence $|V[G_A]| = |V[\phi_s(G_A)]| \leqslant |V[G_B]|$ and $|E[G_A]| = |E[\phi_s(G_A)]| \leqslant |E[G_B]|$. Since $B \preccurlyeq A$, there is a graph embedding $\psi_s : G_B \to G_A$ which satisfies $\psi_s(s_B) = s_A$. So $|V[G_B]| = |V[\psi_s(G_B)]| \leqslant |V[G_A]|$ and $|E[G_B]| = |E[\psi_s(G_B)]| \leqslant |E[G_A]|$. It follows that ϕ_s is actually an isomorphism from G_A to G_B satisfying $\phi_s(s_A) = s_B$. A similar argument shows that there is an isomorphism ϕ_t from G_A to G_B satisfying $\phi_t(t_A) = t_B$. To conclude that $A = B$ requires a single flow graph isomorphism π from A to B, satisfying *both* $\pi(s_A) = s_B$ and $\pi(t_A) = t_B$. Indeed in some cases, no such isomorphism may exist.

Example 5.15. *Let G_A be a directed cycle of length 4, and take s_A, t_A to be any two vertices in $V[G_A]$ that are distance 2 apart. Put G_B isomorphic to G_A, taking s_B, t_B to be two vertices in $V[G_B]$ that are distance 1 apart. Then it is easy to verify that $(G_A, s_A, t_A) = A \preccurlyeq B = (G_B, s_B, t_B)$ and $B \preccurlyeq A$. Clearly, however, $A \neq B$ as flow graphs (see Figure 13).*

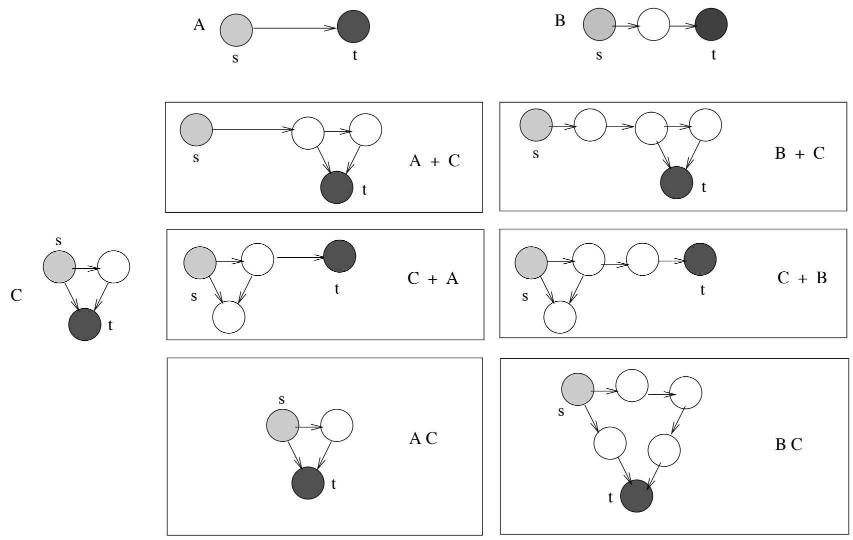

Figure 12: Strong order violations: (i). $A{+}C \not\preccurlyeq B{+}C$, (ii). $C{+}A \not\preccurlyeq C{+}B$, and (iii). $AC \not\preccurlyeq BC$.

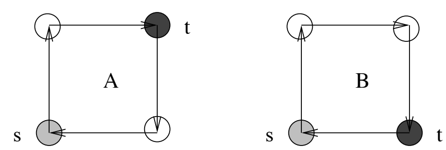

Figure 13: An example which demonstrates that the strong order is not antisymmetric.

The previous example proves the next lemma.

Lemma 5.16 (Non-antisymmetry of strong order $\preccurlyeq$)**.** *There exist flow graphs A and B for which*

$$A \preccurlyeq B \text{ and } B \preccurlyeq A \text{ but } A \neq B.$$

Lemma 5.17 (Transitivity of strong order $\preccurlyeq$)**.** *For all flow graphs A, B, C*

$$A \preccurlyeq B \text{ and } B \preccurlyeq C \text{ implies } A \preccurlyeq C.$$

Proof. $A \preccurlyeq B$: i.e. there are graph embeddings $\phi_s : G_A \to G_B$ and $\phi_t : G_A \to G_B$ which satisfy $\phi_s(s_A) = s_B$ and $\phi_t(t_A) = t_B$. $B \preccurlyeq C$: i.e. there are graph embeddings $\theta_s : G_B \to G_C$ and $\theta_t : G_B \to G_C$ which satisfy $\theta_s(s_B) = s_C$ and $\theta_t(t_B) = t_C$. We want to show $A \preccurlyeq C$: i.e. there are graph embeddings $\alpha_s : G_A \to G_C$ and $\alpha_t : G_A \to G_C$ which satisfy $\alpha_s(s_A) = s_C$ and $\alpha_t(t_A) = t_C$. Put $\alpha_s = \theta_s \circ \phi_s$ and $\alpha_t = \theta_t \circ \phi_t$. □

5.2.2. Induced Order $\ll$

We now investigate the properties of $\mathcal{F}$ under the induced order.

First, note that since flow graphs are connected directed graphs, then any flow graph A satisfies $0 \ll A$.

Lemma 5.18 (Induced Order Preservation)**.** *For flow graphs A, B, C, if $A \ll B$ then*

$$\begin{array}{rccc} (i) & A+C & \ll & B+C \\ (ii) & C+A & \ll & C+B \\ (iii) & CA & \ll & CB \\ (iv) & AC & \ll & BC. \end{array}$$

Proof. (i, ii) Since $A \ll B$, edges of B can be contracted to yield A. When this sequence of contractions is applied to $B+C$, it yields $A+C$. When this sequence of contractions is applied to $C+B$, it yields $C+A$.

(iii, iv) The flow graph CB is obtained by replacing each edge e in B with a graph C_e that is isomorphic to C. Since $A \ll B$, there is a sequence of edges $e_1, e_2, \ldots, e_k$ for which the sequence $B_0 = B$, $B_i = B_{i-1}/e_i$ (for $i = 1, 2, \ldots, k$), ends with $B_k = A$. We shall contract CB in phases, where at phase i, we collapse C_{e_i} to a point. This is possible since $0 \ll C$. At the end of this process, CB has been transformed into CA. The argument which shows (iv) is entirely analogous. □

Lemma 5.19 (Antisymmetry of induced order $\ll$)**.** *For all flow graphs A and B*

$$A \ll B \textit{ and } B \ll A \iff A = B.$$

Proof. Since $A \ll B$, $|E[G_A]| \leqslant |E[G_B]|$ and since $B \ll A$, $|E[G_B]| \leqslant |E[G_A]|$. It follows that $|E[G_B]| = |E[G_A]|$. It follows that no edge contractions are required to transform B into A, hence A and B are isomorphic as flow graphs. □

Lemma 5.20 (Transitivity of induced order $\ll$)**.** *For all flow graphs A, B, C*

$$A \ll B \textit{ and } B \ll C \textit{ implies } A \ll C.$$

Proof. If some sequence of edge contractions transforms C into B, and some sequence of edge contractions transforms B into A, then the concatenation of these two sequences demonstrates that $A \ll C$. □

5.2.3. Summary of Order Properties

Table 1 summarizes properties of the strong $\preccurlyeq$ and induced $\ll$ orders (when substituted for $\lhd$). Note that the induced order satisfies all listed properties, though the significance of this fact should perhaps be mitigated by the fact that both the $=$ and empty relation also satisfy all the properties on the list.

Properties	Strong Order $\lhd = \preccurlyeq$	Induced Order $\lhd = \ll$
$A \lhd B \Longrightarrow (AC) \lhd (BC)$	False	True
$A \lhd B \Longrightarrow (CA) \lhd (CB)$	True	True
$A \lhd B \Longrightarrow (A{+}C) \lhd (B{+}C)$	False	True
$A \lhd B \Longrightarrow (C{+}A) \lhd (C{+}B)$	False	True
$A \lhd B$ and $B \lhd A \Longrightarrow A = B$	False	True
$A \lhd B$ and $B \lhd C \Longrightarrow A \lhd C$	True	True

Table 1: Properties of $\mathcal{F}$ under strong and induced orders.

6. Conclusions and Future Work

Our future research will consider the structural properties of flow graphs and describe $Th(\mathcal{F})$, including for restricted subsets of F that can be defined in terms of structural constraints, e.g. the set of all trees, directed acyclic graphs, etc.

Some questions we are presently considering are listed below.

i. Characterize $+$-commuting pairs, i.e. under what conditions on flow graphs A and B does $A{+}B = B{+}A$?

ii. *Graph $+$-Irreducible Decomposition Conjecture.* Every flow graph is uniquely expressible (up to well-defined reordering) as the sum of $+$-irreducible flow graphs.

iii. Characterize pairs which commute with respect to multiplication, i.e. under what conditions on flow graphs A and B does $AB = BA$?

iv. *Graph Prime Factorization Conjecture.* Every flow graph is uniquely expressible (up to some well-defined reordering and application of unary structural operators) as the product of prime flow graphs.

Acknowledgements

The authors would like to thank the referees for many insightful observations and recommendations that led to considerable refinements in the theory. We are grateful to Martin Kassabov for his incisive comments on an early version of this paper. Aspects of these results were presented at the City University of New York Logic Workshop in September 2004. We thank the logicians present at the seminar, especially Joel Hamkins and Roman Kossak for their constructive comments and suggestions which improved the presentation of the paper. We are especially grateful to Joel Hamkins for his many ideas and contributions which stimulated the development of this theory, and particularly for suggesting the induced order $\ll$.

References

[1] Kiran R. Bhutani and Bilal Khan. A metric on the class of connected simple graphs of given order. *Aequationes Mathematicae*, 66(3):232–240, December 2003.

[2] Kiran R. Bhutani and Bilal Khan. Distance between graphs using graph labelings. *Ars Combinatoria*, 77:45–52, 2005.

[3] George Cantor. *Uber unendliche, lineare Punktmannigfaltigkeiten, Arbeiten zur Mengenlehre aus dem Jahren 1872-1884*. Teubner-Archiv zur Mathematik, Leipzig, Germany, 1884.

[4] K. Ciesielski. *Set Theory for the Working Mathematician*. Cambridge University Press, 1997.

[5] W. Imrich and S. Klavzar. *Product Graphs, Structure and Recognition*. John Wiley & Sons Inc., 2000.

[6] Bilal Khan and Kiran R. Bhutani. A metric representing dilation between connected simple graphs of a given order. preprint.

[7] J. E. Rubin. *Theory for the Mathematician*. Holden-Day, New York, 1967.

[8] P. Suppes. *Axiomatic Set Theory*. Dover, New York, 1972.

[9] Douglas B. West. *Introduction to Graph Theory*. Prentice Hall, 2nd edition, 2001.