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ENUMERATING SEGMENTED PATTERNS IN COMPOSITIONS AND ENCODING BY RESTRICTED PERMUTATIONS

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Abstract

A *composition* of a nonnegative integer n is a sequence of positive integers whose sum is n. A composition is *palindromic* if it is unchanged when its terms are read in reverse order. We provide a generating function for the number of occurrences of arbitrary segmented partially ordered patterns among compositions of n with a prescribed number of parts. These patterns generalize the notions of rises, drops, and levels studied in the literature. We also obtain results enumerating parts with given sizes and locations among compositions and palindromic compositions with a given number of parts. Our results are motivated by "encoding by restricted permutations," a relatively undeveloped method that provides a language for describing many combinatorial objects. We conclude with some examples demonstrating bijections between restricted permutations and other objects.

1. Introduction

A *composition* of a nonnegative integer n is a sequence $\alpha = \alpha_1\alpha_2\cdots\alpha_m$ of positive integers whose sum is n. We consider the empty sequence with no terms to be the unique composition of 0. We will sometimes write compositions as sums rather than as words, as in $\alpha_1 + \alpha_2 +$

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$\cdots + \alpha_m$, though it must be kept in mind that the order of the terms still matters. It is sometimes helpful to think of a composition of n as a sequence of n stones laid in a row, together with a grouping of the stones such that every stone belongs to a group, every group contains a stone, no stone belongs to two groups, and two stones belong to the same group only if every stone between them belongs to that group.

Each term α_i in a composition α is called a *part* of that composition. A part equal to k is called a k-part. A *split* in a composition is an integer that can be expressed as the sum of the first i parts of the composition for some nonnegative integer i. Thus, the composition $3+1+1+2$ has 5 splits: 0, 3, 4, 5, and 7. Using the imagery of stones, the parts of α are the groups of stones, and the splits of α correspond to gaps in the row of stones (including the gap before the first stone and after the last stone) that do not have two stones from the same group on either side. In this context, the split corresponding to such a gap is the sum of the number of stones in the groups to the left of that gap.

We use $\langle\alpha\rangle$ to denote the composition comprising the parts of α written in reverse order. A *palindromic composition*, or a *palindrome*, is a composition for which $\alpha = \langle\alpha\rangle$. A *rise* (resp. *drop*) is a part followed by a larger (resp. smaller) part. A *level* is a part followed by a part equal to itself.

Frequencies of occurrences of k-parts, rises, drops, and levels in (palindromic) compositions, as well as in compositions with additional restrictions, have been studied (e.g., see [4] and [5] and references therein). Heubach and Mansour [5] give a multivariate generating function for joint distribution of parts, rises, levels, and drops in compositions and palindromes. However, using the results from the literature related to the subject, it does not seem to be possible to answer a question like: how many of the double (or triple) levels are followed by rises among all compositions of n? (The question "How many levels are followed by rises among all compositions of n?" is answered in [6, Thm. 2.2].) To consider a more general question, we introduce the notion of a *segmented pattern* in a composition. A segmented pattern is a word $w = w_1w_2\cdots w_k$ in the alphabet of positive integers such that if b is a letter in w and $a < b$, then a is a letter of w. In other words, the letters in w constitute an order ideal. For example, 431242 is a segmented pattern, while 41242 is not. We say that w occurs in a composition $\alpha = \alpha_1\alpha_2\cdots\alpha_m$ if there is a subword $\alpha_i\alpha_{i+1}\ldots\alpha_{i+k-1}$ of α that is order-isomorphic to w. Thus, rises, drops, and levels are occurrences of the patterns 12, 21, and 11, respectively. A level followed by a rise is an occurrence of the pattern 112. The notion of segmented patterns in arbitrary words was studied in [2].

More generally, we study occurrences of so-called *segmented partially ordered patterns* (SPOPs) in compositions. A SPOP w is a word consisting of letters from a partially ordered alphabet $\mathcal{A}$ such that the letters in w constitute an order ideal in $\mathcal{A}$. For instance, if we have a poset on three elements labeled by 1, $1'$, and $2'$ in which the only relation is $1' < 2'$, then the sequence 31254 has two occurrences of $11'2'$, namely 312 and 125. Given a SPOP $w = w_1w_2\cdots w_m$, we say that a segmented pattern $v = v_1v_2\cdots v_m$ is a *linear extension* of w if $w_i < w_j$ implies that $v_i < v_j$. Thus the linear extensions of $11'2'$ are 123, 213, and 312.

This paper is organized as follows. In Section 2, we give our main results. Theorem 2.1 gives a multivariate generating function for the number of occurrences of a given SPOP at a given split among compositions of n with a given number of parts. By specializing variables, we obtain a generating function for the number of occurrences of a given SPOP among all compositions of n (Corollary 2.2). In Theorem 3.1, we enumerate the occurrences of k-parts at a given split in compositions of n with a given number of parts. This generalizes a result in [3]. Our approach to this problem is to use a method which perhaps can be best described as "encoding by restricted permutations." The idea here is to encode a set of objects under consideration as a set of permutations satisfying certain restrictions. Under appropriate encodings, this allows us to transfer the interesting statistics from our original set to the set of permutations, where they are easy to handle. In Section 4, we use restricted permutations to enumerate k-parts with certain statistics in palindromic compositions, refining results in [4]. In Section 5 we provide short bijective encodings of binary bitonic sequences, binary strings without singletons, permutations avoiding 1-3-2-4 and having exactly one descent, and lines drawn through the points of intersections of n straight lines in a plane. Relations of these objects to certain restricted permutations were given in [1], but no bijections were provided. We believe that these examples provide some evidence for the broad applicability of the method of encoding by restricted permutations.

We use the following notations throughout the paper. The set of nonnegative integers is denoted by $\mathbb{N}$, and the set of positive integers is denoted by $\mathbb{P}$. Given $m \le n \in \mathbb{N}$, we write $[m,n] = \{m, m+1, \ldots, n\}$ and $[n] = [1,n]$. The permutations in this paper are written in one-line notation. Given a generating function $G(t)$, we write $[t^n]G(t)$ to denote the coefficient of t^n in $G(t)$. We use $\mathcal{C}(n)$ to denote the set of compositions of n, and we write that $|\alpha| = n$ if $\alpha \in \mathcal{C}(n)$. Finally, let $C(n,\ell)$ be the number of compositions of n with ℓ parts. It is well known and easy to verify that for a fixed non-negative integer ℓ, the generating function for $C(n,\ell)$ is given by

$$\sum_{n=0}^{\infty} C(n,\ell)x^n = \frac{x^\ell}{(1-x)^\ell}. \tag{1}$$

2. Compositions

Given a SPOP $w = w_1 w_2 \cdots w_m$ with m parts, let $c_w(n,\ell,s)$ be the number of occurrences of w among compositions of n with $\ell + m$ parts such that the sum of the parts preceding the occurrence is s. Let $\Omega_w(x,y,z)$ be the generating function for $c_w(n,\ell,s)$:

$$\Omega_w(x,y,z) = \sum_{n,\ell,s\in\mathbb{N}} c_w(n,\ell,s)x^n y^\ell z^s.$$

Our goal is to derive an explicit rational function for $\Omega_w(x,y,z)$.

Before proceeding, we define the following notation. Given a segmented pattern v and $n \in \mathbb{N}$, let $P_v(n)$ denote the number of compositions of n that are order isomorphic to v.

The generating function $\mathcal{P}_v(x)$ for $P_v(n)$ is not difficult to derive. If j is the largest letter of v, then $P_v(n)$ is the number of integral solutions $t_1, \ldots, t_j$ to the system

$$\mu_1 t_1 + \cdots + \mu_j t_j = n, \qquad 0 < t_1 < \cdots < t_j, \tag{2}$$

where μ_k is the number of k's in v. We call $\mu = (\mu_1, \ldots, \mu_j)$ the *content vector* of v. By expanding terms into geometric series, one can see that the number of integral solutions to (2) is the coefficient of x^n in

$$\mathcal{P}_v(x) = \prod_{k=1}^{j} \frac{x^{m_k}}{1 - x^{m_k}}, \tag{3}$$

where $m_k = \mu_{j-k+1} + \cdots + \mu_j$ for $1 \leq k \leq j$.

Theorem 2.1. *Let w be a SPOP. Then*

$$\Omega_w(x, y, z) = \frac{(1-x)(1-xz)}{(1-x-xy)(1-xz-xyz)} \sum_v \mathcal{P}_v(x) \tag{4}$$

where the sum is over all linear extensions v of w.

Proof. We begin by computing $\Omega_v(x, y, z)$ when v is a segmented pattern. We think of an occurrence of v as the triple of compositions (α, β, γ) such that α comprises the parts to the left of the occurrence, β comprises the parts in the occurrence, and γ comprises the parts to the right of the occurrence. Hence, for given $n, \ell, s \in \mathbb{N}$, $c_v(n, \ell, s)$ is the number of triples (α, β, γ) of compositions such that $|\alpha| + |\beta| + |\gamma| = n$, $|\alpha| = s$, β is order isomorphic to v, and α and γ together have ℓ parts. Thus we have that

$$c_v(n, \ell, s) = \sum_{\substack{0 \leq j \leq \ell \\ 0 \leq k \leq n-s}} C(s, j) P_v(k) C(n - s - k, \ell - j).$$

Using this equality, together with the generating function (1) for $C(n, \ell)$, we can factor $\Omega_v(x, y, z)$ into a product of $\mathcal{P}_v(x)$ and two geometric series:

$$\begin{aligned}
\Omega_v(x, y, z) &= \sum_{n, \ell, s \in \mathbb{N}} c_v(n, \ell, s) x^n y^\ell z^s \\
&= \mathcal{P}_v(x) \left(\sum_{n, \ell \in \mathbb{N}} C(n, \ell) x^n y^\ell \right) \left(\sum_{s, \ell \in \mathbb{N}} C(s, \ell) (xz)^s y^\ell \right) \\
&= \mathcal{P}_v(x) \left(\sum_{l \in \mathbb{N}} \frac{x^\ell}{(1-x)^\ell} y^\ell \right) \left(\sum_{l \in \mathbb{N}} \frac{(xz)^\ell}{(1-xz)^\ell} y^\ell \right) \\
&= \mathcal{P}_v(x) \frac{(1-x)(1-xz)}{(1-x-xy)(1-xz-xyz)}.
\end{aligned}$$

Finally, note that if w is a SPOP, then $c_w(x, y, z) = \sum_v c_v(x, y, z)$, where the sum is over all linear extensions v of w. Thus, $\Omega_w(x, y, z) = \sum_v \Omega_v(x, y, z)$, and the theorem follows. $\square$

Setting $y = z = 1$ in equation (4) yields the following.

Corollary 2.2. *Given a segmented pattern w, the number of occurrences of w among compositions of n is equal to*

$$[x^n]\Omega_w(x,1,1) = [x^n]\frac{(1-x)^2}{(1-2x)^2}\sum_v \mathcal{P}_v(x),$$

where the sum is over all linear extensions v of w.

Example 2.3. We compute the number of occurrences of m levels immediately followed by a rise. This is an occurrence of the segmented pattern $w = \underbrace{1\cdots1}_{m+1}2$. The content vector of w is $\mu = (m+1, 1)$, so we have

$$\mathcal{P}_w(x) = \frac{x^{m+3}}{(1-x)(1-x^{m+2})}.$$

Hence, the number of occurrences of w among all compositions of n is

$$[x^n]\Omega_w(x,1,1) = [x^n]\frac{(1-x)^2x^{m+3}}{(1-2x)^2(1-x)(1-x^{m+2})}$$

For fixed m, it is routine to expand the rational function above into partial fractions to obtain a closed-form expression for $[x^n]\Omega_w(x,1,1)$.

Another application of Corollary 2.2 links our work to recent work of Savage and Wilf [7]. Whereas we have focused on the number of *occurrences* of a pattern among all compositions, another natural thing to study is the number of *compositions* containing a given pattern.[4] In general this is a much more difficult question.

Let $a_w(n,k)$ denote the number of compositions of n in which the SPOP w appears k times. Though they had a different definition of pattern, Savage and Wilf [7] studied the number of compositions in which a pattern occurs 0 times, i.e., *pattern avoiding* compositions. More generally, consider the polynomial

$$A_{w,n}(t) := \sum_{k\geq 0} a_w(n,k)t^k.$$

Some easy observations about the polynomial:

$$\begin{aligned} &A_{w,n}(0) = a_w(n,0), \text{ the number of } w\text{-avoiding compositions;}\\ &A_{w,n}(1) = 2^{n-1};\\ &A'_{w,n}(1) = \sum_{k\geq 1} k a_w(n,k) = b_w(n), \end{aligned}$$

[4] Savage and Wilf use the "classical" definition of a pattern in their work, as opposed to the "segmented" one we use.

where $b_w(n)$ is the total number of occurrences of w among all compositions of n. We leave the study of $A_{w,n}(t)$ as an open problem, but show that for now we can compute at least the *average* number of occurrences of w in a given composition of n. Using Corollary 2.2 we find the generating function for the expected number of occurrences of w in a randomly selected composition of n is:

$$2\Omega_w(x/2,1,1) = \sum_{n\geq 0} \frac{b_w(n)}{2^{n-1}} x^n = \frac{(2-x)^2}{2(1-x)^2} \sum_v \mathcal{P}_v(x/2),$$

where the rightmost sum is over all linear extensions v of w.

3. Enumerating k-parts in Compositions

Now we give an enumerative result that describes the number of k-parts located at a given split among compositions of n with a given number of parts. Theorem 3.1 below is our first example of encoding with restricted permutations.[5] For $n, k, \ell, s \in \mathbb{N}$, define $f(n, k, \ell, s)$ to be the number of k-parts occurring among compositions of n with $\ell + 1$ parts such that the sum of the parts preceding the k-part is s. It immediately follows that $f(n, k, \ell, s) = 0$ if either $n = 0$ or $k = 0$. The case when $n = k > 0$ is also clear: $f(n, n, \ell, s) = 1$ if $\ell = s = 0$, and $f(n, n, \ell, s) = 0$ otherwise. The following theorem gives the value of $f(n, k, \ell, s)$ in all remaining cases.

Theorem 3.1. *If $n \in \mathbb{P}$ and $k \in [n-1]$, then*

$$f(n,k,\ell,s) = \begin{cases} \binom{n-k-1}{\ell-1}, & \text{if } s \in \{0, n-k\}, \\ \binom{n-k-2}{\ell-2}, & \text{if } s \in [n-k-1], \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Proof. We give a bijection between the k-parts that we are enumerating and a particular set of restricted permutations. Let S be the set of permutations of the quotient group $Z = \mathbb{Z}/(n-k+1)\mathbb{Z}$ of the form $sw_1w_2\cdots w_{n-k}$, where

$$w_1 > w_2 > \cdots > w_\ell < w_{\ell+1} < \cdots < w_{n-k}.$$

and $s+1 \in \{w_1, \ldots, w_\ell\}$ (where we've identified s with its canonical projection in Z). To see that $|S|$ is given by the right-hand side of equation (5), observe that an element of S is uniquely specified by choosing which elements of Z will be in $\{w_1, \ldots, w_\ell\}$ other than s (which cannot be in there) and $s+1$ and $\min(Z\backslash\{s\})$ (which must be in there, but which are equal when $s \in \{0, n-k\}$).

[5] In fact it is possible to use this result to prove Theorem 2.1, though this approach requires several pages of tedious calculation, and is omitted in favor of the short and self-contained proof given above.

We now show that the elements of S are in bijective correspondence with the k-parts that we wish to enumerate. First, we can think of such a k-part as an element of

$$T = \{(\alpha, \beta) : \alpha \in \mathcal{C}(s),\ \beta \in \mathcal{C}(n-k-s),\ \alpha \text{ and } \beta \text{ together have } \ell \text{ parts } \}.$$

To be precise, a k-part in a composition of n corresponds to the ordered pair (α, β) of compositions such that α comprises the parts to the left of the chosen k-part and β comprises the parts to the right of the chosen k-part.

We now give a bijection $T \leftrightarrow S$. For an explicit example of the bijection we are about to describe, see Example 3.2. Given $(\alpha, \beta) \in T$, we produce a permutation in S as follows. Concatenate the compositions α and β, producing a composition $\gamma \in \mathcal{C}(n-k)$ with ℓ parts. Let $\overline{w}_\ell = 0$ and let

$$\overline{w}_{\ell-i} = \sum_{j=1}^{i} \gamma_j, \quad \text{for } 1 \le i \le \ell - 1.$$

For $1 \le i \le \ell$ let

$$w_i = \begin{cases} \overline{w}_i, & \text{if } \overline{w}_i < s, \\ \overline{w}_i + 1, & \text{if } \overline{w}_i \ge s. \end{cases}$$

Finally, let $w_{\ell+1}, \ldots, w_{n-k}$ be the elements of $[0, n-k] \backslash \{s, w_1, \ldots, w_\ell\}$ written in increasing order (where we've identified $[0, n-k]$ with its canonical projection in Z). It is easy to show that this map yields an element of S.

We show that the map is a bijection by giving its inverse. Given an element of S, one may produce an element of T by letting

$$\overline{w}_i = \begin{cases} w_i, & \text{if } w_i < s, \\ w_i - 1, & \text{if } w_i > s, \end{cases} \qquad \text{for } 1 \le i \le \ell,$$

letting $\gamma_i = \overline{w}_{\ell-i} - \overline{w}_{\ell-i+1}$ for $1 \le i \le \ell - 1$, $\gamma_\ell = n - k - \overline{w}_2$, and letting $\gamma = \gamma_1 \cdots \gamma_\ell$. Because of the requirement that $s + 1 \in \{w_1, \ldots, w_\ell\}$, it follows that, for some $i \in [\ell]$, $\sum_{j=1}^{i} \gamma_j = s$. Let $\alpha = \gamma_1 \cdots \gamma_i$ and let $\beta = \gamma_{i+1} \cdots \gamma_\ell$. We then have that $(\alpha, \beta) \in T$. $\square$

Example 3.2. We choose as our k-part the 6 in the composition 3 1 6 2. Then we have $n = 12$, $k = 6$, $l = 3$, and $s = 4$. The claim is that this corresponds to a permutation of the elements in $\mathbb{Z}/7\mathbb{Z}$.

Applying the maps from the theorem to our chosen k-part yields $\alpha = 3\,1$, and $\beta = 2$. Thus we have $\gamma = 3\,1\,2$. Computing the values of $\overline{w}_i$ yields $\overline{w}_3 = 0$, $\overline{w}_2 = \gamma_1 = 3$, and $\overline{w}_1 = \gamma_1 + \gamma_2 = 4$. Observing that $\overline{w}_1 \ge s = 4$, while $\overline{w}_2, \overline{w}_3 < s$, we compute the w_i's as follows

$$\begin{aligned} w_1 &= \overline{w}_1 + 1 = 5, \\ w_2 &= \overline{w}_2 = 3, \\ w_3 &= \overline{w}_3 = 0. \end{aligned}$$

Finally, we let $w_4\,w_5\,w_6$ be the elements of

$$\begin{aligned}\{0,\dots,6\}\backslash\{s,w_1,w_2,w_3\} &= \{0,\dots,6\}\backslash\{4,5,3,0\}\\ &= \{1,2,6\}\end{aligned}$$

written in increasing order. Therefore, the word corresponding to our original k-part is

$$s\,w_1\,w_2\cdots w_6 = 4\,5\,3\,0\,1\,2\,6.$$

As a corollary to Theorem 3.1, we derive a result that appeared in [3].

Corollary 3.3. *Given $n \in \mathbb{N}$ and $k \in [n-1]$, the number of k-parts among all compositions of n is $2^{n-k-2}(n-k+3)$.*

Proof. The result follows from using equation (5) to compute

$$\begin{aligned}\sum_{\substack{\ell\in[n-k]\\ s\in[0,n-k]}} f(n,k,\ell,s) &= 2\sum_{\ell=1}^{n-k}\binom{n-k-1}{\ell-1} + (n-k-1)\sum_{\ell=2}^{n-k}\binom{n-k-2}{\ell-2}\\ &= 2^{n-k} + 2^{n-k-2}(n-k-1)\\ &= 2^{n-k-2}(n-k+3).\end{aligned}$$

□

4. Palindromic Compositions

We provide two alternative (nonequivalent) encodings by restricted permutations of k-parts in palindromes of N, when N and k have different parity. We give these encodings explicitly in the case of even palindromes of $N = 2(n-1)$ and odd k-parts. These encodings provide bijective proofs of the known result that the number of k-parts in palindromic compositions of $2(n-1)$ is $(n-k+1)2^{n-k-1}$ when k is odd (see [4]). Such k-parts will be encoded as permutations $w_1w_2\cdots w_{n-k+1}$ of $[n-k+1]$ such that, for some $\ell \in \{2,\dots,n-k+1\}$, $w_2 > w_3 > \cdots > w_\ell < w_{\ell+1} < \cdots < w_{n-k+1}$. The enumerative result from [4] then follows immediately because such a restricted permutation is defined by first choosing which of the $n-k+1$ elements will be w_1, and then choosing a subset of the remaining elements (other than the least remaining element) to be $w_2,\dots,w_{\ell-1}$.

In either encoding, the case of odd palindromes $N = 2n-1$ and even k-parts can be obtained using very similar bijections, which we omit for brevity. The case in which k and N have the same parity is only slightly complicated by the possibility that the k-part could be the central term of the palindrome.

Clearly, a palindrome of $2(n-1)$ has either an odd number of parts with an even part in the center or an even number of parts and no central part. To make all palindromes be of odd length, we create a central part "0" for palindromes with an even number of parts. To recover the actual palindromes produced by our bijections, simply drop any 0 terms that arise.

4.1. First Encoding

For the purposes of this encoding, it will be convenient to re-index the set whose permutations will encode k-parts in palindromes. We will encode each k-part as a permutation $w_0 w_1 \cdots w_{n-k}$ of $[0, n-k]$ with

$$w_1 > w_2 > \cdots > w_\ell < w_{\ell+1} < \cdots < w_{n-k}.$$

First, observe that such a permutation corresponds to an ordered pair (w_0, α) with $w_0 \in [0, n-k]$ and $\alpha = \alpha_1 \cdots \alpha_\ell \in \mathcal{C}(n-k)$ as follows. If $\ell = 1$, let $\alpha_1 = n-k$. Otherwise, let

$$\overline{w}_i = \begin{cases} w_i, & \text{if } w_i < w_0, \\ w_i - 1, & \text{if } w_i > w_0, \end{cases} \qquad \text{for } 1 \leq i \leq l,$$

and put

$$\begin{aligned} \alpha_i &= \overline{w}_{\ell-i} - \overline{w}_{\ell-i+1}, \quad \text{for } 1 \leq i \leq l-1, \\ \alpha_\ell &= n - k - \overline{w}_1. \end{aligned}$$

We now explicitly describe the correspondence between pairs (w_0, α) and odd k's in palindromic compositions of $2(n-1)$. It may be helpful to use the imagery of stones discussed in Section 1. In this context, w_0 can be thought of as distinguishing a *gap* in the sequence of stones, where the gaps are the spaces between any two adjacent stones (whether they belong to the same group or not), as well the space before the first stone and after the last stone. Hence, a sequence of $n-k$ stones has $n-k+1$ gaps, which we index with the set $[0, n-k]$. Note that splits are special cases of gaps.

Case I: $w_0 \in \{0, n-k\}$. These pairs (w_0, α) correspond to the k's that are either the left-most or right-most terms in the compositions containing them. In particular, $(0, \alpha)$ corresponds to the left-most k in the composition

$$k + \sum_{i=1}^{\ell-1} \alpha_i + 2(\alpha_\ell - 1) + \left\langle \sum_{i=1}^{\ell-1} \alpha_i \right\rangle + k,$$

while $(n-k, \alpha)$ corresponds to the right-most k.

Case II: $w_0 \in [n-k-1]$, and w_0 is a split in α. These pairs (w_0, α) correspond to k's that are on the left-hand side of the palindromic compositions containing them, but which are not the left-most terms. In these cases, (w_0, α) corresponds to the indicated k on the left-hand side of the palindromic composition

$$\sum_{i=1}^{j} \alpha_i + k + \sum_{i=j+1}^{\ell-1} \alpha_i + 2(\alpha_\ell - 1) + \left\langle \sum_{i=1}^{j} \alpha_i + k + \sum_{i=j+1}^{\ell-1} \alpha_i \right\rangle,$$

where j is such that $\sum_{i=1}^{j} \alpha_i = w_0$. (The reader might note that the palindrome defined above is not well-defined if $j = \ell$. However, that does not happen for the pairs (w_0, α) considered in this case. This is because j is recovered from (w_0, α) by finding the number of terms of α that must be added together to equal w_0. If $j = \ell$, then we would have $w_0 = \sum_{i=1}^{\ell} \alpha_i = n - k$, which would put us in Case I.)

Case III: $w_0 \in [n-k-1]$, and w_0 is not a split in α. These pairs (w_0, α) correspond to the k's which are on the right-hand side of the palindromic compositions containing them, but which are not the right-most terms. These cases break into two subordinate cases:

Case IIIA: w_0 is a gap in the last term of α. These pairs (w_0, α) correspond to k's that are immediately to the right of the center term of the palindromic compositions containing them. In particular, such a (w_0, α) corresponds to the indicated k on the right-hand side of the palindromic composition

$$\sum_{i=1}^{\ell-1} \alpha_i + \alpha'_\ell + k + 2(\alpha''_\ell - 1) + k + \alpha'_\ell + \left\langle \sum_{i=1}^{\ell-1} \alpha_i \right\rangle,$$

where we use the identities $\alpha'_\ell + \alpha''_\ell = \alpha_\ell$ and $\sum_{i=1}^{\ell-1} \alpha_i + \alpha'_\ell = w_0$.

Case IIIB: w_0 is a gap in some term of α other than the last one. These pairs (w_0, α) correspond to k's that are on the right-hand side of the palindromic compositions containing them, but which are neither the right-most nor the left-most terms on the right-hand side. In particular, such a (w_0, α) corresponds to the indicated k on the right-hand side of the palindromic composition

$$\sum_{i=1}^{j-1} \alpha_i + \alpha'_j + k + \alpha''_j + \sum_{i=j+1}^{\ell-1} \alpha_i + 2(\alpha_\ell - 1) + \left\langle \sum_{i=1}^{j-1} \alpha_i + \alpha'_j + k + \alpha''_j + \sum_{i=j+1}^{\ell-1} \alpha_i \right\rangle,$$

where we use the identities $\alpha'_j + \alpha''_j = \alpha_j$ and $\sum_{i=1}^{j-1} \alpha_i + \alpha'_j = w_0$.

This completes the bijection between k-parts of palindromes of $2(n-1)$ and permutations $w_0 w_1 \cdots w_{n-k}$ of $[0, n-k]$ with $w_1 > w_2 > \cdots > w_\ell < w_{\ell+1} < \cdots < w_{n-k}$. By adding 1 to each term w_i, we have the desired bijection between k-parts of palindromes of $2(n-1)$ and permutations $w_1 w_2 \cdots w_{n-k+1}$ of $[n-k+1]$ such that $w_2 > w_3 > \cdots > w_\ell < w_{\ell+1} < \cdots < w_{n-k+1}$. Table 1 illustrates the present encoding of the 3s in all the palindromic compositions of 10 (*i.e.*, the case $n = 6$ and $k = 3$).

Permutation	(w_1, α)	Case	Palindromic composition
(1,2,3,4)	(1,3)	I	**3**+4+3
(1,3,2,4)	(1,12)	I	**3**+1+2+1+3
(1,4,2,3)	(1,21)	I	**3**+2+0+2+3
(1,4,3,2)	(1,111)	I	**3**+1+1+0+1+1+3
(4,1,2,3)	(1,3)	I	3+4+**3**
(4,2,1,3)	(1,12)	I	3+1+2+1+**3**
(4,3,1,2)	(1,21)	I	3+2+0+2+**3**
(4,3,2,1)	(1,111)	I	3+1+1+0+1+1+**3**
(2,3,1,4)	(2,12)	II	1+**3**+2+3+1
(2,4,3,1)	(2,111)	II	1+**3**+1+0+1+3+1
(3,4,1,2)	(3,21)	II	2+**3**+0+3+2
(3,4,2,1)	(3,111)	II	1+1+**3**+0+3+1+1
(2,1,3,4)	(2,3)	IIIA	1+3+2+**3**+1
(2,4,1,3)	(2,21)	IIIB	1+3+1+0+1+**3**+1
(3,1,2,4)	(3,3)	IIIA	2+3+0+**3**+2
(3,2,1,4)	(3,12)	IIIA	1+1+3+0+**3**+1+1

Table 1: The first encoding in the case $n = 6$ and $k = 3$.

4.2. Second Encoding

We present an algorithm to produce a "good" permutation given an underlined k-part in a palindrome P. (Recall that the permutations of interest are the permutations $w_1w_2 \cdots w_{n-k+1}$ of $[n-k+1]$ such that, for some $\ell \in \{2, \ldots, n-k+1\}$, $w_2 > w_3 > \cdots > w_\ell < w_{\ell+1} < \cdots < w_{n-k+1}$.) We only consider the case when the chosen part is to the left of the center in P; for a part from the right-hand side, we proceed with the part symmetric to it, and we switch 1 and 2 in the obtained permutation. In the bijection below, a part is to the left of the center if and only if in the corresponding permutation, 1 precedes 2. In general, we find the permutation corresponding to $\underline{k}$ by inserting the numbers $n-k+1$, $n-k$, $n-k-1$, and so on, into initially empty slots $w_1, w_2 \ldots, w_{n-k+1}$.

Suppose $P = C\underline{k}Dx\bar{D}k\bar{C}$ where $x = 2t$ for $t \geq 0$, and $\bar{W}$ is the reverse of W. In what follows, if at any step we get $w_1 = 2$, set $w_1 = 1$ and place 2 in the (only) remaining slot.

(1) If D is empty and $x = 0$, we set $w_1 = (n-k+1)$ and proceed with (2) below. Otherwise, we set $w_{n-k-t+2}w_{n-k-t+3} \cdots w_{n-k+1} = (n-k-t+2)(n-k-t+3) \cdots (n-k+1)$. Now if D is empty, set $w_1 = (n-k-t+1)$ and proceed with step (2). If D is not empty, $w_2 = (n-k-t+1)$. We read the parts in D from right to left and fill in the slots $w_3, w_4, \ldots$ by placing $n-k-t$, then $n-k-t-1$, and so on (we refer to the empty slots in $w_3, w_4, \ldots$ as the *placement area*): if a current part is a, then we place $a-1$ of the largest unplaced numbers at the right end of the placement area in increasing order, and we place the largest number out of the remaining numbers at the left end

of the placement area. We then proceed with the next part in D reading from right to left. The only exception is the part immediately to the right of $\underline{k}$. In this case, we place $a-1$ of the largest unplaced numbers at the right end of the placement area in increasing order, and then we set w_1 be the largest of the remaining unplaced numbers. Continue with step (2).

(2) If C is empty or if $C = 1$, place the unplaced numbers in increasing order into the empty slots. Otherwise, suppose $C = a_1a_2\cdots a_k$. Then we consider the binary vector $0^{a_1-1}10^{a_2-1}1\cdots 0^{a_k-1}$ (each block of 0's except the last one is followed by a 1). We read this binary vector from right to left and whenever we meet a 0, we place the largest unplaced number into the leftmost available slot; otherwise, we place this number into the rightmost available slot. If this procedure can no longer be continued, and 1 or 2 have not yet been placed, place them so that 1 precedes 2.

Note that the procedure above produces an $(n-k+1)$-permutation. Indeed, $X = C\underline{k}Dt$ is a composition of $(n-1)$, and each part a of X, with the exception of $\underline{k}$ and t, contributes placement of a elements in the permutation, whereas $\underline{k}$ contributes one element and t contributes $t-1$ elements. The requirement that 1 precedes 2 in the permutation places one last element. Clearly the algorithm constructs a "good" permutation.

We provide some examples. Suppose we are interested in $\underline{1}$ in the following palindrome of 16: $2\underline{1}2141212$. The steps of our recursive bijection are as follows:

$$*7*****89 \to *76****89 \to 476***589 \to 4763**589 \to 476312589.$$

As further examples, one can check that the underlined 5's in $\underline{5}115$, $15\underline{5}1$, and $\underline{5}25$ correspond to 231, 321, and 123 respectively. More examples can be found in Table 2 illustrating our second encoding of the 3s in all the palindromic compositions of $N = 10$ (the case $n = 6$ and $k = 3$). Comparing Tables 1 and 2 shows that our first and second encodings are nonequivalent.

The inverse of this algorithm is easy to find: we use the positions of $1, 2, \ldots, w_1 - 1$ to invert (2) and the positions of $w_1 + 1, w_1 + 2, \ldots, n - k + 1$ to invert (1). In particular, if $w_1 = 1$ (resp. $w_1 = 2$) then the corresponding k-part is the leftmost (resp. rightmost) one in a composition.

5. Additional Encodings with Restricted Permutations

We now provide some additional examples of encodings of combinatorial objects by restricted permutations to demonstrate various approaches to bijective enumeration. But first, some definitions.

Permutation	(C, D, t)	Palindromic composition
(1,2,3,4)	$(\emptyset, \emptyset, 2)$	**3**+4+3
(1,3,2,4)	$(\emptyset, 1, 1)$	**3**+1+2+1+3
(1,4,2,3)	$(\emptyset, 2, 0)$	**3**+2+0+2+3
(1,4,3,2)	$(\emptyset, 11, 0)$	**3**+1+1+0+1+1+3
(2,1,3,4)	$(\emptyset, \emptyset, 2)$	3+4+**3**
(2,3,1,4)	$(\emptyset, 1, 1)$	3+1+2+1+**3**
(2,4,1,3)	$(\emptyset, 2, 0)$	3+2+0+2+**3**
(2,4,3,1)	$(\emptyset, 11, 0)$	3+1+1+0+1+1+**3**
(3,1,2,4)	$(1, \emptyset, 1)$	1+**3**+2+3+1
(3,4,1,2)	$(1, 1, 0)$	1+**3**+1+0+1+3+1
(4,3,1,2)	$(2, \emptyset, 0)$	2+**3**+0+3+2
(4,2,1,3)	$(11, \emptyset, 0)$	1+1+**3**+0+3+1+1
(3,2,1,4)	$(1, \emptyset, 1)$	1+3+2+**3**+1
(3,4,2,1)	$(1, 1, 0)$	1+3+1+0+1+**3**+1
(4,3,2,1)	$(2, \emptyset, 0)$	2+3+0+**3**+2
(4,1,2,3)	$(11, \emptyset, 0)$	1+1+3+0+**3**+1+1

Table 2: The second encoding in the case $n = 6$ and $k = 3$.

A sequence $a_1, a_2, \ldots, a_n$ is *bitonic* if for some h, $1 \le h \le n$, we have that $a_1 \le a_2 \le \cdots \le a_h \ge a_{h+1} \ge \cdots \ge a_{n-1} \ge a_n$ or $a_1 \ge a_2 \ge \cdots \ge a_h \le a_{h+1} \le \cdots \le a_{n-1} \le a_n$. A binary string x is said to be *without singletons* if the words 010 and 101 are not factors of x.

Let S_1 (resp. S_2) be the set of $(n+2)$-permutations $w_1 w_2 \cdots w_{n+2}$ such that $w_1 w_2 = (n+1)(n+2)$ or $w_1 w_2 = (n+2)(n+1)$, and $w_3 w_4 \cdots w_{n+2}$ avoids simultaneously the patterns 1-2-3 and 2-3-1 (resp. 1-2-3, 1-3-2, and 2-1-3). According to [1], $|S_1| = n^2 - n + 2$ and $|S_2| = 2F_n$, where F_n is the n-th Fibonacci number with $F_0 = F_1 = 1$.

Let S_3 be the set of $(n+3)$-permutations $w_1 w_2 \cdots w_{n+3}$ such that, $w_1 < w_2 < w_3$ and $w_4 w_5 \cdots w_{n+3}$ is in decreasing order. Clearly, $|S_3| = \binom{n+3}{3}$.

Let S_4 be the set of n-permutations $w_1 w_2 \cdots w_n$ such that w_1 is the largest letter among the four leftmost letters, $w_3 < w_4$ and $w_5 w_6 \cdots w_n$ is in decreasing order. One can see that $|S_4| = 3\binom{n}{4}$.

Bijection 1. *The elements of S_1 are in one-to-one correspondence with binary bitonic sequences of length $n-1$.*

Proof. In order to avoid the restrictions, $w_3 w_4 \cdots w_{n+2}$ must be either of the form $i(i-1)\cdots 1 n(n-1)\cdots(i+1)$ for $i \ge 0$ or of the form

$$n(n-1)\cdots(n-i+1)(j+1)j\cdots 1(n-i)(n-i-1)\cdots(j+2)$$

for some $i > 0$ and $j \ge 0$.

We describe our bijection in the case $w_1w_2 = (n+1)(n+2)$. We then use the same bijection for $w_1w_2 = (n+2)(n+1)$ and replace 0's by 1's and 1's by 0's in the corresponding sequences.

To the permutation $(n+1)(n+2)i(i-1)\cdots 1n(n-1)\cdots(i+1)$ there corresponds the bitonic sequence 01^i0^{n-i-2} where $i > 0$; to the permutation

$$(n+1)(n+2)n(n-1)\cdots(n-i+1)(j+1)j\cdots 1(n-i)(n-i-1)\cdots(j+2)$$

there corresponds the sequence $00^i1^{n-i-j-2}0^j$ where $i \geq 0$ and $j \geq 0$. Clearly, our map involves all the binary bitonic sequences starting with 0 exactly once and the reverse to this map is easy to see. Together with the case $w_1w_2 = (n+2)(n+1)$ we have a bijection. $\square$

Bijection 2. *The elements of S_2 are in one-to-one correspondence with binary strings of length $n+2$ without singletons.*

Proof. Clearly, any string under consideration ends with either 00 or with 11. We match the strings ending with 00 with the permutations beginning with $w_1w_2 = (n+1)(n+2)$. It will suffice to consider this case. The remaining case is handled by replacing 0's by 1's and 1's by 0's, proceeding with the first case, and then replacing $(n+1)(n+2)$ with $(n+2)(n+1)$ in the resulting permutation.

We begin with a procedure for constructing permutations $w_3w_4\cdots w_{n+2}$ that avoid the specified patterns. Insert the numbers $1, 2, \ldots, n$, in that order, into n slots corresponding to the letters w_i, $3 \leq i \leq n+2$ according to the procedure below. Note that we must either set $w_{n+2} = 1$ or set $w_{n+1}w_{n+2} = 12$, since otherwise we get an occurrence of a prohibited pattern. We proceed by induction. If the rightmost i slots have been filled and w_{n-i} is empty, then we only have two choices: either set $w_{n-i} = i+1$ or set $w_{n-i-1}w_{n-i} = (i+1)(i+2)$. From this assignment pattern it is easy to see that a permutation $w_3w_4\cdots w_{n+2}$ that avoids the restricted patterns may be thought of as a tiling of a $1 \times n$ board by monominos and dominos.

Now, given such a tiling, we construct a binary string $b_1b_2\cdots b_n00$ corresponding to that tiling. Read the tiling from right to left. If the rightmost tile is a monomino, set $b_n = 0$. Otherwise, set $b_{n-1}b_n = 11$. In general, if the last digit placed in the binary string was $b_i = x \in \{0, 1\}$, and the next unread tile is a monomino, read this tile and set $b_{i-1} = x$. Otherwise, if the next unread tile is a domino, read this domino and set $b_{i-2}b_{i-1} = \bar{x}\bar{x}$, where $\bar{x}$ is the binary complement of x. In this way, we avoid the possibility of creating singletons.

This process is reversible: we read a binary word without singletons from right to left while tiling a $1 \times n$ board with monominos and dominos. Whenever we meet $\bar{x}\bar{x}$ after passing x in the binary string, we place a domino on the board. Otherwise, we place a monomino. The resulting tiling defines the corresponding permutation according to the construction described above. Performing all of these correspondences yields the desired bijection. For example, if $w_1w_2\cdots w_9 = 896753412$, then we produce $b_1b_2\cdots b_700 = 110001100$. $\square$

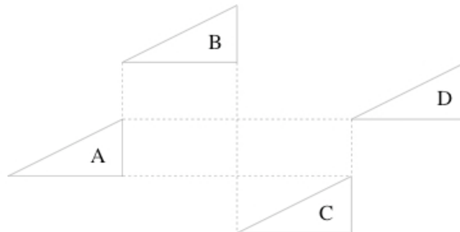

Figure 1: The structure of permutations avoiding 1-3-2-4 and having exactly one descent.

Bijection 3. *The elements of S_3 are in one-to-one correspondence with $(n+2)$-permutations avoiding* 1-3-2-4 *and having exactly one descent (a descent is an i such that $w_i > w_{i+1}$).*

Proof. Any $(n+2)$-permutation avoiding 1-3-2-4 and having exactly one descent has the structure $ABCD$, where $A = (i_1+1)(i_1+2)\cdots i_2$, $B = (i_3+1)(i_3+2)\cdots(n+2)$, $C = 12\cdots i_1$, $D = (i_2+1)(i_2+2)\cdots i_3$ (see Figure 1), and one of the following four mutually exclusive possibilities occurs:

1. none of A, B, C, and D is empty: there are $\binom{n+1}{3}$ such permutations, given by the number of ways to choose the least elements in A, B, and D (we know that 1 belongs to C);

2. C is empty: there are $\binom{n+1}{2}$ such permutations, since 1 belongs to A and we choose the least elements in B and D;

3. B is empty: there are $\binom{n+1}{2}$ such permutations, since 1 belongs to C and we choose the least elements in A and D;

4. A and C are empty: there are $(n+1)$ such permutations, since 1 is in D and we need to choose the length of D (B is not empty).

No other cases can exist since requiring that only A is empty, or requiring that only D is empty, gives Case 4 above. Note that summing over all the cases gives us exactly $\binom{n+3}{3}$ permutations. Once the permutations avoiding 1-3-2-4 with exactly one descent have been partitioned into the four cases above, it is easy to find bijections in each case with permutations in S_3 as follows.

1. $abc(n+3)(n+2)w_6w_7\cdots w_{n+3}$, where $a < b < c < n+2$ and $w_6w_7\cdots w_{n+3}$ is decreasing. Choosing a, b, and c corresponds to choosing i_1, i_2, and i_3;

2. $ab(n+3)(n+2)w_5w_6\cdots w_{n+3}$, where $a < b < n+2$ and $w_5w_6\cdots w_{n+3}$ is decreasing. Choosing a and b corresponds to choosing i_2 and i_3;

3. $ab(n+2)(n+3)w_5w_6\cdots w_{n+3}$, where $a < b < n+2$ and $w_5w_6\cdots w_{n+3}$ is decreasing. Choosing a and b corresponds to choosing i_1 and i_2;

4. $a(n+2)(n+3)w_4w_5\cdots w_{n+3}$, where $a < n+2$ and $w_4w_5\cdots w_{n+3}$ is decreasing. The length of D corresponds to a.

Since these cases provide a partition of permutations in S_3, the bijection is complete. □

Bijection 4. *Given n lines in the plane such that no two are parallel, no three lines share a point, and no three points of intersection are collinear, let P be the set of points at which pairs of these lines intersect. Then the elements of S_4 are in one-to-one correspondence with the set of all lines drawn through pairs of points in P.*

Proof. Labeling the lines by $1, 2, \ldots, n$, each intersection point can be represented by a pair of numbers (x, y) corresponding to the intersecting lines. Any line containing a pair of intersecting points can be described by a pair $((x, y), (z, v))$ where x, y, z, and v are different. Assuming that $x < y$, $z < v$, and $y < v$ we construct the corresponding permutation $vzxyw_5w_6\cdots w_n$ where $w_5w_6\cdots w_n$ is decreasing. Clearly this map is a bijection. □

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