THE SIZE OF THE LARGEST PART OF RANDOM PLANE PARTITIONS OF LARGE INTEGERS

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Abstract

We study the asymptotic behavior of the largest part size of a plane partition ω of the positive integer n, assuming that ω is chosen uniformly at random from the set of all such partitions. We prove that this characteristic, appropriately normalized, tends weakly, as $n \to \infty$, to a random variable having an extreme value probability distribution with distribution function, equal to $e^{-e^{-z}}, -\infty < z < \infty$. The representation of a plane partition as a solid diagram shows that the same limit theorem holds for the numbers of rows and columns of a random plane partition of n.

1. Introduction

A plane partition ω of the positive integer n is an array of non-negative integers

$$\begin{array}{cccc} \omega_{1,1} & \omega_{1,2} & \omega_{1,3} & \cdots \\ \omega_{2,1} & \omega_{2,2} & \omega_{2,3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array} \tag{1.1}$$

for which $\sum_{i,j} \omega_{i,j} = n$ and the rows and columns are arranged in decreasing order:

$$\omega_{i,j} \geq \omega_{i+1,j}, \omega_{i,j} \geq \omega_{i,j+1}$$

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for all $i, j \geq 1$. The non-zero entries $\omega_{i,j} > 0$ are called parts of ω. If there are λ_i parts in the ith row of ω, so that for some r

$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r > \lambda_{r+1} = 0,$$

then the partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ of the integer $p = \lambda_1 + \lambda_2 + ... + \lambda_r$ is called the shape of ω, denoted by λ. We also say that ω has r rows and p parts. Sometimes, for the sake of brevity, the zeroes in the array (1.1) are deleted. For instance, the abbreviation

$$\begin{matrix} 3 & 2 & 1 \\ 1 & 1 & \end{matrix}$$

is assumed to present a plane partition of $n = 8$ having $r = 2$ rows and $p = 5$ parts.

It seems that MacMahon was the first who introduced the idea of a plane partition; see [9]. He deals with the general problem of such partitions, enumerating them by the size of each part, number of rows and number of columns. These problems have been subsequently reconsidered by other authors who have developed methods, entirely different from those of MacMahon. For important references and more details in this direction, we refer the reader to the monographs of Andrews [4; Chap. 11] and Stanley [15; Chap. 7], as well as to the survey paper [14; Chap. V].

Let $q(n)$ denote the total number of plane partitions of the integer $n \geq 1$. It turns out that

$$Q(x) = 1 + \sum_{n=1}^{\infty} q(n)x^n = \prod_{j=1}^{\infty} (1 - x^j)^{-j} \tag{1.2}$$

(see [4; Corollary 11.3] or [14; Corollary 18.2]). The asymptotic of $q(n)$ has been obtained by Wright [17]. It is given by the following formula:

$$q(n) \sim \frac{[\zeta(3)]^{7/36}}{2^{11/36} 3^{1/2} \pi^{1/2}} n^{-25/36} \exp\{3[\zeta(3)]^{1/3}(n/2)^{2/3} + 2\gamma\}, \tag{1.3}$$

where

$$\zeta(s) = \sum_{j=1}^{\infty} j^{-s} \tag{1.4}$$

is the Riemann's zeta function and

$$\gamma = \int_0^{\infty} \frac{y \log y}{e^{2\pi y} - 1} dy.$$

The asymptotic of the coefficients in the power series expansion of $\prod_{j=1}^{\infty}(1 - x^j)^{-a_j}$ under a general scheme of assumptions on the sequence of non-negative numbers $\{a_j\}_{j \geq 1}$ was obtained by Meinardus [10]. For plane partitions, we have $a_j = j, j = 1, 2, ...$ Meinardus' theorem confirms Wright's formula (1.3). (In the original statement of (1.3) [17; p. 179] the constant $3^{1/2}$ in the denominator is missing. However, it appears at the end of the proof on p. 189. Further results on the asymptotic expansion of $q(n)$ may be found in [2,3].)

We introduce the uniform probability measure P on the set of all plane partitions of n, assuming that the probability $1/q(n)$ is assigned to each plane partition.

Let L_n, C_n and R_n denote the size of the largest part, the number of columns and number of rows in a plane partition of n, respectively. With respect to the probability measure P, L_n, C_n and R_n become random variables defined on the set of plane partitions of n. Our aim in this paper is to study their limiting distributions as $n \to \infty$.

We notice that any plane partition ω, defined by array (1.1), has an associated diagram $D(\omega) = \{(i,j,k) \in \mathbf{N}^3 : 1 \le k \le \omega_{i,j}\}$. Here $\mathbf{N}$ denotes the set of the positive integers. Any permutation σ of the three coordinate axis (i,j,k), different from the identical one, transforms $D(\omega)$ in a diagram that uniquely determines another plane partition $\sigma \circ \omega$. The permutation σ also permutes the three statistics (L_n, C_n, R_n). Then, if one of these statistics is restricted by certain inequality, the same restriction occurs on the statistics permuted by σ. The one to one correspondence between ω and $\sigma \circ \omega$ implies that L_n, C_n and R_n have one and the same probability distribution (for more details see also [15; p. 371]).

The starting point in our asymptotic analysis is the following generating function identity, which follows from a stronger result of MacMahon [9; Section 495]:

$$1 + \sum_{n=1}^{\infty} P(L_n \le m, R_n \le r) q(n) x^n = \prod_{k=1}^{m} \prod_{j=1}^{r} (1 - x^{j+k-1})^{-1}, m, r = 1, 2, \ldots.$$

(For more details and other proofs of this result we refer the reader to [14; Chap. V]). If we keep either of the parameters m and r fixed, setting the other one $:= \infty$, we obtain

$$1 + \sum_{n=1}^{\infty} P(X_n \le m) q(n) x^n = \prod_{k=1}^{m} (1 - x^k)^{-k} \prod_{j=m+1}^{\infty} (1 - x^j)^{-m}$$
$$= Q(x) \prod_{j=m+1}^{\infty} (1 - x^j)^{j-m}, X_n = L_n, C_n, R_n. \tag{1.5}$$

Here $Q(x)$ was defined by (1.2). Note that C_n is also included in (1.5) since its distribution coincides with those of L_n and R_n.

The object of this paper is to provide an asymptotic analysis for the Cauchy integrals stemming from (1.5) in order to prove the following theorem.

Theorem 1. For any real and finite z, we have

$$\lim_{n \to \infty} P \left\{ \left[\frac{2\zeta(3)}{n} \right]^{1/3} X_n - \log \left[\frac{n}{2\zeta(3)} \right]^{2/3} - \log \left[\frac{1}{2} \log^2 \left(n^{2/3} \right) \right] \le z \right\} = e^{-e^{-z}}, \tag{1.6}$$

where $\zeta(s)$ denotes Riemann's zeta function (1.4) and $X_n = L_n, C_n, R_n$.

The proof of this result is based on a method developed by Hayman [7]. The idea comes from the fact that $Q(x)$, the generating function enumerating the numbers $q(n)$ of plane partitions of n, satisfies Hayman's admissibilty properties in a neighborhood of its main singularity $x = 1$ and outside it (see Lemmas 1 and 2 of the next section, respectively). Hence, Hayman's asymptotic result for the coefficients $q(n)$ compensates, roughly speaking, their contribution to the general asymptotic of the coefficients in (1.5). A careful analysis of the product

$$\prod_{j=m+1}^{\infty} (1 - x^j)^{j-m},$$

around $x = 1$ then yields the weak convergence to the extreme value distribution (1.6). A relevant analytic approach to problems concerning random integer partitions is presented in [11,12]. A different probabilistic method was previously suggested by Fristedt [6], who used a conditioning device to transfer the problems related to integer partition statistics to problems dealing with functionals of independent and geometrically distributed random variables.

We organize the paper as follows. Section 2 contains auxiliary facts on the admissibility of $Q(x)$ and on the asymptotic behavior of the coefficients in its power series expansion. In Section 3 we prove our main result.

We conclude this section noticing that plane partitions have many applications to such diverse topics as ballot problems (see [14; Chap. IV] and [5]), symmetric functions (see [14; Chap. II - IV], [15; Chap. 7]) and the references therein), the representation theory of the symmetric group (see [14; Chap III] and [15; Chap. 7]), and analysis of algorithms (see [8; Section 5.2.4] and [13; Chap. 12]).

2. Preliminary Asymptotics

In order to get an asymptotic estimate for $Q(x)$ around its main singularity $x = 1$, we will use a general result due to Meinardus [10] (see also [4; Chap. 6]) on the asymptotic behavior of the nth coefficient in the expansion of

$$G(x) = \prod_{j=1}^{\infty} (1 - x^j)^{-a_j},$$

where $\{a_j\}_{j=1}^{\infty}$ is a sequence of given non-negative numbers. He introduced a set of assumptions on the Dirichlet's series

$$D(s) = \sum_{j=1}^{\infty} a_j j^{-s}, \tag{2.1}$$

generated by the sequence $\{a_j\}_{j=1}^{\infty}$. A basic restriction in his scheme states that $D(s)$ has to converge in the half-plane $\Re e(s) > \alpha > 0$ and can be analytically continued into the half-plane $\Re e(s) \geq -\alpha_0$ for some $\alpha_0 \in (0, 1)$. Moreover, in $\Re e(s) \geq -\alpha_0$, $D(s)$ is analytic except for a simple pole at $s = \alpha$ with residue A. Under these and certain other restrictions on $D(s)$, for $v = y + 2\pi i w$, Meinardus [10] (see also [4; Lemma 6.1]) proved that

$$G(e^{-v}) = \exp\{A\Gamma(\alpha)\zeta(\alpha+1)v^{-\alpha} - D(0)\log v + D'(0) + O(y^{\alpha_0})\} \tag{2.2}$$

as $y \to 0$ uniformly for $|\arg v| \leq \pi/4$ and $|w| \leq 1/2$. (Here $\Gamma(\alpha)$ denotes Euler's gamma function and $\log(.)$ presents the main branch of the logarithmic function satisfying $\log v < 0$ for $0 < v < 1$.)

Let us take now a sequence $\{r_n\}_{n=1}^{\infty}$ which, as $n \to \infty$, satisfies

$$r_n = 1 - \frac{[2\zeta(3)]^{1/3}}{n^{1/3}} + \frac{[2\zeta(3)]^{2/3}}{2n^{2/3}} - \frac{\zeta(3)}{n} + O(n^{-4/3}). \tag{2.3}$$

For the sake of brevity, for $0 < r < 1$, we also set

$$b(r) = \frac{6\zeta(3)}{(1-r)^4}. \tag{2.4}$$

It is easy to check that (2.3) and (2.4) imply

$$b(r_n) = \frac{3n^{4/3}}{[2\zeta(3)]^{1/3}} + O(n) \tag{2.5}$$

as $n \to \infty$.

The next lemma suggests a tool that we will subsequently use in Section 3 to obtain the main terms in our asymptotic.

Lemma 1. If r_n satisfies (2.3) for large n, then

$$Q(r_n e^{i\theta})e^{-i\theta n} = Q(r_n)e^{-\theta^2 b(r_n)/2}[1 + O(1/\log^3 n)]$$

as $n \to \infty$ uniformly for $|\theta| \leq \delta_n$, where

$$\delta_n = \frac{n^{-5/9}}{\log n} \tag{2.6}$$

and $b(r_n)$ is determined by (2.4).

Proof. We apply first Meinardus' asymptotic formula (2.2) to the plane partition generating function $Q(x)$ (see (1.2)). Since in this case we have $a_j = j, j = 1, 2, ...$ and $D(s) = \zeta(s-1)$ (see (2.1)), we find that $\alpha = 2, A = 1$ and $D(0) = -1/12$. Classical results on the ζ function (see [16; Section 13.51]) imply that (2.2) is valid with $v = y + 2\pi i w$. Therefore, we can write

$$Q(e^{-v}) = \exp\{\zeta(3)v^{-2} + \frac{1}{12}\log v + D'(0) + O(y^{\alpha_0})\} \tag{2.7}$$

as $y \to 0$ uniformly for $|w| \leq 1/2$ and $|\arg(v)| \leq \pi/4$ (here $\alpha_0 \in (0,1)$ is the constant specified in Meinardus assumption [10] for the analytical continuation of $D(s) = \zeta(s-1)$). Setting

$$e^{-v} = r_n e^{i\theta}, \tag{2.8}$$

we see that $y = y_n = -\log r_n$ and $w = -\theta/2\pi$. An asymptotic expansion for $-\log r_n$, as $n \to \infty$, can be found using (2.3) as follows:

$$\begin{aligned} y_n = -\log r_n &= \frac{[2\zeta(3)]^{1/3}}{n^{1/3}} - \frac{[2\zeta(3)]^{2/3}}{2n^{2/3}} + \frac{\zeta(3)}{n} \\ &+ \frac{[2\zeta(3)]^{2/3}}{n^{2/3}} - \frac{2^{4/3}[2\zeta(3)]^{2/3}[\zeta(3)]^{1/3}}{4n} + O(n^{-4/3}) \\ &= \frac{[2\zeta(3)]^{1/3}}{n^{1/3}} + O(n^{-4/3}). \end{aligned} \tag{2.9}$$

Combining (2.7) - (2.9), we observe that

$$\frac{Q(r_n e^{i\theta})}{Q(r_n)} e^{-i\theta n} = \left(\frac{y_n - i\theta}{y_n}\right)^{1/12} \exp\{\zeta(3)[(y_n - i\theta)^{-2} - y_n^{-2}] - i\theta n + O(y_n^{\alpha_0})\}. \tag{2.10}$$

A Taylor's formula expansion for $|\, \theta \,| \leq \delta_n$ yields

$$\begin{aligned} &(y_n - i\theta)^{-2} - y_n^{-2} = 2i\theta y_n^{-3} - 6\frac{\theta^2}{2} y_n^{-4} + O(|\, \theta \,|^3 \, y_n^{-5}) \\ &= 2i\theta y_n^{-3} - 3\theta^2 y_n^{-4} + O(\delta_n^3 y_n^{-5}). \end{aligned} \tag{2.11}$$

Using (2.9), we also get the following estimate for the factor outside the exponent in (2.10):

$$\left(\frac{y_n - i\theta}{y_n}\right)^{1/12} = \left\{\frac{[2\zeta(3)]^{1/3} - i\theta n^{1/3} + O(n^{-1})}{[2\zeta(3)]^{1/3} + O(n^{-1})}\right\}^{1/12} = 1 + O(\delta_n n^{1/3}), n \to \infty. \tag{2.12}$$

Finally we notice that (2.9) implies the bound

$$y_n^{\alpha_0} = O(n^{-\alpha_0/3}), n \to \infty, \tag{2.13}$$

where $\alpha_0 \in (0, 1)$. Hence, inserting (2.5), (2.9), (2.11) - (2.13) into (2.10), we obtain

$$\begin{aligned} &\frac{Q(r_n e^{i\theta})}{Q(r_n)} e^{-i\theta n} = [1 + O(\delta_n n^{1/3})] \exp\{\zeta(3)[\frac{2i\theta n}{2\zeta(3)[1 + O(n^{-1})]^3} \\ &- \frac{\theta^2}{2} \frac{6n^{4/3}}{2^{4/3}[\zeta(3)]^{4/3}[1 + O(n^{-1})]^4} + O(\delta_n^3 y_n^{-5})] - i\theta n + O(n^{-\alpha_0/3})\} \\ &= [1 + O(n^{-2/9}/\log n)] \exp\{\zeta(3)[\frac{i\theta n}{\zeta(3)}(1 + O(n^{-1})) \\ &- \frac{\theta^2}{2}[b(r_n) + O(n)](1 + O(n^{-1})) + O(\delta_n n^{5/3})] - i\theta n + O(n^{-\alpha_0/3})\} \\ &= [1 + O(n^{-2/9}/\log n)] \exp\{i\theta n + O(\delta_n) - \theta^2 b(r_n)/2 + O(n\delta_n^2) \\ &+ O(\delta_n^2 b(r_n) n^{-1}) + O(\delta_n^3 n^{5/3}) - i\theta n + O(n^{-\alpha_0/3})\} \\ &= [1 + O(n^{-2/9}/\log n)] \exp\{-\theta^2 b(r_n)/2 + O(1/\log^3 n)\} \\ &= e^{-\theta^2 b(r_n)/2}[1 + O(1/\log^3 n)], n \to \infty. \end{aligned}$$

This completes the proof. □

We also need another lemma that will establish a uniform estimate for

$$Q_m(x) = \prod_{j=1}^{m}(1-x^j)^{-j}, m = 1, 2, ..., \tag{2.14}$$

where $\mid x \mid = r_n$ and $\arg(x)$ is outside the range $(-\delta_n, \delta_n)$. For the sake of convenience, we also let

$$f_{m,n}(\theta) = \sum_{j>m} j r_n^j [\cos(j\theta) - 1]. \tag{2.15}$$

Lemma 2. If r_n and δ_n satisfy (2.3) and (2.6), respectively, then there exist two positive constants ϵ and n_0 such that

$$\mid Q_m(r_n e^{i\theta}) \mid \le Q_m(r_n) \exp\{\epsilon - (2.5)n^{2/9}/[2\zeta(3)]^{4/3} \log^2 n - f_{m,n}(\theta)\}$$

uniformly for $m = 1, 2, ..., \pi \ge \mid \theta \mid \ge \delta_n$ and $n \ge n_0$.

Proof. By taking logarithms in (2.14), for $\mid x \mid < 1$, we get

$$\log Q_m(x) = \log[\prod_{j=1}^{m}(1-x^j)^{-j}] = -\sum_{j=1}^{m} j \log(1-x^j)$$
$$= \sum_{j=1}^{m} j \sum_{l=1}^{\infty} \frac{x^{jl}}{l} = \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=1}^{m} j (x^l)^j.$$

Thus, substituting $x = r_n e^{i\theta}$, we obtain

$$\mid Q_m(r_n e^{i\theta}) \mid = \mid \exp\{\log Q_m(r_n e^{i\theta})\} \mid = \mid \exp\{\sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=1}^{m} j r_n^{lj} e^{ijl\theta}\} \mid$$
$$= \mid \exp\{\sum_{j=1}^{m} j r_n^j e^{ij\theta} + \sum_{l=2}^{\infty} \frac{1}{l} \sum_{j=1}^{m} j r_n^{lj} e^{ijl\theta}\} \mid$$
$$= \exp\{\sum_{j=1}^{m} j r_n^j \Re e(e^{ij\theta}) + \sum_{l=2}^{\infty} \frac{1}{l} \sum_{j=1}^{m} j r_n^{lj} \Re e(e^{ijl\theta})\}$$
$$= \exp\{\sum_{j=1}^{m} j r_n^j \cos(j\theta) + \sum_{l=2}^{\infty} \frac{1}{l} \sum_{j=1}^{m} j r_n^{lj} \cos(lj\theta)\}$$
$$\le \exp\{\sum_{j=1}^{m} j r_n^j \cos(j\theta) + \sum_{l=2}^{\infty} \frac{1}{l} \sum_{j=1}^{m} j r_n^{lj}\}$$
$$= \exp\{\sum_{j=1}^{m} j r_n^j [\cos(j\theta) - 1] + \sum_{l=1}^{\infty} \frac{1}{l} \sum_{j=1}^{m} j r_n^{lj}\}$$
$$= Q_m(r_n) \exp\{H_{m,n}(\theta)\}, \tag{2.16}$$

where

$$\begin{aligned} H_{m,n}(\theta) &= \sum_{j=1}^{m} j r_n^j [\cos(j\theta) - 1] \\ &= \Re e \left[\frac{r_n e^{i\theta}}{(1 - r_n e^{i\theta})^2} \right] - \frac{r_n}{(1-r_n)^2} - f_{m,n}(\theta) \\ &= \frac{r_n(\cos\theta + r_n^2 \cos\theta - 2r_n)}{(1 - 2r_n\cos\theta + r_n^2)^2} - \frac{r_n}{(1-r_n)^2} - f_{m,n}(\theta). \end{aligned} \tag{2.17}$$

The function $\cos\theta + r_n^2 \cos\theta$ attains its maximum in the range $\delta_n \leq | \theta | \leq \pi$ at $\theta = \delta_n$ and $\theta = -\delta_n$. Note that

$$\cos\delta_n = 1 - \frac{\delta_n^2}{2} + O(\delta_n^4) = 1 - \frac{n^{-10/9}}{2\log^2 n} + O(n^{-20/9}/\log^4 n)$$

and

$$(1 - r_n)^{-2} = \left[\frac{n}{2\zeta(3)}\right]^{2/3} + O(n^{1/3}).$$

Substituting these estimates into the right-hand side of (2.17), after some manipulations, we obtain

$$\begin{aligned} H_{m,n}(\theta) &\leq \frac{r_n[1 - n^{-4/9}/2[2\zeta(3)]^{2/3}\log^2 n + O(n^{-7/9}/\log^2 n)]}{(1-r_n)^2\left[1 + \frac{r_n n^{-10/9}}{(1-r_n)^2\log^2 n} + O\left(\frac{n^{-20/9}}{(1-r_n)^2\log^4 n}\right)\right]^2} \\ &-\frac{r_n}{(1-r_n)^2} - f_{m,n}(\theta) \\ &= \frac{r_n[1 - n^{-4/9}/2[2\zeta(3)]^{2/3}\log^2 n + O(n^{-7/9}/\log^2 n)]}{(1-r_n)^2[1 + 2n^{-4/9}/[2\zeta(3)]^{2/3}\log^2 n + O(n^{-7/9}/\log^2 n)} \\ &-\frac{r_n}{(1-r_n)^2} - f_{m,n}(\theta) \\ &= \frac{r_n[1 - n^{-4/9}/2[2\zeta(3)]^{2/3}\log^2 n + O(n^{-7/9}/\log^2 n)]}{(1-r_n)^2} \\ &\times \left\{1 - \frac{2n^{-4/9}}{[2\zeta(3)]^{2/3}\log^2 n} + O\left(\frac{n^{-7/9}}{\log^2 n}\right)\right\} - \frac{r_n}{(1-r_n)^2} - f_{m,n}(\theta) \\ &= -r_n\left\{\left[\frac{n}{2\zeta(3)}\right]^{2/3} + O(n^{1/3})\right\}\frac{5n^{-4/9}}{2[2\zeta(3)]^{2/3}\log^2 n} + o(1) - f_{m,n}(\theta) \\ &= -\frac{(2.5)n^{2/9}}{[2\zeta(3)]^{4/3}\log^2 n} + o(1) - f_{m,n}(\theta). \end{aligned}$$

Inserting this estimate into (2.16), we obtain the required result. □

Further, we will essentially use the asymptotic form of the numbers $q(n)$. Although this is given by Wright's formula (1.3), we need this result in a slightly different form. Since Lemmas 1 and 2, together with (2.3) - (2.5), provide Hayman's admissiblity [7] for the generating function $Q(x)$ (see (1.2)), we can apply his general asymptotic formula to the coefficients $q(n)$. The next lemma encompasses these results. We only sketch its proof and insert a remark explaining the role of the asymptotic expansion (2.3) there.

Lemma 3. For $Q(x)$, the generating function of the numbers $q(n)$ of plane partitions of n defined by (1.2), we have

$$q(n) \sim Q(r_n) r_n^{-n} / \sqrt{2\pi b(r_n)} \tag{2.18}$$

as $n \to \infty$, where r_n satisfies the equation

$$rQ'(r)/Q(r) = n \tag{2.19}$$

for sufficiently large n and $b(r_n)$ is defined by (2.4).

Sketch of the proof. It is clear that $\mid x \mid= 1$ is a natural boundary for $Q(x)$. Lemma 1 shows the behavior of $Q(x)$ around its main singularity $x = 1$ (condition I of Hayman's definition [7]); Lemma 2 establishes that the growth of $Q(x)$ as $x \to x_0, \mid x_0 \mid= 1$ and $x_0 \neq 1$ is negligible (condition II of [7]). It is then easily seen that

$$\begin{aligned} \frac{rQ'(r)}{Q(r)} &= 2\sum_{j=1}^{\infty} \frac{r^{2j}}{(1-r^j)^3} + \sum_{j=1}^{\infty} \frac{r^j}{(1-r^j)^2} \\ &= \frac{2\zeta(3)}{(1-r)^3} + o((1-r)^{-1}) \end{aligned}$$

as $r \to 1^-$. This enables one to conclude that r_n, determined for sufficiently large n by (2.19), can be substituted by the asymptotic expansion (2.3). Thus, one can obtain (2.18) after direct application of Hayman's theorem [7]. □

Finally, notice that (2.3) - (2.5) and (2.7) imply the coincidence of the right sides of (1.3) and (2.18).

3. Proof of the Main Result

We apply first Cauchy's coefficient formula to (1.5) using the circle $x = r_n e^{i\theta}, -\pi < \theta \leq \pi$ as a contour of integration, where r_n is determined by (2.3). Thus, for $X_n = L_n$, we obtain

$$P(L_n \leq m)q(n) = \frac{r_n^{-n}}{2\pi} \int_{-\pi}^{\pi} Q(r_n e^{i\theta}) e^{-i\theta n} \prod_{j=m+1}^{\infty} (1 - r_n e^{ij\theta})^{j-m} d\theta.$$

Then, we break up the range of integration as follows:

$$P(L_n \leq m)q(n) = J_1(m,n) + J_2(m,n), \tag{3.1}$$

where

$$J_1(m,n) = \frac{r_n^{-n}}{2\pi} \int_{-\delta_n}^{\delta_n} Q(r_n e^{i\theta}) e^{-i\theta n} \prod_{j=m+1}^{\infty} (1 - r_n^j e^{ij\theta})^{j-m} d\theta, \tag{3.2}$$

$$J_2(m,n) = \frac{r_n^{-n}}{2\pi} \int_{\delta_n \le |\theta| \le \pi} Q(r_n e^{i\theta}) e^{-i\theta n} \prod_{j=m+1}^{\infty} (1 - r_n^j e^{ij\theta})^{j-m} d\theta. \tag{3.3}$$

We next let

$$e^{F_m(x)} = \prod_{j=m+1}^{\infty} (1 - x^j)^{j-m},$$

that is

$$F_m(x) = \sum_{j=m+1}^{\infty} (j-m) \log (1 - x^j). \tag{3.4}$$

3.1. An Asymptotic Estimate for $J_1(m,n)$

We will employ a more compact form for $J_1(m,n)$. It is based on the notation given in (3.4). We can write

$$J_1(m,n) = \frac{r_n^{-n}}{2\pi} \int_{-\delta_n}^{\delta_n} Q(r_n e^{i\theta}) \exp\{F_m(r_n e^{i\theta}) - i\theta n\} d\theta. \tag{3.5}$$

Using Taylor's formula expansion, we obtain

$$F_m(r_n e^{i\theta}) = F_m(r_n) + r_n(e^{i\theta} - 1) F_m'(r_n) + \frac{r_n^2}{2} (e^{i\theta} - 1)^2 F_m''(r_n)$$
$$+ O(|\theta|^3 | F_m'''(r_n) |).$$

Substituting this expression into (3.5) and applying the result of Lemma 1 to $Q(r_n e^{i\theta})$, we get a new expression for $J_1(m,n)$, namely:

$$J_1(m,n) = \frac{r_n^{-n}}{2\pi} Q(r_n) e^{F_m(r_n)} [1 + O(1/\log^3 n)]$$
$$\times \int_{-\delta_n}^{\delta_n} \exp\{(e^{i\theta} - 1) r_n F_m'(r_n) + \frac{1}{2}(e^{i\theta} - 1)^2 r_n^2 F_m''(r_n)$$
$$+ O(\delta_n^3 \mid F_m'''(r_n) \mid)\} d\theta. \tag{3.6}$$

Next, we will obtain asymptotic expressions for $F_m(r_n), F_m'(r_n), F_m''(r_n)$ and $F_m'''(r_n)$ in terms of the functions

$$\psi_j(v) = \int_0^{\infty} \frac{u^j du}{e^{u+v} - 1}, j = 0, 1, 2, 3, \tag{3.7}$$

where the parameter $v > 0$ will be further substituted by

$$v = v_n = m y_n \tag{3.8}$$

(see again (2.9) for the asymptotic expansion of y_n). The functions $\psi_j(v)$ are closely related to the Debye functions (see, e.g., [1; Section 27.1]). We will interpret the sums that we will obtain as Riemann ones with step sizes equal to $y_n = -\log r_n$.

Starting with $F_m(r_n)$, we find, as $n \to \infty$, that

$$\begin{aligned} F_m(r_n) &= \sum_{j>m} (j-m) \log(1-r_n^j) \\ &= y_n^{-2} \sum_{jy_n > v_n} (jy_n) y_n \log(1-e^{-jy_n}) - m y_n^{-1} \sum_{jy_n > v_n} y_n \log(1-e^{-jy_n}) \\ &\sim y_n^{-2} \int_{v_n}^{\infty} (u-v_n) \log(1-e^{-u}) du \end{aligned} \tag{3.9}$$

with v_n defined by (3.8). Integrating by parts the integral in (3.9) and using (3.7) with $j = 2$, we find that

$$e^{F_m(r_n)} = [1+o(1)] \exp\left\{-\frac{y_n^{-2}}{2} \psi_2(v_n)\right\} \tag{3.10}$$

as $n \to \infty$. In the same way, we show that

$$\begin{aligned} r_n F_m'(r_n) &= \sum_{j>m} \frac{(j-m) j r_n^j}{1-r_n^j} \\ &= -y_n^{-3} \sum_{jy_n > v_n} \frac{(jy_n)^2 y_n e^{-jy_n}}{1-e^{-jy_n}} + m y_n^{-2} \sum_{jy_n > v_n} \frac{(jy_n) y_n e^{-jy_n}}{1-e^{-jy_n}} \\ &= [1+o(1)](-y_n^{-3}) \int_{v_n}^{\infty} \frac{u(u-v_n)}{e^u-1} du = -y_n^{-3} [\psi_2(v_n) + v_n \psi_1(v_n)]. \end{aligned} \tag{3.11}$$

For the asymptotic of the second derivative of F_m we have

$$\begin{aligned} r_n^2 F_m''(r_n) &= -\sum_{j>m} (j-m) j \left[\frac{j r_n^j}{1-r_n^j} - \frac{r_n^j}{1-r_n^j} + \frac{j r_n^{2j}}{(1-r_n^j)^2}\right] \\ &= -\sum_{j>m} j^3 \left[\frac{r_n^j}{1-r_n^j} + \frac{r_n^{2j}}{(1-r_n^j)^2}\right] + m \sum_{j>m} j^2 \left[\frac{r_n^j}{1-r_n^j} + \frac{r_n^{2j}}{(1-r_n^j)^2}\right] \\ &+ \sum_{j>m} j^2 \frac{r_n^j}{1-r_n^j} - m \sum_{j>m} j \frac{r_n^j}{1-r_n^j} \\ &= [1+o(1)] \left\{-y_n^{-4} \int_{v_n}^{\infty} u^2 (u-v_n) \left[\frac{1}{e^u-1} + \frac{1}{(e^u-1)^2}\right] du + y_n^{-3} \int_{v_n}^{\infty} u(u-v_n) \frac{du}{e^u-1}\right\} \\ &= -[1+o(1)] y_n^{-4} \int_{v_n}^{\infty} u^2 (u-v_n) \left[\frac{1}{e^u-1} + \frac{1}{(e^u-1)^2}\right] du + O(y_n^{-3}) \end{aligned} \tag{3.12}$$

as $n \to \infty$. Since the integrand in (3.12) includes $(e^u - 1)^2$ in its denominator, we need to make an auxiliary calculation. Thus, for any $a \geq 1$, we have

$$I_a = \int_{v_n}^{\infty} \frac{u^a}{(e^u-1)^2} du = \frac{v_n^a}{e^{v_n}-1} + a \int_0^{\infty} \frac{(z+v_n)^{a-1}}{e^{z+v_n}-1} dz - \int_0^{\infty} \frac{(z+v_n)^a}{e^{z+v_n}-1} dz.$$

Setting $a = 3$ and $a = 2$, after simple manipulations, we easily obtain

$$\int_{v_n}^{\infty} \frac{u^2(u-v_n)}{(e^u-1)^2} du = I_3 - v_n I_2 = \int_0^{\infty} \frac{(3-2v_n) z^2 + v_n z + v_n^2 - z^3}{e^{z+v_n}-1} dz$$

$$= -\psi_3(v_n) + (3 - 2v_n)\psi_2(v_n) + v_n\psi_1(v_n) + v_n^2\psi_0(v_n)$$
$$= -\psi_3(v_n) + (3 - 2v_n)\psi_2(v_n) + v_n\psi_1(v_n) - \log(1 - e^{-v_n}). \quad (3.13)$$

On the other hand,

$$\int_{v_n}^{\infty} \frac{u^2(u - v_n)}{e^u - 1} du = \int_0^{\infty} \frac{(z^3 + 2v_n z^2 + v_n^2 z)dz}{e^{z+v_n} - 1} = \psi_3(v_n) + 2v_n\psi_2(v_n) + v_n^2\psi_1(v_n). \quad (3.14)$$

Upon replacing (3.13) and (3.14) into (3.12), we get the required asymptotic of the second derivative of F_m:

$$r_n^2 F_m''(r_n) = -[1 + o(1)]y_n^{-4}[3\psi_2(v_n) + (v_n^2 + v_n)\psi_1(v_n) + v_n^2\psi_0(v_n)], n \to \infty. \quad (3.15)$$

Finally, following similar but simpler analysis, we obtain the following O-bound for the third derivative of F_m:

$$F_m'''(r_n) = O\left(y_n^{-5} \int_{v_n}^{\infty} u^4 \left[\frac{1}{e^u - 1} + \frac{1}{(e^u - 1)^2} + \frac{1}{(e^u - 1)^3}\right] du\right) = O(y_n^{-5}), n \to \infty. \quad (3.16)$$

To find a complete asymptotic estimate for the integrand in (3.6), we approximate $e^{i\theta} - 1$ and $(e^{i\theta} - 1)^2$ by $i\theta + O(\delta_n^2)$ and $-\theta^2 + O(\delta_n^3)$, respectively. Hence, for $|\theta| \le \delta_n$, by (2.6) we get

$$e^{i\theta} - 1 = i\theta + O(n^{-10/9}/\log^2 n), \quad (3.17)$$
$$(e^{i\theta} - 1)^2 = -\theta^2 + O(n^{-5/3}/\log^3 n) \quad (3.18)$$

as $n \to \infty$. Furthermore, we notice that (2.9) implies, for large n, that

$$y_n^{-k} = \left[\frac{n}{2\zeta(3)}\right]^{k/3} [1 + O(1/n)] = \left[\frac{n}{2\zeta(3)}\right]^{k/3} + O(n^{k/3-1}), k = 2, 3, 4, 5.$$

We substitute these estimates in the right sides of (3.10), (3.11), (3.15) and (3.16). Combining the results obtained after these substitutions with those of (3.17) and (3.18), we see now that (3.6) becomes

$$J_1(m, n) = \frac{r_n^{-n}}{2\pi} Q(r_n)[1 + o(1)] \exp\left\{-\frac{1}{2}\left[\frac{n}{2\zeta(3)}\right]^{2/3} \psi_2(v_n)\right\}$$
$$\times \int_{-\delta_n}^{\delta_n} \exp\{-\frac{i\theta n}{2\zeta(3)}[\psi_2(v_n) + v_n\psi_1(v_n)] - \frac{\theta^2 b(r_n)}{2}$$
$$+\frac{\theta^2}{2}\left[\frac{n}{2\zeta(3)}\right]^{4/3} [3\psi_2(v_n) + (v_n^2 + v_n)\psi_1(v_n) - v_n^2 \log(1 - e^{-v_n})]\} d\theta, n \to \infty.$$

By letting $\theta = t/b^{1/2}(r_n), -\infty < t < \infty$, in this integral with $b(r_n)$ defined by (2.4) (see also (2.5)), we get

$$J_1(m, n) = \frac{r_n^{-n}}{2\pi b^{1/2}(r_n)} Q(r_n)[1 + o(1)] exp\left\{-\frac{1}{2}\left[\frac{n}{2\zeta(3)}\right]^{2/3} \psi_2(v_n)\right\}$$
$$\times \int_{-\delta_n b^{1/2}(r_n)}^{\delta_n b^{1/2}(r_n)} \exp\{-itA_n - B_n t^2/2\} dt, n \to \infty, \quad (3.19)$$

where

$$A_n = \frac{n}{2\zeta(3)b^{1/2}(r_n)}[\psi_2(v_n) + v_n\psi_1(v_n)], \tag{3.20}$$

$$B_n = 1 - \left[\frac{n}{2\zeta(3)}\right]^{4/3}[3\psi_2(v_n) + (v_n^2 + v_n)\psi_1(v_n) - v_n^2 \log(1 - e^{-v_n})]/b(r_n). \tag{3.21}$$

On the other hand, (2.5) and (2.6) imply that

$$\delta_n b^{1/2}(r_n) \sim dn^{1/9} \log n, d = \sqrt{3}/[2\zeta(3)]^{1/6}. \tag{3.22}$$

Further, we also need some asymptotic estimates for $\psi_j(v_n), j = 1, 2$, as $v_n \to \infty$, presented in a slightly different form. Integrating by parts the right side of $\psi_2(v_n)$ (see (3.7)), we get

$$\psi_2(v_n) = -v_n^2 \log(1 - e^{-v_n}) - 2\int_{v_n}^{\infty} u \log(1 - e^{-u})du. \tag{3.23}$$

For $u \in [v_n, \infty)$, we have

$$-\log(1 - e^{-u}) = e^{-u} + O(e^{-2u}) = e^{-u}[1 + O(e^{-v_n})],$$

and therefore,

$$\begin{aligned} -\int_{v_n}^{\infty} u \log(1 - e^{-u})du &= [1 + O(e^{-v_n})]\int_{v_n}^{\infty} ue^{-u}du \\ &= [1 + O(e^{-v_n})](v_n e^{-v_n} + e^{-v_n}) = O(v_n e^{-v_n}), v_n \to \infty. \end{aligned}$$

Substituting this into (3.23), we conclude that

$$\psi_2(v_n) = -v_n^2 \log(1 - e^{-v_n}) + O(v_n e^{-v_n}), v_n \to \infty. \tag{3.24}$$

In the same way for $\psi_1(v_n)$ we obtain the following asymptotic:

$$\psi_1(v_n) = -v_n \log(1 - e^{-v_n}) + O(e^{-v_n}), v_n \to \infty. \tag{3.25}$$

We are now ready to specify the sequence $\{v_n\}_{n=1}^{\infty}$ that will produce a non-degenerate limiting distribution. We set

$$v_n = \frac{2}{3}\log n + \log\log n + z, -\infty < z < \infty, n = 1, 2, ... \tag{3.26}$$

Obviously, assumption (3.26) implies that

$$-\log(1 - e^{-v_n}) = e^{-v_n} + O(e^{-2v_n}) = \frac{e^{-z}}{n^{2/3}\log^2 n} + O(n^{-4/3}/\log^4 n),$$

and therefore,

$$-v_n^2 \log(1 - e^{-v_n}) = \frac{4e^{-z}}{9n^{2/3}}\left[1 + O\left(\frac{\log\log n}{\log n}\right)\right] \tag{3.27}$$

as $v_n \to \infty$. Therefore, the exponent outside the integral of the right side of (3.19) becomes

$$\exp\left\{-\frac{1}{2}\left[\frac{n}{2\zeta(3)}\right]^{2/3}\psi_2(v_n)\right\} = \exp\left\{-\frac{2e^{-z}}{9[2\zeta(3)]^{2/3}}\right\} + O\left(\frac{\log\log n}{\log n}\right) \tag{3.28}$$

as $n \to \infty$. Turning to the integrand of (3.19), note that (3.24) - (3.27) imply somewhat less precise but shorter estimates for ψ_1 and ψ_2. We have

$$\psi_2(v_n) = O(n^{-2/3}), v_n\psi_1(v_n) = O(n^{-2/3}), n \to \infty.$$

Inserting them into (3.20) and taking into account the asymptotic form of $b(r_n)$ from (2.5), we get

$$A_n = O(n^{-2/3}), n \to \infty. \tag{3.29}$$

A similar analysis of (3.21) shows that

$$B_n = 1 + O(n^{-2/3}\log n). \tag{3.30}$$

Therefore the integrand of (3.19) tends to $e^{-t^2/2}$ as $n \to \infty$.

Our final remark concerns the bounds of the integral in (3.19). They can be substituted by $-\infty$ and ∞ with an error term that is estimated, e.g., by using the asymptotic expansion of the function $(2\pi)^{-1/2}\int_v^\infty e^{-t^2}dt$ as $v \to \infty$ (see [1; Chap. 7]). Hence, by (3.22) we find that its order, as $n \to \infty$, is at most $O(n^{-1/9}(\log n)\exp\{-d^2 n^{2/9}/\log^2 n\}) = o(1)$. This, in combination with (3.28) - (3.30) and the Lebesgue dominated convergence theorem, shows that

$$J_1(m,n) = \frac{r_n^{-n}}{2\pi b^{1/2}(r_n)}Q(r_n)[1+o(1)]\exp\left\{-\frac{2e^{-z}}{9[2\zeta(3)]^{2/3}}\right\}$$
$$\times\int_{-\infty}^{\infty} e^{-t^2/2}dt = \frac{r_n^{-n}}{\sqrt{2\pi b(r_n)}}Q(r_n)[1+o(1)]\exp\left\{-\frac{2e^{-z}}{9[2\zeta(3)]^{2/3}}\right\} \tag{3.31}$$

as $n \to \infty$. Thus the estimate for $J_1(m,n)$ is completed.

3.2. An Asymptotic estimate for $J_2(m,n)$

First, we recall (2.14) and (3.3). It is also clear that

$$Q(r_n e^{i\theta})\prod_{j=m+1}^{\infty}(1 - r_n^j e^{ij\theta})^{j-m} = Q_m(r_n e^{i\theta})\prod_{j=m+1}^{\infty}\frac{1}{(1 - r_n^j e^{ij\theta})^m}. \tag{3.32}$$

The infinite product in the right side of (3.32) can be estimated as follows:

$$\left|\prod_{j=m+1}^{\infty}\frac{1}{(1 - r_n^j e^{ij\theta})^m}\right| \le \prod_{j=m+1}^{\infty}\frac{1}{(1 - r_n^j)^m} < \prod_{j=m+1}^{\infty}\frac{1}{(1 - r_n^j)^j}. \tag{3.33}$$

This bound is uniform for $m = 1, 2, ...$ Now, to estimate the integrand in $J_2(m, n)$, we apply successively Lemma 2 and (3.33). We obtain

$$\begin{aligned} &\mid Q(r_n e^{i\theta}) e^{-i\theta n} \prod_{j=m+1}^{\infty} (1 - r_n^j e^{ij\theta})^{j-m} \mid \\ &\leq Q_m(r_n) \exp\{\epsilon - (2.5) n^{2/9}/[2\zeta(3)]^{4/3} \log^2 n - f_{m,n}(\theta)\} \prod_{j=m+1}^{\infty} \frac{1}{(1 - r_n^j)^j} \\ &= Q(r_n) \exp\{\epsilon - (2.5) n^{2/9}/[2\zeta(3)]^{4/3} \log^2 n - f_{m,n}(\theta)\}, \end{aligned} \tag{3.34}$$

since by (2.14)

$$Q(r_n) = Q_m(r_n) \prod_{j=m+1}^{\infty} \frac{1}{(1 - r_n^j)^j}.$$

Using (2.15), (2.9) and (3.26), we find that

$$\begin{aligned} &\mid f_{m,n}(\theta) \mid \leq 2 \sum_{j>m} j r_n^j = 2 y_n^{-2} \sum_{j y_n > m y_n} y_n (j y_n) e^{-j y_n} \\ &= O\left(n^{2/3} \int_{v_n}^{\infty} u e^{-u} du\right) = O(1). \end{aligned}$$

In order to get an estimate which contains a term that is asymptotically equivalent to $q(n)$, we apply first the bound from (3.34), and then multiply and divide the whole expression by $\sqrt{2\pi b(r_n)}$. Thus, using (2.4) again, we finally get

$$\begin{aligned} &\mid J_2(m, n) \mid \leq r_n^{-n} Q(r_n) \exp\{-(2.5) n^{2/9}/[2\zeta(3)]^{4/3} \log^2 n + O(1)\} \\ &= \frac{r_n^{-n} Q(r_n)}{\sqrt{2\pi b(r_n)}} exp\{-(2.5) n^{2/9}/[2\zeta(3)]^{4/3} \log^2 n + \log \sqrt{2\pi b(r_n)} + O(1)\} \\ &= \frac{r_n^{-n} Q(r_n)}{\sqrt{2\pi b(r_n)}} \exp\{-(2.5) n^{2/9}/[2\zeta(3)]^{4/3} \log^2 n + \frac{4}{3} \log n + O(1)\} \end{aligned} \tag{3.35}$$

as $n \to \infty$.

3.3. End of the Proof

Applying the result of Lemma 3 to (3.31) and (3.35), we obtain

$$J_1(m, n) = \exp\left\{-\frac{2e^{-z}}{9[2\zeta(3)]^{2/3}}\right\} q(n)[1 + o(1)]$$

and

$$J_2(m, n) = o(q(n)).$$

Replacing these estimates into (3.1) - (3.3), after appropriate cancellations we find that

$$P(L_n \leq m) = \exp\left\{-\frac{2e^{-z}}{9[2\zeta(3)]^{2/3}}\right\} + o(1). \tag{3.36}$$

Here $m = v_n/y_n$ is determined by (3.8). Using estimate (2.9) for y_n and expression (3.26) for v_n, we conclude that

$$m = \frac{\frac{2}{3}\log n + 2\log\log n + z}{y_n} = \left[\frac{n}{2\zeta(3)}\right]^{1/3}\left(\frac{2}{3}\log n + 2\log\log n + z\right)[1 + O(1/n)]$$
$$= \left[\frac{n}{2\zeta(3)}\right]^{1/3}\left(\frac{2}{3}\log n + 2\log\log n + z\right) + O(n^{-2/3}\log n)$$

as $n \to \infty$. Substituting this value into (3.36), we obtain

$$P\left\{\left[\frac{2\zeta(3)}{n}\right]^{1/3} L_n - \log(n^{2/3}) - \log(\log^2 n) \leq z + o(1)\right\}$$
$$= \exp\left\{-\frac{2e^{-z}}{9[2\zeta(3)]^{2/3}}\right\} + o(1). \tag{3.37}$$

Now, it remains to replace z with $z + \log\{2/9[2\zeta(3)]^{2/3}\}$ and apply simple algebra to see that

$$-\log(n^{2/3}) - \log(\log^2 n) - \log\{2/9[2\zeta(3)]^{2/3}\}$$
$$= -\log\left[\frac{n}{2\zeta(3)}\right]^{2/3} - \log\left[\frac{1}{2}\log^2(n^{2/3})\right]. \tag{3.38}$$

The limiting distribution result (1.6) now follows from (3.37), (3.38) and the fact that L_n, C_n and R_n are equidistributed. □

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