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SUMBERS – SUMS OF UPS AND DOWNS

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Abstract

A sumber is a sum of ups, downs and star. We provide a simplification rule that can determine whether a game G is a sumber or not, and if it is, determine the exact sumber of G from its left and right options, G^L and G^R. Once one knows a game is a sumber, the outcome of the game can be determined by a simple rule.

1. Introduction

We are concerned with combinatorial games and follow the notations and conventions of *Winning Ways* [1]. A game G is an ordered pair of sets of games,

$$G = \{G^L | G^R\},$$

where G^L and G^R are called the left and right options of G. We assume the readers are familiar with the *birthdays* [2] of games, and say a game is *simpler* if it was *born* earlier. The simplest game is the game 0, defined as

$$0 = \{\ |\ \}.$$

The following are the basic definitions of combinatorial games:

$$-G = \{-G^R | -G^L\}.$$

$$G \geq H \text{ iff no element in } G^R \leq H.$$

$$G \leq H \text{ iff no element in } G^L \geq H.$$

$$G = H \text{ iff } G \geq H \text{ and } G \leq H.$$

$$G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}.$$

These few lines already define a group with a very rich structure. The properties of zero, negation, order, equality and addition, can be proven to have the *usual* [1][2] behaviors.

There is a natural interpretation of the order defined above. Consider a game G played by two players, L and R, who move alternatively. The player who makes the last legal move is the winner. The outcome of the game can be describe as:

$$G \geq 0, \text{ if } R \text{ cannot win the game playing first,}$$

$$G \leq 0, \text{ if } L \text{ cannot win the game playing first,}$$

$$G = 0, \text{ if the first player cannot win the game, and}$$

$$G \not\geq H \text{ and } G \not\leq H, \text{ if the first player can win the game.}$$

To ease the discussion, we include the following definitions.

$$G \mid\mid H \text{ iff } G \not\geq H \text{ and } G \not\leq H,$$

$$G > H \text{ iff } G \geq H \text{ and } G \neq H,$$

$$G < H \text{ iff } G \leq H \text{ and } G \neq H,$$

$$G \mid > H \text{ iff } G \not\leq H,$$

$$G < \mid H \text{ iff } G \not\geq H.$$

For any game G, the outcome is in one of the four cases: $G > 0$, $G < 0$, $G = 0$, or $G \mid\mid 0$.

In general, it is not an easy task to determine the outcome of a complex game. We are interested in subgroups of games with the following properties:

1. There exists a simple rule to determine the outcome of any game in the subgroup.
2. There exists a simplification rule that can simplify games in the subgroup.

For examples, *numbers* [2] and *nimbers* [3] are well known subgroups with the above properties. The *simplest number rule* [2] and the *minimal excluded rule* [4] can simplify numbers and nimbers respectively.

In this article, we study another subgroup of games called *sumbers*. Sumbers also have the properties mentioned above. The rest of this article is organized as follows. In section 2, we introduce the constituting elements of a sumber : ups, downs and star. We define the *minimum cut* of a sumber and the *critical section* of a game whose options are sumbers. The critical section plays an important role in determining whether a game is a sumber or not. In section 3, we present the major result for single-option games, together with the proofs. In section 4, the result is extended to cover multi-option games. Examples are provided in section 5.

2. Sumbers

2.1 Ups and Downs

For any number d, define

$$\uparrow(d) = \{\uparrow(d^L), *|*, \uparrow(d^R)\}$$

where $* = \{0|0\}$ (pronounced *star*). When $d = 0$, we have

$$\uparrow(0) = \{*|*\} = 0.$$

By simple induction, one can show that, for any number d,

$$\uparrow(-d) = \{\uparrow(-d^R), *|*, \uparrow(-d^L)\} = \{-\uparrow(d^R), *|*, -\uparrow(d^L)\} = -\uparrow(d).$$

Each $\uparrow(d), d > 0$, is called an *up* and has atomic weight 1 [2] (chapter 16, page 218). The negation of an up is called a *down*. The number d is called the *order* of $\uparrow(d)$.

We use the notation $n.\uparrow(d)$ to denote the sum of n copies of $\uparrow(d)$:

$$0.\uparrow(d) = 0$$

$$n.\uparrow(d) = (n-1).\uparrow(d) + \uparrow(d), n > 0.$$

The ups defined above have the following properties:

Proposition 1: $\uparrow(d_2) > \uparrow(d_1)$, for all $d_2 > d_1$.

Proof: We prove by induction. When $d_1 = 0$, the claim is clearly true. Otherwise, suppose the claim is true for simpler games. Since $d_2 > d_1$, we have $d_2 > d_1^L$ and $d_2^R > d_1$. By induction hypothesis, we have $\uparrow(d_2) > \uparrow(d_1^L)$ and $\uparrow(d_2^R) > \uparrow(d_1)$. This implies R cannot win the game $\uparrow(d_2) - \uparrow(d_1)$ playing first. Thus $\uparrow(d_2) > \uparrow(d_1)$. □

Proposition 2: $\uparrow(d_2) + \uparrow(d_2) - \uparrow(d_1) + * > 0$, for all $d_2 > d_1 > 0$.

Proof: Consider the game $G = \uparrow(d_2) + \uparrow(d_2) - \uparrow(d_1) + *$. L can win G by moving to $\uparrow(d_2) + \uparrow(d_2)$. R cannot win G by moving to $\uparrow(d_2^R) + \uparrow(d_2) - \uparrow(d_1) + *$, $\uparrow(d_2) - \uparrow(d_1)$, $\uparrow(d_2) + \uparrow(d_2) - \uparrow(d_1^L) + *$, $\uparrow(d_2) + \uparrow(d_2)$, $\uparrow(d_2) + \uparrow(d_2) - \uparrow(d_1)$. Thus $\uparrow(d_2) + \uparrow(d_2) - \uparrow(d_1) + * > 0$. □

Proposition 3: $\uparrow(d_2) - \uparrow(d_1) > n.(\uparrow(d_3) - \uparrow(d_2))$, for all $d_3 > d_2 > d_1 > 0$ and $n \geq 0$.

Proof: We prove by induction on n. When $n = 0$, the claim is clearly true. Otherwise, suppose the claim is true when $n = k$. Let d be a number such that $d_2 > d > d_1$. By induction hypothesis, we have $\uparrow(d_2) - \uparrow(d) > k.(\uparrow(d_3) - \uparrow(d_2))$ and $\uparrow(d) - \uparrow(d_1) > \uparrow(d_2) - \uparrow(d) > \uparrow(d_3) - \uparrow(d_2)$. Thus $\uparrow(d_2) - \uparrow(d_1) > (k+1).(\uparrow(d_3) - \uparrow(d_2))$. □

A sum of ups, downs and star is called a *sumber*. A sumber S can be written in the standard form:

$$S = s_0. * + \sum_{k=1,n} s_k. \uparrow (d_k),$$

where $s_0 = 0$ or 1, $s_k \neq 0, 0 < k \leq n$ and $0 < d_k < d_{k+1}, 0 < k < n$.
$\sum_{k=1,n} s_k$ is called the *net weight* of S.

Sumbers are closed under addition. Conway first found this interesting subgroup of games. He also found a rule [2](theorem 88, page 194) similar to the theorem below to determine the outcome of a sumber.

Theorem 1: *Sumber outcome rule.* Let S be a sumber in the above standard form.

- $S > 0$ if and only if $(\sum_{k=1,n} s_k > s_0)$ or $(\sum_{k=1,n} s_k = s_0$ and $s_1 < 0)$.
- $S < 0$ if and only if $-S > 0$
- $S = 0$ if and only if $n = 0$ and $s_0 = 0$.
- $S \mid 0$, otherwise.

Proof: We prove the *if* part of the first case. The rest is trival once the first case been proven. If $\sum_{k=1,n} s_k > s_0$, let d be a number such that $d_1 > d > 0$, then

$$S = s_0. * + \sum_{k=1,n} s_k. \uparrow (d_k)$$

$$> s_0. * + \sum_{k=1,n} |s_k|.(\uparrow (d_1) - \uparrow (d_n)) + \sum_{k=1,n} s_k. \uparrow (d_1)$$

$$> s_0. * + \uparrow (d) - \uparrow (d_1) + (s_0 + 1). \uparrow (d_1)$$

$$= s_0. * + \uparrow (d) + s_0. \uparrow (d_1) > 0.$$

If $\sum_{k=1,n} s_k = s_0$ and $s_1 < 0$, let d be a number such that $d_2 > d > d_1$, then

$$S = s_0. * + \sum_{k=1,n} s_k. \uparrow (d_k)$$

$$> s_0. * + \sum_{k=2,n} |s_k|.(\uparrow (d_2) - \uparrow (d_n)) + \sum_{k=2,n} s_k. \uparrow (d_2) + s_1. \uparrow (d_1)$$

$$> s_0. * + \uparrow (d) - \uparrow (d_2) + (s_0 - s_1). \uparrow (d_2) + s_1. \uparrow (d_1)$$

$$> s_0. * + \uparrow (d) + s_0. \uparrow (d_2) - \uparrow (d_1) > 0.$$

□

2.2. Minimum Cut and Critical Section

Definition 1: *Cut of a sumber*. Let S be a sumber written in the standard form:

$$S = s_0. * + \sum_{k=1,n} s_k. \uparrow (d_k),$$

where $s_0 = 0$ or 1, $s_k \neq 0, 0 < k \leq n$ and $0 < d_k < d_{k+1}, 0 < k < n$.

For $m \in \{0, d_k : 1 \leq k \leq n\}$, define

$$S^m = \sum_{k=1,n;\ d_k \geq m} s_k. \uparrow (d_k) - \sum_{k=1,n;\ d_k \geq m} s_k. \uparrow (m).$$

We say S has a *cut* at m if $S^m \leq 0$.

Example 1: Consider the sumber $S = 4. \uparrow (1) - 2. \uparrow (2) + \uparrow (3) + \uparrow (5) - \uparrow (6)$.

$$\begin{aligned}
S^0 &= 4. \uparrow (1) - 2. \uparrow (2) + \uparrow (3) + \uparrow (5) - \uparrow (6) &&- 3. \uparrow (0) > 0,\\
S^1 &= 4. \uparrow (1) - 2. \uparrow (2) + \uparrow (3) + \uparrow (5) - \uparrow (6) &&- 3. \uparrow (1) < 0,\\
S^2 &= -2. \uparrow (2) + \uparrow (3) + \uparrow (5) - \uparrow (6) &&+ 1. \uparrow (2) > 0,\\
S^3 &= \uparrow (3) + \uparrow (5) - \uparrow (6) &&- 1. \uparrow (3) < 0,\\
S^5 &= \uparrow (5) - \uparrow (6) &&+ 0. \uparrow (5) < 0,\\
S^6 &= - \uparrow (6) &&+ 1. \uparrow (6) = 0.
\end{aligned}$$

Thus, S has cuts at 1, 3, 5 and 6. □

We only concerned with the *minimum cut*: the smallest number in $\{0, d_k : 1 \leq k \leq n\}$ which is a cut. When S is a sumber, and m is the minimum cut of S, we call S^m the *upper section* and $S_m = S - S^m$ the *lower section* of S.

Example 2: Consider the sumber $S = 4. \uparrow (1) - 2. \uparrow (2) + \uparrow (3) + \uparrow (5) - \uparrow (6)$.

From example 1, we know S has cuts at 1, 3, 5 and 6. The minimum cut of S is at 1. The upper section of S is $S^1 = \uparrow (1) - 2. \uparrow (2) + \uparrow (3) + \uparrow (5) - \uparrow (6)$, and the lower section is $S_1 = 3. \uparrow (1)$. □

Definition 2: *Critical section of* $\{A|B\}$. Let A, B be two sumbers, define

$$X(A|B) = \{x \geq m : A - B_m + * <|\ \uparrow (x)\ <|\ B^m + *,\ m \text{ is the minimum cut of } B\}.$$

$X(A|B)$ is called the *critical section* of the game $\{A|B\}$.

The calculation of $X(A|B)$ involves solving inequalities of sumbers. Theorem 1 can help in solving these inequalities.

Example 3: Consider the game $\{0|S\} = \{0 \mid 4.\uparrow(1) - 2.\uparrow(2) + \uparrow(3) + \uparrow(5) - \uparrow(6)\}$.

From example 2, we know the minimum cut of S is at 1.

$$X(0|S) = \{x \geq 1 : 0 - S_1 + * <| \uparrow(x) <| S^1 + *\}$$

$$= \{x \geq 1 : -3.\uparrow(1) + * <| \uparrow(x) <| \uparrow(1) - 2.\uparrow(2) + \uparrow(3) + \uparrow(5) - \uparrow(6) + *\}$$

$$= \{1\}.$$

□

One of the interesting results about ups is the following up-star equality [2]:

$$\{0| \uparrow(1)\} = \uparrow(1) + \uparrow(1) + *.$$

The interesting point is that certain games can be decomposed as sums of ups, downs and star.

Kao [5] also designed a game played on a number of heaps of colored counters. Each counter is colored either black or white. Left and Right move alternatively and their legal moves are different:

- When it is L's turn to move, he can choose any one of the heaps and repeatly removes the top counter of that heap until either he removed a white counter or the heap has become empty.
- When it is R's turn to move, he can choose any one of the heaps and repeatly removes the top counter of that heap until either he removed a black counter or the heap has become empty.

The player who remove the last counter is the winner. Kao [5] found that each of the colored heaps can be decomposed as a sum of ups, downs and star.

These exciting results raise the questions: *what is the general rule that a game can be decomposed as a sum of ups, downs and star? Can we obtain a result similar to the simplest number rule for numbers or the minimal excluded value rule for nimbers?* As we shall see in the next section, the answer is yes and $X(A|B)$ plays an important role in determining whether $\{A|B\}$ is a sumber or not.

3. Sumber Simplification Rule

If A and B are sumbers then the following rule can determine whether $\{A|B\}$ is a sumber or not.

Theorem 2: *Sumber simplification rule* (single-option games).

Let A, B be two sumbers.

1. If $A <\mid 0$ and $B \mid> 0$ then $\{A|B\} = 0$.
2. Otherwise, either $A \geq 0$ or $B \leq 0$. Without loss of generality, we may assume $A \geq 0$ and the net weight of $A + B$ is greater than or equal to 0 (otherise, apply this rule to $-\{A|B\}$). By assumption, the net weight of B is greater than or equal to -1 (otherise, $\{A|B\}$ is not a sumber).
   (a) If B has non-negative net weight then $\{A|B\}$ is a sumber if and only if $X(A|B)$ is not empty.
   (b) If B has net weight -1 and $B \mid\mid 0$ then $\{A|B\}$ is a sumber if and only if $X(A|B)$ is not empty.
   (c) Otherwise, B has net weight -1, and $B < 0$. If $A||*$ and $B||*$, then $\{A|B\} = *$.
   (d) Otherwise, B has net weight -1, either $A > *$ or $B < *$. Without loss of generality, we may assume $A > *$ (otherise, apply this rule to $-\{A|B\}$). $\{A|B\}$ is a sumber if and only if $X(A|B)$ is not empty.

In cases 2(a), 2(b), and 2(d), If $\{A|B\}$ is a sumber then $\{A|B\} = B_m + * + \uparrow(p)$ where m is the minimum cut of B and p is the simplest number in $X(A|B)$.

The proofs for the above cases are provided in propositions 4, 5, and 6.

Lemma 1: *If B is a sumber with non-negative net weight and $m \geq 0$ is the minimum cut of B, then, whenever L removes a negative term from B_m, R can find a higher or equal order positive term from the remaining sum to remove.*

Proof: When $m = 0$, since B has non-negative net weight, we have either $B_m = 0$ or $B_m = *$. In either case, there is no negative term in B_m and the claim is obviously true.

When $m > 0$, we prove the claim by contradiction. Consider the game B_m.
Suppose, after some pair(s) of moves, L chooses $-\uparrow(b)$ in the remaining sum and the most positive term left is $\uparrow(a)$, $0 < a < b < m$. By assumption, B_m must be of the form:

$$B_m = (\text{terms with order} \leq a) + \uparrow(a) - \uparrow(b) + (\text{terms with order} \geq b, \text{ net weight} = 0).$$

But this implies $a < m$ is a cut of B and m is not the minimum cut, a contradiction. □

Lemma 2: *If B is a sumber with non-negative net weight, and $m > 0$ is the minimum cut of B, then $\{0|B\} - B_m + \uparrow(m) - \uparrow(n) \geq 0$, for all $n < m$.*

Proof: It is sufficient to show that L cannot win the sum $S = \{-B|\ 0\} + B_m - \uparrow(m) + \uparrow(n)$. Without loss of generality, we may assume n is a number greater than the order of the most negative term in B_m.

1. L cannot win S by choosing the $\{-B|\ 0\}$ option, because
$-B + B_m - \uparrow(m) + \uparrow(n) = -B^m - \uparrow(m) + \uparrow(n) < 0$
(the sum has net weight 0, without $*$, and the smallest term, $\uparrow(n)$, is positive,).

2. L cannot win S by choosing any negative term from $B_m - \uparrow(m) + \uparrow(n)$, because, by lemma 1, whenever L removes a negative term from $B_m - \uparrow(m) + \uparrow(n)$, R can find a higher or equal order positive term from the remaining sum to remove. ($B_m - \uparrow(m) + \uparrow(n)$ has non-negative net weight and it equals its lower section.)

3. L cannot win S by choosing any positive term from $B_m - \uparrow(m) + \uparrow(n)$. From *atomic weight calculus* [1], we know the atomic weight of $\{0|B\}$ eqauls -1, 0 or [1 + the atomic weight of B], depending on $\{0|B\}$ is less than, confused with, or greater than the remote star. Since B has non-negative atomic weight and $m > 0$, we know $B \not\leq 0$, and $B \not\leq *$, thus $\{0|B\} > 0$ and $\{0|B\} > *$, the atomic weight of $\{0|B\}$ must equals [1 + the atomic weight of B]. This implies the atomic weight of S is -1. So, L cannot win S by choosing some positive term from $B_m - \uparrow(m) + \uparrow(n)$.

From 1, 2 and 3, we know L cannot win S. □

Lemma 3: *If $A \geq 0$ is a sumber, B is a sumber with non-negative net weight, $m \geq 0$ is the minimum cut of B, then $\{A|B\} - B_m + * - \uparrow(n) > 0$, for all $n < m$.*

Proof: Consider the sum $S = \{A|B\} - B_m + * - \uparrow(n)$.

When $m = 0$, since B has non-negative net weight, we have either $B_m = 0$ or $B_m = *$. Note that in this case $n < 0$. If $B_m = 0$ then $S = \{A|B^m\} + * - \uparrow(n) > 0$. If $B_m = *$ then $S = \{A|B^m + *\} - \uparrow(n) > 0$. In either case, we have $S > 0$.

When $m > 0$, we need to show that L can and R cannot win S. Without loss of generality, we may assume n is a number greater than the order of the most negative term in B_m. L can win S by choosing the $-\uparrow(m)$ term in $-B_m$, because, after L's move, the sum becomes $\{A|B\} - B_m + \uparrow(m) - \uparrow(n) \geq \{0|B\} - B_m + \uparrow(m) - \uparrow(n) \geq 0$ (by lemma 2). R cannot win S by choosing the $\{A|B\}$ option, because, after R's move, the sum becomes $B - B_m + * - \uparrow(n) || 0$ (there exists $*$, net weight $= -1$, and the least order term is negative). R cannot win S by choosing any positive term from $-B_m$, because, by lemma 1, whenever R removes a positive term from $-B_m$, L can find a higher order negative term in the remaining sum to remove. Thus, $\{A|B\} - B_m + * - \uparrow(n) > 0$. □

Proposition 4: *If $A \geq 0$ is a sumber, B is a sumber with non-negative net weight and $X(A|B)$ is not empty, then $\{A|B\}$ is a sumber. Moreover $\{A|B\} = B_m + * + \uparrow(p)$ where m is the minimum cut of B and p is the simplest number in $X(A|B)$.*

Proof: To prove $\{A|B\} = B_m + * + \uparrow(p)$, it is sufficient to show that the first player cannot win the game $G = \{A|B\} - B_m + * - \uparrow(p)$.

Claim 1: L cannot win the game G.

L can choose terms from $\{A|B\}$, $-\uparrow(p)$, $-B_m$ or $*$. Since the later two are dominated options, we only need to consider the first two terms.

1. L cannot win G by choosing $\{A|B\}$, because, after L's move, the remaining game will be $A - B_m + * - \uparrow(p) <| \, 0$ (because $p \in X(A|B)$).
2. L cannot win G by choosing $-\uparrow(p)$, because, after L's move, the remaining game will be $\{A|B\} - B_m <| \, 0$ (because $B \leq B_m$).

In either case, L cannot win the game G.

Claim 2: R cannot win the game G.

R can choose terms from $\{A|B\}$, $-\uparrow(p)$, $-B_m$ or $*$.

1. R cannot win G by choosing $\{A|B\}$, because, after R's move, the remaining game will be $B - B_m + * - \uparrow(p) |> 0$ (because $p \in X(A|B)$).
2. If R chooses the $-\uparrow(p)$ term then the remaining game will be $\{A|B\} - B_m + * - \uparrow(p^L)$. Since p is the simplest number in $X(A|B)$, we have either $p^L \in X(A|B)$ or $p^L \notin X(A|B)$.
   (a) If $p^L \notin X(A|B)$ then $A - B_m + * - \uparrow(p^L) \geq 0$, thus $\{A|B\} - B_m + * - \uparrow(p^L) |> 0$.
   (b) If $p^L \in X(A|B)$ then $p = m$, thus $\{A|B\} - B_m + * - \uparrow(p^L) = \{A|B\} - B_m + * - \uparrow(m^L) > 0$ (by lemma 3).
3. If R chooses a positive term, say $\uparrow(n)$, from $-B_m$ then L can respond a move at $-\uparrow(p)$. After the exchange, the remaining game becomes $\{A|B\} - B_m + * - \uparrow(n) > 0$ (by lemma 3). R should never consider choosing any negative terms from $-B_m$ because these options are dominated by the $-\uparrow(p)$ option.
4. If R chooses the $*$ term then L can respond a move at $-\uparrow(p)$. After the exchange, the remaining game becomes $\{A|B\} - B_m + * > 0$.

In any of the 4 cases, R cannot win the game G. □

Proposition 5: *If $A \geq 0$, $B \,||\, 0$ are sumbers, B has net weight -1 and $X(A|B)$ is not empty, then $\{A|B\} = B_m + * + \uparrow(p)$, where m is the minimum cut of B and p is the simplest number in $X(A|B)$.*

Proof: Since $B \,||\, 0$ and B's net weight is -1, we know the least order term in B is negative. Let $-\uparrow(q)$ be the least order term in B. Now, consider the sum $S = \{A + \uparrow(q) | B + \uparrow(q)\} - \uparrow(q) + \{-B | -A\}$. We want to show that the first player cannot win S.

By proposition 4, we know $\{A+\uparrow(q)|B+\uparrow(q)\} = B_m+\uparrow(q)+*+\uparrow(p)$, where m is the minimum cut of $B+\uparrow(q)$ and p is the simplest number in $X(A+\uparrow(q)|B+\uparrow(q))$. Note that the minimum cut of $B+\uparrow(q)$ equals the minimum cut of B and $X(A+\uparrow(q)|B+\uparrow(q)) = X(A|B)$. We can rewrite S as $S = B_m+\uparrow(q)+*+\uparrow(p)-\uparrow(q)+\{-B|-A\}$. Thus, $-\uparrow(q)$ is a dominated option for both players. If a player can win S, then the player must have a winning move other than $-\uparrow(q)$. Since neither of the players can win S by moving to $\{A+\uparrow(q)|B+\uparrow(q)\}$ or $\{-B|-A\}$, $S=0$. Also note that the minimum cut if B equals the minimum cut of $B+\uparrow(q)$. We have $\{A|B\} = B_m+*+\uparrow(p)$, where m is the minimum cut of B and p is the simplest number in $X(A|B)$. □

Proposition 6: *If $A > 0$ is a sumber with net weight 1, $B < 0$ is a sumber with net weight -1, and $X(A|B)$ is not empty, then $\{A|B\}$ is a sumber. Moreover, if $A > *$ then $\{A|B\} = B_m+*+\uparrow(p)$, where m is the minimum cut of B and p is the simplest number in $X(A|B)$.*

Proof: Since $X(A|B)$ is not empty, we have either $A \,||\, *$ or $B \,||\, *$ (If $A > *$ and B $< *$ then $X(A|B)$ is empty.) If $A \,||\, *$ and $B \,||\, *$ then $\{A|B\} = *$. Otherwise, either ($A > *$ and $B||*$) or ($A||*$ and $B < *$). Without loss of generality, we may assume $A > *$ and $B||*$. Consider the sum $S = \{A+*|B+*\}+*+\{-B|-A\}$. We want to show that the first player cannot win S. Since $A+* > 0$, $B+*$ has net weight -1 and $B+* \,||\, 0$, by proposition 5, we have $\{A+*|B+*\} = B_m+\uparrow(p)$ where m is the minimum cut of B and p is the simplest number in $X(A|B)$. We can rewrite S as $S = B_m+\uparrow(p)+*+\{-B|-A\}$. Note that $*$ is a dominated option for both players. If a player can win S, then the player must have a winning move other than $*$. Since neither of the players can win S by moving to $\{A+*|B+*\}$ or $\{-B|-A\}$, we have $S=0$. Thus, $\{A|B\} = B_m+*+\uparrow(p)$, where m is the minimum cut of B and p is the simplest number in $X(A|B)$. □

4. Games with Multiple Options

Section 3 deals with games where each player has a unique option. In this section, we extend the theorem to deal with multiple-option games. Note that, sums of ups and downs (excluding *) are totally ordered. If G^L and G^R are sets of sumbers, then G has at most two non-dominated options, one contains $*$ and the other does not, in each of G^L and G^R. In other words, G can be simplified as $G = \{A, B|C, D\}$ where A, B are the non-dominated options in G^L and C, D are the non-dominated options in G^R. Without loss of generality, we may assume the net weight of C is less than or equal to the net weight of D.

The critical section $X(G)$ of $G = \{A, B|C, D\}$ is defined as empty set, if C and D has the same net weight. Otherwise (the net weight of C is less than the net weight of D),

$X(G)$ is defined as the set of numbers $x \geq m$ satisfying all the following inequalities:

$$\uparrow(x) \mid> A - C_m + *,$$

$$\uparrow(x) \mid> B - C_m + *,$$

$$\uparrow(x) <\mid C - C_m + *,$$

$$\uparrow(x) <\mid D - C_m + *,$$

where m is the minimum cut of C.

Theorem 3: *Sumber simplification rule* (multi-option games).

Let $G = \{A, B|C, D\}$, where A, B, C and D are sumbers. The net weight of C is less than or equal to the net weight of D.

1. If $A <\mid 0$, $B <\mid 0$ and $C \mid> 0$, $D \mid> 0$ then $G = 0$.
2. Otherwise, $A \geq 0$, $B \geq 0$, $C \leq 0$ or $D \leq 0$. Without loss of generality, we may assume $A \geq 0$ or $B \geq 0$ and the net weight of $A + C$ or $B + C$ is greater than or equal to 0 (otherise, apply this rule to $-G$). By assumption, the net weight of C is greater than or equal to -1.
   (a) If C has non-negative net weight then G is a sumber if and only if $X(G)$ is not empty.
   (b) If C has net weight -1 and $C \| 0$ then G is a sumber if and only if $X(G)$ is not empty.
   (c) Otherwise, C has net weight -1, and $C < 0$. If $A\|*$, $B\|*$, and $C\|*$, $D\|*$ then $G = *$.
   (d) Otherwise, C has net weight -1, either ($A > *$ or $B > *$) or ($C < *$ or $D < *$). Without loss of generality, we may assume $A > *$ or $B > *$ (otherise, apply this rule to $-G$). G is a sumber if and only if $X(G)$ is not empty.

In cases 2(a), 2(b), and 2(d), If G is a sumber then $G = C_m + * + \uparrow(p)$ where m is the minimum cut of C and p is the simplest number in $X(G)$.

The proofs for the above cases are similar to the ones provided in section 3.

5. Examples

Example 4: Consider the game $\{A|B\} = \{0| \uparrow(1)\}$

The minimum cut of B occurs at 1. $B_1 = \uparrow(1)$ and $B^1 = 0$.
$X(A|B) = \{x \geq 1 : A - B_1 + * <\mid \uparrow(x) <\mid B^1 + *\} = \{x \geq 1\}$

The simplest number in $X(A|B)$ is 1.
By rule 2(a), we have $\{A|B\} = B_1 + * + \uparrow(1) = \uparrow(1) + \uparrow(1) + *$. □

Example 4 is the up-star equality first found by Conway[2]. Indeed, the equality still holds if 1 is replaced by any positive integer n.

$$\{0|\uparrow(n)\} = \uparrow(n) + \uparrow(n) + *.$$

When n is not an integer, the result is interesting.

Example 5: Consider the game $\{A|B\} = \{0|\uparrow(\frac{1}{2})\}$

The minimum cut of B occurs at $\frac{1}{2}$. $B_{\frac{1}{2}} = \uparrow(\frac{1}{2})$ and $B^{\frac{1}{2}} = 0$.
$X(A|B) = \{x \geq \frac{1}{2} : A - B_{\frac{1}{2}} + * <|\uparrow(x) <| B^{\frac{1}{2}} + *\} = \{x \geq \frac{1}{2}\}$
The simplest number in $X(A|B)$ is 1.
By rule 2(a), we have $\{A|B\} = B_{\frac{1}{2}} + * + \uparrow(1) = \uparrow(\frac{1}{2}) + \uparrow(1) + *$. □

There are cases where A and B are sumbers with term(s) of integer order(s), but $\{A|B\}$ is a sumber with term(s) of non-integer order(s).

Example 6: Consider the game $\{A|B\} = \{\uparrow(1)| * + 2.\uparrow(2) - 2.\uparrow(3)\}$

The minimum cut of B occurs at 0. $B_0 = *$ and $B^0 = 2.\uparrow(2) - 2.\uparrow(3)$.
$X(A|B) = \{x \geq 0 : A - B_0 + * <|\uparrow(x) <| B^0 + *\} = \{1 < x < 2\}$
The simplest number in $X(A|B)$ greater than or equal to 0 is $1\frac{1}{2}$.
By rule 2(a), we have $\{A|B\} = B_0 + * + \uparrow(1\frac{1}{2}) = \uparrow(1\frac{1}{2})$. □

There are cases where A and B are sumbers, but $\{A|B\}$ is not a sumber. In general, if $A - B > \uparrow(n) + *$, for any n, then $\{A|B\}$ is not a sumber.

Example 7: Consider the game $\{A|B\} = \{\uparrow(1) + \uparrow(1)|*\}$

The minimum cut of B occurs at 0. $B_0 = *$ and $B^0 = 0$.
$X(G) = \{x \geq 0 : A - B_0 + * <|\uparrow(x) <| B^0 + *\} = \{\}$
$X(A|B)$ is an empty set, so $\{A|B\}$ is not a sumber. □

There are cases where all the games in G^L and G^R are sumbers, the difference between G^L and G^R is less than $\uparrow(n) + *$ for some n, but G is not a sumber.

Example 8: Consider the game $G = \{A|B, C\} = \{\uparrow(1)|*, \uparrow(1)\}$

The minimum cut of B occurs at 0. $B_0 = *$ and $B^0 = 0$.
$X(G) = \{x \geq 0 : A - B_0 + * <|\uparrow(x) <| B^0 + *$ and $\uparrow(x) <| C - B_0 + * \} = \{\}$
$X(G)$ is an empty set, so G is not a sumber. □

6. Conclusion

This article introduces a subgroup of games called *sumbers*. Similar to numbers and nimbers, sumbers have the following properties:

1. Sumbers are closed under addition.
2. There exists a simple rule to determine the outcome of a sumber.
3. There exists a simplification rule that can, when G is a sumber, determine the exact sumber of G from G^L and G^R.

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