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# SOME FOURTH DEGREE DIOPHANTINE EQUATIONS IN GAUSSIAN INTEGERS

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**Abstract**

For certain choices of the coefficients $a$, $b$, $c$ the solutions of the Diophantine equation $ax^4+by^4 = cz^2$ in Gaussian integers satisfy $xy = 0$.

## 1. Introduction

The solution $(x_0, y_0, z_0)$ of the equation $ax^4 + by^4 = cz^2$ is called *trivial* if $x_0 = 0$ or $y_0 = 0$. P. Fermat showed that the equation $x^4 + y^4 = z^2$ has only trivial solutions in integers. D. Hilbert [2] extended this result by showing that the equation $x^4 + y^4 = z^2$ has only trivial solutions in a larger domain, namely in the integers of $Q(\sqrt{-1})$. In fact from his proof, it follows that the equation $x^4 - y^4 = z^2$ also has only trivial solutions. J. T. Cross [1] gave a new proof for Hilbert's result. We consider the following eight equations $x^4 + my^4 = z^2$, where $m = \pm 2^n$, $0 \leq n \leq 3$. The equations $x^4 - 2y^4 = z^2$, $y^4 + 8y^4 = z^2$ have nontrivial solutions in integers as shown by the solutions $(3, 2, 7)$, $(1, 1, 3)$, respectively. We will show that the remaining six equations have only trivial solutions in the integers of the quadratic field $Q(\sqrt{-1})$. The $m = \pm 1$ case is covered by Hilbert's result, so we will deal only with four cases. It is worthwhile to point out that the equation $x^4 + 2y^4 = z^2$ has nontrivial solution in $Z[\sqrt{\pm 2}]$, as the solution $(1, \sqrt{\pm 2}, 3)$ shows.

It is proved in [3], among various similar results, that the equation $x^4 - py^4 = z^2$ has only trivial solutions in integers, where $p$ is a prime $p \equiv \pm 3, -5 \pmod{16}$. We will show that the equations $x^4 - py^4 = z^2$, $x^4 - p^3y^4 = z^2$ have only trivial solutions in the Gaussian integers, where $p$ is a prime $p \equiv 3 \pmod 8$. We would like to point out that the equations $x^4+py^4 = z^2$, $x^4 + p^2y^4 = z^2$ have nontrivial integer solutions when $p = 3$ as shown by the solutions $(1, 1, 2)$, $(2, 1, 5)$, respectively.

It is shown in [4] (Theorem 117, p. 230), that the equation $x^4 - y^4 = pz^2$ has only trivial solutions in integers, where $p$ is a prime $p \equiv 3 \pmod 8$. It is shown in [3] (p. 23) that the equation $x^4 - py^4 = z^2$ has only trivial solutions in integers, where $p$ is a prime $p \equiv \pm 3, -5 \pmod{16}$. Motivated by these results, we will show that the equations $x^4 - y^4 = pz^2$, $x^4 - p^2y^4 = z^2$ have only trivial solutions in Gaussian integers if $p$ is a rational prime $p \equiv 3 \pmod 8$.

We list the properties of $Q(\sqrt{-1})$ which play part later. Let $i = \sqrt{-1}$ and $\omega = 1 + i$. The ring of integers of $Q(i)$ is $Z[i] = \{u + vi : u, v \in Z\}$ which is a unique factorization domain. The units of $Z[i]$ are 1, $i$, $-1$, $-i$. The norm of $\omega$ is 2 and consequently $\omega$ is a prime in $Z[i]$. The prime factorization of 2 is $(-i)\omega^2$. We will use the ideals formed by the Gaussian integer multiples of $\omega^n$, $1 \le n \le 6$. Note that $\omega^2$, $\omega^4$, $\omega^6$ are associates of 2, 4, 8, respectively, and so they span the same ideals. We will prefer to use the terminology connected with congruences instead of with ideals.

We will use the next observation several times. If $\alpha$ is an integer in $Q(\sqrt{-1})$ and $\alpha \equiv 1 \pmod{\omega}$, then $\alpha^2 \equiv 1 \pmod{\omega^2}$ and $\alpha^4 \equiv 1 \pmod{\omega^6}$. In order to verify the first claim, write $\alpha$ in the form $\alpha = k\omega + 1$, where $k \in Z[i]$ and compute $\alpha^2$. Since $\alpha^2 = k^2\omega^2 + 2k\omega + 1$, it follows that $\alpha^2 \equiv 1 \pmod{\omega^2}$. In order to verify the second claim write $\alpha$ in the form $\alpha = k\omega^2 + l$, $k, l \in Z[i]$ and compute $\alpha^4$.

$$\alpha^4 = (k\omega^2)^4 + 4(k\omega^2)^3 l + 6(k\omega^2)^2 l^2 + 4(k\omega^2) l^3 + l^4$$

This shows that $\alpha^4 \equiv l^4 \pmod{\omega^6}$. Since 0, 1, $i$, $1 + i$ form a complete set of representatives modulo $\omega^2$ and since $\alpha \equiv 1 \pmod{\omega}$ we can choose $l$ to be either 1 or $i$. Therefore $\alpha^4 \equiv 1 \pmod{\omega^6}$.

## 2. The equation $x^4 - dy^4 = z^2$

**Theorem 1.** Let $p$ be a rational prime $p \equiv 3 \pmod 8$ and let $d = p$ or $d = p^3$. The equation $x^4 - dy^4 = z^2$ has only trivial solution in $Z[i]$.

*Proof.* We divide the proof into 5 smaller steps.

(1) If $(x_0, y_0, z_0)$ is a nontrivial solution of the equation $x^4 - dy^4 = z^2$, then we may assume that $x_0$, $y_0$, $z_0$ are pairwise relatively primes.

Let $(x_0, y_0, z_0)$ be a solution of the equation $x^4 - dy^4 = z^2$. We will call the quantity $N(x_0)N(y_0)N(z_0)$ the *height* of the solution. Here $N(u)$ is the norm of $u$. Choose a nontrivial solution $(x_0, y_0, z_0)$ such that the height of the solution is minimal.

Suppose first that $x_0$ and $y_0$ are not relatively prime. Let $g$ be the greatest common divisor of $x_0$ and $y_0$ in $Z[i]$. As $x_0 \neq 0$, it follows that $g \neq 0$. Dividing $x_0^4 - dy_0^4 = z_0^2$ by $g^4$ we get

$(x_0/g)^4 - d(y_0/g)^4 = (z_0/g^2)^2$. This equation holds in $Q(i)$. The left hand side of the equation is an element of $Z[i]$. Consequently the right hand side of the equation belongs to $Z[i]$. Thus, $(x_0/g, y_0/g, z_0/g^2)$ is also a nontrivial solution of the equation $x^4 - dy^4 = z^2$ in $Z[i]$. Clearly the height of this solution is smaller than the height of $(x_0, y_0, z_0)$. Hence, we may assume that $x_0$ and $y_0$ are relatively prime in $Z[i]$. We claim that if there is a prime $q$ of $Z[i]$ such that $q|x_0$ and $q|z_0$, then $q|y_0$. In order to verify the claim assume that $q$ is prime that divides $x_0$ and $z_0$. From the equation $x_0^4 - dy_0^4 = z_0^2$ it follows that $q^2|dy_0^4$. If $d = p$, then $q|y_0$ because $p$ itself is a prime in $Z[i]$. If $d = p^3$, then it may be the case that $q = p$. But in this case $p^3|x_0^4$ and $p^3|dy_0^4$ so $p^3|z_0^2$, therefore $p^2|z_0$; hence, $p^4|z_0^2$ and since $p^4|x_0^4$, it follows that $p^4|dy_0^4$ and we can conclude that $p|y_0$.

This violates that $x_0$ and $y_0$ are relatively prime. Similarly, if $q|y_0$ and $q|z_0$, then $q|x_0$ violating again that $x_0$ and $y_0$ are relatively prime. Thus we may assume that $x_0$, $y_0$, $z_0$ are pairwise relatively prime.

(2) Let $(x_0, y_0, z_0)$ be a nontrivial solution of the equation $x^4 - dy^4 = z^2$ in $Z[i]$ such that $x_0$, $y_0$, $z_0$ are pairwise relatively prime. Note that at most one of $x_0$, $y_0$, $z_0$ can be congruent to 0 modulo $\omega$. We consider the following four cases. None of $x_0$, $y_0$, $z_0$ is congruent to 0 modulo $\omega$ and three cases depending on $x_0$, $y_0$, $z_0$ congruent to 0 modulo $\omega$ respectively. Table 1 summarizes the cases.

| | $x_0 \equiv$ | $y_0 \equiv$ | $z_0 \equiv$ | |
|---|---|---|---|---|
| case 1 | 1 | 1 | 1 | $\pmod{\omega}$ |
| case 2 | 0 | 1 | 1 | $\pmod{\omega}$ |
| case 3 | 1 | 0 | 1 | $\pmod{\omega}$ |
| case 4 | 1 | 1 | 0 | $\pmod{\omega}$ |

Table 1

In case 1, the equation $x_0^4 - dy_0^4 = z_0^2$ leads to the contradiction $1 - 1 \equiv 1 \pmod{\omega}$. In other words case 1 is merely the fact that "a sum of two odd numbers cannot be odd", where here, an odd number is a number which is not a multiple of $\omega$. We will use this fact several times later without mentioning it explicitely.

In case 2, write the equation $x_0^4 - dy_0^4 = z_0^2$ in the equivalent form $dy_0^4 = (x_0^2 - z_0)(x_0^2 + z_0)$ and compute the greatest common divisor of $(x_0^2 - z_0)$ and $(x_0^2 + z_0)$. Let $g$ be this greatest common divisor. Then $g|(x_0^2 - z_0)$, $g|(x_0^2 + z_0)$ implies $g|(2x_0^2)$, $g|(2z_0)$. As $g|(dy_0^4)$, it follows that $g \neq 0$. If $q$ is a prime divisor of $g$, then $q \not| \omega$ and so $q|x_0$, $q|z_0$. But this cannot happen as $x_0$ and $z_0$ are relatively prime. Thus, $g = 1$. The unique factorization property in $Z[i]$ gives that there are elements $a$, $b$, $A$, $B$ and a unit $\varepsilon$ of $Z[i]$ such that

$$x_0^2 - z_0 = \varepsilon a^4 A, \qquad x_0^2 + z_0 = \varepsilon^{-1} b^4 B,$$

where

$$AB = d, \qquad a^4 b^4 = y_0^4.$$

Further, $A$ and $B$ are relatively prime, and so we may assume that either $A = 1$, $B = d$, or $A = d$, $B = 1$. By addition, we get that

$$2x_0^2 = \varepsilon a^4 A + \varepsilon^{-1} b^4 B.$$

Let $x_0 = \omega^t x_1$, $t \geq 1$, $x_1 \equiv 1 \pmod{\omega}$. Writing $x_1$ in the form $x_1 = k\omega + 1$ and computing $x_1^2$,

$$x_1^2 = k^2\omega^2 + 2k\omega + 1,$$

we get that $x_0^2 = \omega^{2t}(m\omega^2 + 1)$ with a suitable $m \in Z[i]$.

As $a|y_0^4$, it follows that $a \equiv 1 \pmod{\omega}$. We then get that $a^4 \equiv 1 \pmod{\omega^6}$. Similarly, $b^4 \equiv 1 \pmod{\omega^6}$.

We focus our attention on the equation

$$(-i)\omega^2\omega^{2t}(m\omega^2 + 1) = \varepsilon a^4 A + \varepsilon^{-1} b^4 B$$

modulo $\omega^6$. (We remind the reader that $2 = (-i)\omega^2$.) Let us first deal with the case when $A = d$ and $B = 1$. If $t = 1$, then the equation reduces to

$$-i\omega^4 \equiv \varepsilon(3) + \varepsilon^{-1} \pmod{\omega^6}.$$

As $\varepsilon$ varies over the units of $Z[i]$, we get the following contradictions.

$$\begin{aligned}
(-i)\omega^4 &\equiv (1)(3) + (1) \pmod{\omega^6}, \\
(-i)\omega^4 &\equiv (i)(3) + (-i) \pmod{\omega^6}, \\
(-i)\omega^4 &\equiv (-1)(3) + (-1) \pmod{\omega^6}, \\
(-i)\omega^4 &\equiv (-i)(3) + (i) \pmod{\omega^6}.
\end{aligned}$$

If $t \geq 2$, then the equation reduces to

$$0 \equiv \varepsilon(3) + \varepsilon^{-1} \pmod{\omega^6}.$$

As $\varepsilon$ varies over the units of $Z[i]$, we get the following contradictions.

$$\begin{aligned}
0 &\equiv (1)(3) + (1) \pmod{\omega^6}, \\
0 &\equiv (i)(3) + (-i) \pmod{\omega^6}, \\
0 &\equiv (-1)(3) + (-1) \pmod{\omega^6}, \\
0 &\equiv (-i)(3) + (i) \pmod{\omega^6}.
\end{aligned}$$

The case when $A = 1$ and $B = d$ can be settled in a similar way. This shows that case 2 is not possible.

Next, we verify that case 4 is not possible either. We write $z_0$ in the form $z_0 = \omega^t z_1$, $t \geq 1$, $z_1 \equiv 1 \pmod{\omega}$. We focus our attention on the equation

$$x_0^4 - dy_0^4 = \omega^{2t} z_1^2$$

modulo $\omega^4$. It leads to the contradictions $(1) - (3)(1) \equiv \omega^2$, $(1) - (3)(1) \equiv 0 \pmod{\omega^4}$ corresponding to $t = 1$ or $t \geq 2$.

(3) In case 3, let $(x_1, \omega^t y_1, z_1)$ be a solution of the equation $x^4 - dy^4 = z^2$, where $t \geq 1$, $x_1 \equiv y_1 \equiv z_1 \equiv 1 \pmod{\omega}$ and $x_1$, $y_1$, $z_1$ are pairwise relatively prime. We will show that $z_1 \equiv 1 \pmod{\omega^2}$.

In order to prove this claim, write $z_1$ in the form $z_1 = k\omega^2 + l$, $k, l \in Z[i]$, and compute $z_1^2$.

$$z_1^2 = k^2\omega^4 + 2k\omega^2 l + l^2.$$

From this it follows that $z_1^2 \equiv l^2 \pmod{\omega^4}$. Since the elements 0, 1, $i$, $1 + i$ form a complete set of representatives modulo $\omega^2$, and since $z_1 \equiv 1 \pmod{\omega}$, we may choose $l$ to be 1, or $i$. Consequently, $z_1^2$ is congruent to 1 or $-1$ modulo $\omega^4$. The equation $x_1^4 - d\omega^{4t} y_1^4 = z_1^2$ gives that $1 \equiv z_1^2 \pmod{\omega^4}$, and so $z_1 \equiv 1 \pmod{\omega^2}$.

(4) We will show that there are pairwise relatively prime elements $x_2$, $y_2$, $z_2$ of $Z[i]$, such that $x_2 \equiv y_2 \equiv z_2 \equiv 1 \pmod{\omega}$ and $(x_2, \omega^{t-1} y_2, z_2)$ is a solution of the equation $x^4 - dy^4 = z^2$.

In order to verify the claim, write the equation $x_1^4 - d\omega^{4t} y_1^4 = z_1^2$ in the form $d\omega^{4t} y_1^4 = (x_1^2 - z_1)(x_1^2 + z_1)$, and compute the greatest common divisor of $(x_1^2 - z_1)$ and $(x_1^2 + z_1)$. Let $g$ be this greatest common divisor. As $g | d\omega^{4t} y_1^4$, it follows that $g \neq 0$. Now $g | (x_1^2 - z_1)$, $g | (x_1^2 + z_1)$ implies that $g | 2x_1^2$, $g | 2z_1$. If $q$ is a prime divisor of $g$ with $q \nmid \omega$, we then get $q | x_1$, $q | z_1$. But we know that this is not the case as $x_1$ and $z_1$ are relatively prime. Thus, $g = \omega^s$, and $0 \leq s \leq 2$ since $g | 2$. By (3) $z_1 \equiv 1 \pmod{\omega^2}$. This together with $x_1^2 \equiv 1 \pmod{\omega^2}$ gives that $(x_1^2 - z_1) \equiv 0 \pmod{\omega^2}$, $(x_1^2 + z_1) \equiv 0 \pmod{\omega^2}$. Therefore, $g = \omega^2$. The unique factorization property in $Z[i]$ gives that there are relatively prime elements $a, b \in Z[i]$ such that

$$x_1^2 - z_1 = \omega^2 a, \qquad x_1^2 + z_1 = \omega^2 b.$$

Let $a = \omega^u a_1$, $b = \omega^v b_1$. So $d\omega^{4t} y_1^4 = \omega^{u+v+4} a_1 b_1$. By the unique factorization property in $Z[i]$, there are elements $a_2$, $b_2$, $A$, $B$ and a unit $\varepsilon$ in $Z[i]$ for which

$$x_1^2 - z_1 = \omega^{u+2} \varepsilon a_2^4 A, \qquad x_1^2 + z_1 = \omega^{v+2} \varepsilon^{-1} b_2^4 B,$$

$$4t = u + v + 4, \qquad a_2^4 b_2^4 = y_1^4, \qquad AB = d.$$

Here, $a_2$, $b_2$ are prime to $\omega$, and $A$ is prime to $B$. It follows that $a_2 \equiv 1 \pmod{\omega}$, $b_2 \equiv 1 \pmod{\omega}$. We may choose $A$, $B$ such that either $A = d$, $B = 1$, or $A = 1$, $B = d$. By addition, we get

$$2x_1^2 = \omega^{v+2}\varepsilon^{-1}b_2^4 B + \omega^{u+2}\varepsilon a_2^4 A.$$

After dividing by $\omega^2$, we get

$$-ix_1^2 = \omega^{v}\varepsilon^{-1}b_2^4 B + \omega^{u}\varepsilon a_2^4 A.$$

In the remaining part of the proof, we distinguish two cases depending on whether $A = 1$, $B = d$, or $A = d$, $B = 1$.

Let us deal with the case $A = 1$, $B = d$ first. We distinguish two subcases depending on whether $u = 0$, $v = 4t - 4$, or $v = 0$, $u = 4t - 4$. When $u = 0$, $v = 4t - 4$, we get

$$-ix_1^2 = \omega^{4t-4}\varepsilon^{-1}b_2^4 d + \varepsilon a_2^4.$$

If $4t - 4 = 0$, then this relation reduces to

$$-i \equiv \varepsilon^{-1} + \varepsilon \pmod{\omega^2}.$$

But this is not possible as $\varepsilon^{-1} + \varepsilon \equiv 0 \pmod{\omega^2}$. Thus, $4t - 4 \neq 0$. Now

$$-i \equiv \varepsilon \pmod{\omega^2}.$$

From this, it follows that $\varepsilon = \pm i$. By multiplying it by $-\varepsilon$ we get

$$(i\varepsilon)x_1^2 = \omega^{4t-4}(-\varepsilon^{-1}\varepsilon)b_2^4 d + (-\varepsilon^2)a_2^4.$$

Note that $i\varepsilon$ is a square of an element of $Z[i]$, say $i\varepsilon = \sigma^2$. Thus $(a_2, \omega^{t-1}b_2, \sigma x_1)$, $t \geq 2$ is a nontrivial solution of the equation $x^4 - dy^4 = z^2$.

When $v = 0$, $u = 4t - 4$, we get

$$-ix_1^2 = \varepsilon^{-1}b_2^4 d + \omega^{4t-4}\varepsilon a_2^4.$$

If $4t - 4 = 0$, then this reduces to

$$-i \equiv \varepsilon^{-1} + \varepsilon \pmod{\omega^2}.$$

But this is not possible as $\varepsilon^{-1} + \varepsilon \equiv 0 \pmod{\omega^2}$. Thus $4t - 4 \neq 0$. Now

$$-i \equiv \varepsilon \pmod{\omega^2}.$$

From this, it follows that $\varepsilon = \pm i$. By multiplying it by $\varepsilon^{-1}$, we get

$$(-i\varepsilon^{-1})x_1^2 = (\varepsilon^{-2})b_2^4 d + \omega^{4t-4}(\varepsilon^{-1}\varepsilon)a_2^4.$$

Note that $-i\varepsilon^{-1}$ is a square of an element of $Z[i]$, say $-i\varepsilon^{-1} = \sigma^2$. Thus $(\omega^{t-1}a_2, b_2, \sigma x_1)$, $t \geq 2$ is a nontrivial solution of the equation $x^4 - dy^4 = z^2$. By (2), this is not possible. The case $A = d$, $B = 1$ can be settled in a similar way.

(5) Let $(x_0, y_0, z_0)$ be a nontrivial solution of the equation $x^4 - dy^4 = z^2$ in $Z[i]$. By (2), there is a solution $(x_1, \omega^t y_1, z_1)$ with $x_1, y_1, z_1 \equiv 1 \pmod{\omega}$, $t \geq 1$. Choose a solution for which $t$ is minimal. According to (4), there is a solution $(x_2, \omega^{t-1}y_2, z_2)$, where $x_2, y_2, z_1 \equiv 1 \pmod{\omega}$, $t \geq 2$. This contradicts the choice of $t$, and so completes the proof.

## 3. The equations $x^4 - y^4 = pz^2$ and $x^4 - p^2y^4 = z^2$

**Theorem 2.** Let $p$ be a rational prime $p \equiv 3 \pmod{8}$. The equations $x^4 - y^4 = pz^2$ and $x^4 - p^2y^4 = z^2$ have only trivial solutions in $Z[i]$.

*Proof.* We divide the proof into 11 steps. The first 3 steps deal with the equation $x^4 - y^4 = pz^2$, and the next 7 steps deal with the equation $x^4 - p^2y^4 = z^2$.

(1) If $(x_0, y_0, z_0)$ is a nontrivial solution of the equation $x^4 - y^4 = pz^2$, we may then assume that $x_0$, $y_0$, $z_0$ are pairwise relatively prime.

Choose a nontrivial solution $(x_0, y_0, z_0)$ of the equation $x^4 - y^4 = pz^2$ with minimal height. Suppose first that there is a prime $q$ in $Z[i]$ such that $q|x_0$, $q|y_0$. From $x_0^4 - y_0^4 = pz_0^2$ it follows that $q^4|pz_0^2$. If $q \not| p$, then $q^4|z_0^2$. Now $(x_0/q)^4 - (y_0/q)^4 = p(z_0/q^2)^2$ shows that $(x_0/q, y_0/q, z_0/q^2)$ is a nontrivial solution of the equation $x^4 - y^4 = pz^2$. The height of this solution is smaller than the height of $(x_0, y_0, z_0)$. This is a contradiction. If $q|p$, then $q^3|z_0^2$. Again we conclude that that $q^4|z_0^2$ and then $(x_0/q, y_0/q, z_0/q^2)$ is a nontrivial solution of the equation $x^4 - y^4 = pz^2$. The height again decreased. Thus we may assume that if $(x_0, y_0, z_0)$ is a nontrivial solution of the equation $x^4 - y^4 = pz^2$, then $x_0$ and $y_0$ are relatively prime.

Next suppose that there is a prime $q$ in $Z[i]$ such that $q|x_0$, $q|z_0$. It follows that $q|y_0$. This violates that $x_0$ and $y_0$ are relatively prime.

Finally suppose that there is a prime $q$ in $Z[i]$ such that $q|y_0$, $q|z_0$. We get that $q|x_0$. This is a contradiction as $x_0$ and $y_0$ are relatively prime.

(2) Let $(x_0, y_0, z_0)$ be a nontrivial solution of the equation $x^4 - y^4 = pz^2$ in $Z[i]$ such that $x_0$, $y_0$, $z_0$ are pairwise relatively primes. Note that at most one of $x_0$, $y_0$, $z_0$ can be congruent to 0 modulo $\omega$. We consider the four cases summarized in Table 1.

We first show that case 1 is not possible. To do this, write $z_0$ in the form $z_0 = k\omega^2 + l$, $k, l \in Z[i]$. Computing $z_0^2$, $z_0^2 = k^2\omega^4 + 2k\omega^2 l + l^2$ shows that $z_0^2 \equiv l^2 \pmod{\omega^4}$. As 0, 1, $i$, $1+i$ is a complete set of representatives modulo $\omega^2$, it follows that $l$ can be chosen to be 1 or $i$.

From the equation $x_0^4 - y_0^4 = pz_0^2$, we get that $0 \equiv 3l^2 \pmod{\omega^4}$. But this is a contradiction as $l^2 = \pm 1$.

We next show that case 3 is not possible either. Let $y_0 = \omega^t y_1$, where $t \geq 1$ and $y_1$ is prime to $\omega$. Writing $z_0$ in the form $z_0 = k\omega^2 + l$, $k, l \in Z[i]$, from the equation $x_0^4 - \omega^{4t} y_1^4 = pz_0^2$, we get that $1 \equiv 3l^2 \pmod{\omega^4}$. In the case $l = 1$, this leads to the contradiction $1 \equiv 3 \pmod{\omega^4}$, and so we left with the $l = i$ choice. Now writing $z_0$ in the form $z_0 = r\omega^4 + s$, $r, s \in Z[i]$ and computing $z_0^2$, $z_0^2 = r^2\omega^8 + 2r\omega^4 s + s^2$ gives that $z_0^2 \equiv s^2 \pmod{\omega^6}$. From $z_0 \equiv i \pmod{\omega^2}$, it follows that we can choose $s$ and $s^2$ in the way summarized by Table 2.

| | | | | |
|---|---|---|---|---|
| $s$ | $i$ | $2+i$ | $3i$ | $2+3i$ |
| $s^2$ | $-1$ | $3+4i$ | $-9$ | $-5+12i$ |

Table 2

From the equation $x_0^4 - \omega^{4t} y_1^4 = z_0^2$, we get $1 + \omega^{4t} \equiv s^2 \pmod{\omega^6}$. In the case $t = 1$, this leads to the contradictions

$$1 - 4 \equiv -1 \pmod{\omega^6}, \qquad 1 - 4 \equiv 3 + 4i \pmod{\omega^6}.$$

In the case $t \geq 2$, we arrive at the contradictions

$$1 \equiv -1 \pmod{\omega^6}, \qquad 1 \equiv 3 + 4i \pmod{\omega^6}.$$

(Note that Table 2 shows four possibilities but modulo $\omega^6 = 8$ there are only two posibilities.)

Finally, notice that multiplying the equation $x_0^4 - y_0^4 = pz_0^2$ by $(-1)$ gives $y_0^4 - x_0^4 = p(iz_0)^2$, and so case 2 reduces to case 3.

(3) In case 4, let $(x_1, y_1, \omega^t z_1)$ be a solution of the equation $x^4 - y^4 = pz^2$, where $t \geq 1$, $x_1 \equiv y_1 \equiv z_1 \equiv 1 \pmod{\omega}$, and $x_1$, $y_1$, $z_1$ are pairwise relatively primes. We will show that there are pairwise relatively prime elements $x_2$, $y_2$, $z_2$ of $Z[i]$ such that $x_2 \equiv y_2 \equiv z_2 \equiv 1 \pmod{\omega}$, and either $(\omega^{t-2}x_2, y_2, z_2)$ or $(x_2, \omega^{t-2}y_2, z_2)$ is a solution of the equation $x^4 - p^2y^4 = z^2$.

In order to verify the claim, write the equation $x_1^4 - y_1^4 = p\omega^{2t}z_1^2$ in the form $p\omega^{2t}z_1^2 = (x_1^2 - y_1^2)(x_1^2 + y_1^2)$, and compute the greatest common divisor of $(x_1^2 - y_1^2)$ and $(x_1^2 + y_1^2)$. Let $g$ be this greatest common divisor. As $g | p\omega^{2t}z_1^2$ it follows that $d \neq 0$. $g | (x_1^2 - y_1^2)$, $g | (x_1^2 + y_1^2)$ implies that $g | 2x_1^2$, $g | 2y_1^2$. If $q$ is a prime divisor of $g$ with $q \not| \omega$, we then get $q | x_1$, $q | y_1$. But we know that this is not the case as $x_1$ and $y_1$ are relatively prime. Thus, $g = \omega^s$ and $0 \leq s \leq 2$ since $g | 2$. As $(x_1^2 - y_1^2) \equiv (x_1^2 + y_1^2) \equiv 0 \pmod{\omega^2}$, it follows that $g = \omega^2$. The unique factorization property in $Z[i]$ gives that there are relatively prime elements $a, b \in Z[i]$ such that

$$x_1^2 - y_1^2 = \omega^2 a, \qquad x_1^2 + y_1^2 = \omega^2 b.$$

Let $a = \omega^u a_1$, $b = \omega^v b_1$. So, $p\omega^{2t} z_1^2 = \omega^{u+v+4} a_1 b_1$. By the unique factorization property in $Z[i]$, there are elements $a_2$, $b_2$, $a_3$, $b_3$ and a unit $\varepsilon$ in $Z[i]$ for which

$$x_1^2 - y_1^2 = \omega^{u+2}\varepsilon a_2^2 a_3, \qquad x_1^2 + y_1^2 = \omega^{v+2}\varepsilon^{-1} b_2^2 b_3,$$

$$2t = u + v + 4, \qquad a_2^2 b_2^2 = z_1^2, \qquad a_3 b_3 = p.$$

Here $a_2$, $b_2$ are prime to $\omega$. It follows that $a_2 \equiv b_2 \equiv 1 \pmod{\omega}$. By addition and subtraction, we get

$$2x_1^2 = \omega^{v+2}\varepsilon^{-1} b_2^2 b_3 + \omega^{u+2}\varepsilon a_2^2 a_3,$$
$$2y_1^2 = \omega^{v+2}\varepsilon^{-1} b_2^2 b_3 - \omega^{u+2}\varepsilon a_2^2 a_3.$$

After dividing by $\omega^2$, they give

$$(-i)x_1^2 = \omega^{v}\varepsilon^{-1} b_2^2 b_3 + \omega^{u}\varepsilon a_2^2 a_3,$$
$$(-i)y_1^2 = \omega^{v}\varepsilon^{-1} b_2^2 b_3 - \omega^{u}\varepsilon a_2^2 a_3.$$

By multiplying the two equations together and multiplying the result by $\varepsilon^2$, we get

$$-\varepsilon^2 x_1^2 y_1^2 = \omega^{2v} b_2^4 b_3^2 - \omega^{2u} a_2^4 a_3^2.$$

We distinguish two cases depending on whether $a_3 = 1$, $b_3 = p$, or $a_3 = p$, $b_3 = 1$. In the case $a_3 = 1$, $b_3 = p$ $-\varepsilon^2 x_1^2 y_1^2 = \omega^{2v} b_2^4 p^2 - \omega^{2u} a_2^4$ we distinguish two subcases depending on whether $u = 0$, $v = 2t-4$, or $u = 2t-4$, $v = 0$. When $u = 0$, $v = 2t-4$, we get $-\varepsilon^2 x_1^2 y_1^2 = \omega^{4t-8} b_2^4 p^2 - a_2^4$. Thus, $(a_2, \omega^{t-2} b_2, \varepsilon x_1 y_1)$, is a nontrivial solution of the equation $x^4 - p^2 y^4 = z^2$.

When $u = 2t - 4$, $v = 0$, we get $-\varepsilon^2 x_1^2 y_1^2 = b_2^4 p^2 - \omega^{4t-8} a_2^4$. Thus, $(\omega^{t-2} a_2, b_2, \varepsilon x_1 y_1)$, is a nontrivial solution of the equation $x^4 - p^2 y^4 = z^2$. The case $a_3 = p$, $b_3 = 1$ can be settled in a similar way.

We now turn our attention to the equation $x^4 - p^2 y^4 = z^2$.

(4) If $(x_0, y_0, z_0)$ is a nontrivial solution of the equation $x^4 - p^2 y^4 = z^2$, we may then assume that $x_0$, $y_0$, $z_0$ are pairwise relatively prime.

Choose a nontrivial solution $(x_0, y_0, z_0)$ of the equation $x^4 - p^2 y^4 = z^2$ with minimal height. First suppose that $x_0$ and $y_0$ are not relatively prime. Let $g$ be a greatest common divisor of $x_0$ and $y_0$ in $Z[i]$. The left hand side of the equation $(x_0/g)^4 - p^2(y_0/g)^4 = (z_0/g^2)^2$ is an element of $Z[i]$ so the right hand side of the equation is an element of $Z[i]$. Therefore $(x_0/g, y_0/g, z_0/g^2)$ is a nontrivial solution of $x^4 - p^2 y^4 = z^2$. By the minimality of the height we may assume that $x_0$ and $y_0$ are relatively prime.

Next suppose that there is a prime $q$ in $Z[i]$ such that $q|y_0$, $q|z_0$. In this case we get $q|x_0$. This is a contradiction since $x_0$ and $y_0$ are relatively prime.

Finally suppose that there is a prime $q$ in $Z[i]$ such that $q|x_0$, $q|z_0$. It follows that $q^2|p^2y_0^4$. If $q|y_0$, then $x_0$ and $y_0$ are not relatively prime. This is not the case so $q \not| y_0$. It follows that $q^2|p^2$. Hence $q$ and $p$ are associates. Set $x_0 = px_1$, $z_0 = pz_1$. From $p^4x_1 - p^2y_0^4 = p^2z_1$ we get

$$p^2x_1^2 - y_0^4 = z_1^2,$$
$$y_0^4 - p^2x_1^4 = -z_1^2.$$

Therefore $(y_0, x_1, iz_1)$ is a nontrivial solution of the equation $x^4 - p^2y^4 = z^2$. By the minimality of the height we may assume that $x_0$ and $z_0$ are relatively prime.

(5) Let $(x_0, y_0, z_0)$ be a nontrivial solution of the equation $x^4 - p^2y^4 = z^2$ in $Z[i]$ such that $x_0$, $y_0$, $z_0$ are pairwise relatively prime. We consider the four cases listed in Table 1. In case 1, the equation $x_0^4 - p^2y_0^4 = z_0^2$ gives the contradiction $1 - 1 \equiv 1 \pmod{\omega}$.

(6) In case 2, multiply the equation $x^4 - p^2y^4 = z^2$ by $(-1)$ to get $-x^4 + p^2y^4 = (iz)^2$. Let $(\omega^t x_1, y_1, z_1)$ be a solution of the equation $-x^4 + p^2y^4 = z^2$, where $t \geq 1$, $x_1 \equiv y_1 \equiv z_1 \equiv 1 \pmod{\omega}$ and $x_1$, $y_1$, $z_1$ are pairwise relatively prime. We will show that $z_1 \equiv 1 \pmod{\omega^2}$.

In order to prove this claim, write $z_1$ in the form $z_1 = k\omega^2 + l$, $k, l \in Z[i]$. It follows that $z_1^2 \equiv l^2 \pmod{\omega^4}$, and we may choose $l$ to be 1 or $i$. Consequently, $z_1^2$ is congruent to 1 or $-1$ modulo $\omega^4$. The equation $-\omega^{4r}x_1^4 + p^2y_1^4 = z_1^2$ gives that $1 \equiv z_1^2 \pmod{\omega^4}$, and so $z_1 \equiv 1 \pmod{\omega^2}$.

(7) In case 2 let $(\omega^r x_1, y_1, z_1)$ be a solution of the equation $-x^4 + p^2y^4 = z^2$, where $r \geq 1$, $x_1 \equiv y_1 \equiv z_1 \equiv 1 \pmod{\omega}$ and $x_1$, $y_1$, $z_1$ are pairwise relatively prime. We will show that there are pairwise relatively prime elements $x_2$, $y_2$, $z_2$ of $Z[i]$ such that $x_2 \equiv y_2 \equiv z_2 \equiv 1 \pmod{\omega}$ and $(x_2, \omega^{r-1}y_2, z_2)$ is a solution of the equation $x^4 - y^4 = pz^2$.

In short, case 2 leads to a nontrivial solution of the equation $x^4 - y^4 = pz^2$ corresponding to cases 1–3. By step (2), no such solution exists, and so case 2 of equation $x^4 - p^2y^4 = z^2$ is not possible.

From the equation $\omega^{4t}x_1^4 = (py_1^2 - z_1)(py_1^2 + z_1)$, we can deduce that

$$(-i)py_1^2 = \omega^v \varepsilon^{-1} b_2^4 + \omega^u \varepsilon a_2^4,$$

$$4t = u + v + 4, \qquad a_2b_2 = x_1^4.$$

We distinguish two cases depending on whether $u = 0$, $v = 4t - 4$, or $u = 4t - 4$, $v = 0$. In the case $u = 0$, $v = 4t - 4$, $(-i)py_1^2 = \omega^{4t-4}\varepsilon^{-1}b_2^4 + \varepsilon a_2^4$. If $4t - 4 = 0$, then it reduces to $-i \equiv \varepsilon^{-1} + \varepsilon \pmod{\omega^2}$. But this a contradiction as $\varepsilon^{-1} + \varepsilon \equiv 0 \pmod{\omega^2}$. The details are given in Table 3.

| | | | | |
|---|---|---|---|---|
| $\varepsilon$ | 1 | $i$ | $-1$ | $-i$ |
| $\varepsilon^{-1}$ | 1 | $-i$ | $-1$ | $i$ |
| $\varepsilon+\varepsilon^{-1}$ | 2 | 0 | $-2$ | 0 |

Table 3

Thus $4t-4 \geq 2$, and $-i \equiv \varepsilon \pmod{\omega^2}$. From this, it follows that $\varepsilon = \pm i$, that is, either $\varepsilon = i$, $\varepsilon^{-1} = -i$, or $\varepsilon = -i$, $\varepsilon^{-1} = i$. In the first case

$$(-i)py_1^2 = \omega^{4t-4}(-i)b_2^4 + (i)a_2^4,$$
$$py_1^2 = \omega^{4t-4}b_2^4 - a_2^4,$$

and so $(\omega^{t-1}b_2, a_2, y_1)$ is a solution of the equation $x^4 - y^4 = pz^2$. In the second case

$$(-i)py_1^2 = \omega^{4t-4}(i)b_2^4 + (-i)a_2^4,$$
$$py_1^2 = -\omega^{4t-4}b_2^4 + a_2^4;$$

hence $(a_2, \omega^{t-1}b_2, y_1)$ is a solution of the equation $x^4 - y^4 = pz^2$. The case $u = 4t-4$, $v = 0$ can be settled in a similar way.

(8) In case 4, let $(x_1, y_1, \omega^t z_1)$ be a solution of the equation $x^4 - p^2y^4 = z^2$, where $t \geq 1$, $x_1 \equiv y_1 \equiv z_1 \equiv 1 \pmod{\omega}$ and $x_1$, $y_1$, $z_1$ are pairwise relatively prime. It follows that $z_1 \equiv 1 \pmod{\omega^2}$.

(9) In case 3, let $(x_1, \omega^t y_1, z_1)$ be a solution of the equation $x^4 - p^2y^4 = z^2$, where $r \geq 1$, $x_1 \equiv y_1 \equiv z_1 \equiv 1 \pmod{\omega}$, and $x_1$, $y_1$, $z_1$ are pairwise relatively prime. We will show that there are pairwise relatively prime elements $x_2$, $y_2$, $z_2$ of $Z[i]$ such that $x_2 \equiv y_2 \equiv z_2 \equiv 1 \pmod{\omega}$, and either $(\omega^{t-1}x_2, y_2, z_2)$, or $(x_2, \omega^{t-1}y_2, z_2)$ is a solution of the equation $x^4 - p^2y^4 = z^2$.

Form the equation $p^2\omega^{4t}y_1^4 = (x_1^2 - z_1)(x_1^2 + z_1)$, we deduce again that

$$(-i)x_1^2 = \omega^v \varepsilon^{-1} b_2^4 b_3 + \omega^u \varepsilon a_2^4 a_3,$$

$$4t = u + v + 4, \qquad a_2^4 b_2^4 = y_1^4, \qquad a_3 b_3 = p^2.$$

As $a_3$ and $b_3$ are relatively prime, we may assume that there are two cases depending on whether $a_3 = 1$, $b_3 = p^2$, or $a_3 = p^2$, $b_3 = 1$. In the case $a_3 = 1$, $b_3 = p^2$,

$$(-i)x_1^2 = \omega^v \varepsilon^{-1} b_2^4 p^2 + \omega^u \varepsilon a_2^4.$$

We distinguish two subcases depending on whether $u = 0$, $v = 4t-4$, or $u = 4t-4$, $v = 0$. When $u = 0$, $v = 4t-4$, we get $(-i)x_1^2 = \omega^{4t-4}\varepsilon^{-1}b_2^4p^2 + \varepsilon a_2^4$. If $4t-4 = 0$, then this reduces to $-i \equiv \varepsilon^{-1} + \varepsilon \pmod{\omega^2}$, which is not possible. Thus $4t-4 \geq 2$ and $(-i) \equiv -\varepsilon \pmod{\omega^2}$.

From this, it follows that $\varepsilon = \pm i$, that is, either $\varepsilon = i$, $\varepsilon^{-1} = -i$ or $\varepsilon = -i$, $\varepsilon^{-1} = i$. In the first case, we get

$$(-i)x_1^2 = \omega^{4t-4}(-i)b_2^4p^2 + (i)a_2^4,$$
$$-x_1^2 = -\omega^{4t-4}b_2^4p^2 + a_2^4.$$

Thus $(a_2, \omega^{t-1}b_2, ix_1)$, is a nontrivial solution of the equation $x^4 - p^2y^4 = z^2$. In the second case, we get

$$(-i)x_1^2 = \omega^{4t-2}(i)b_2^4p^2 + (-i)a_2^4,$$
$$x_1^2 = -\omega^{4t-4}b_2^4p^2 + a_2^4.$$

Therefore $(a_2, \omega^{t-1}b_2, x_1)$ is a nontrivial solution of the equation $x^4 - p^2y^4 = z^2$.

When $u = 4t-4$, $v = 0$, we get $(-i)x_1^2 = \varepsilon^{-1}b_2^4p^2 + \omega^{4t-4}\varepsilon a_2^4$. If $4t-4 = 0$, then this reduces to $-i \equiv \varepsilon^{-1} + \varepsilon \pmod{\omega^2}$, which is not possible. Thus $4t-4 \geq 2$ and $(-i) \equiv \varepsilon^{-1} \pmod{\omega^2}$. From this, it follows that $\varepsilon = \pm i$, that is, either $\varepsilon = i$, $\varepsilon^{-1} = -i$, or $\varepsilon = -i$, $\varepsilon^{-1} = -i$. In the first case, we get

$$(-i)x_1^2 = (-i)b_2^4p^2 + \omega^{4t-4}(i)a_2^4,$$
$$-x_1^2 = -b_2^4p^2 + \omega^{4t-4}a_2^4.$$

Thus $(\omega^{t-1}a_2, b_2, ix_1)$ is a nontrivial solution of the equation $x^4 - p^2y^4 = z^2$. In the second case, we get

$$(-i)x_1^2 = (i)b_2^4p^2 + \omega^{4t-4}(-i)a_2^4,$$
$$x_1^2 = -b_2^4p^2 + \omega^{4t-4}a_2^4,$$

and so $(\omega^{t-1}a_2, b_2, x_1)$ is a nontrivial solution of the equation $x^4 - p^2y^4 = z^2$.

The case when $a_3 = p^2$, $b_3 = 1$ can be completed in a similar way.

(10) In case 4, let $(x_1, y_1, \omega^t z_1)$ be a solution of the equation $x^4 - p^2y^4 = z^2$, where $t \geq 1$, $x_1 \equiv y_1 \equiv z_1 \equiv 1 \pmod{\omega}$, and $x_1$, $y_1$, $z_1$ are pairwise relatively prime. We will show that there are pairwise relatively prime elements $x_2$, $y_2$, $z_2$ of $Z[i]$ such that $x_2 \equiv y_2 \equiv z_2 \equiv 1 \pmod{\omega}$ and either $(\omega^{t-2}x_2, y_2, z_2)$ or $(x_2, \omega^{t-2}y_2, z_2)$ is a solution of the equation $x^4 - y^4 = pz^2$.

The conclusion of these steps is that in case 4 we end up with a nontrivial solution of the equation $x^4 - y^4 = pz^2$ corresponding to one of cases 1–3. Since by step (2) this is not possible, it follows that case 4 of equation $x^4 - p^2y^4 = z^2$ is not possible either.

From the equation $\omega^{2t}z_1^2 = (x_1^2 - py_1^2)(x_1^2 + py_1^2)$, we can deduce that

$$(-i)x_1^2 = \omega^v\varepsilon^{-1}b_2^2 + \omega^u\varepsilon a_2^2,$$
$$(-i)py_1^2 = \omega^v\varepsilon^{-1}b_2^2 - \omega^u\varepsilon a_2^2,$$
$$2t = u + v + 4, \qquad a_2^2b_2^2 = z_1^2.$$

By multiplying the first two equations above together and multiplying the result by $\varepsilon^2$, we get

$$-\varepsilon^2px_1^2y_1^2 = \omega^{2v}b_2^4 - \omega^{2u}a_2^4.$$

We distinguish two cases depending on whether $u = 0$, $v = 2t-4$, or $u = 2t-4$, $v = 0$. When $u = 0$, $v = 2t-4$, we get $-\varepsilon^2 p x_1^2 y_1^2 = \omega^{4t-8} b_2^4 - a_2^4$. Thus $(a_2, \omega^{t-2} b_2, \varepsilon x_1 y_1)$, is a nontrivial solution of the equation $x^4 - y^4 = pz^2$.

When $u = 2t-4$, $v = 0$, we get $-\varepsilon^2 p x_1^2 y_1^2 = b_2^4 - \omega^{4t-8} a_2^4$. Thus, $(\omega^{t-2} a_2, b_2, \varepsilon x_1 y_1)$ is a nontrivial solution of the equation $x^4 - y^4 = pz^2$.

(11) Let $(x_0, y_0, z_0)$ be a nontrivial solution of the equation $x^4 - p^2 y^4 = z^2$ in $Z[i]$. By steps (5), (7) and (10), cases 1, 2 and 4 are not possible, and so there is a solution $(x_1, \omega^t y_1, z_1)$ with $x_1, y_1, z_1 \equiv 1 \pmod{\omega}$, $t \geq 1$. Choose a solution for which $t$ is minimal. According to step (9), there is a solution either of the form $(\omega^{t-1} x_2, y_2, z_2)$, or of the form $(x_2, \omega^{t-1} y_2, z_2)$, where $x_2, y_2, z_2 \equiv 1 \pmod{\omega}$, $t \geq 2$. The first case is not possible. The second case contradicts the choice of $t$, and so we conclude that the equation $x^4 - p^2 y^4 = z^2$ has no nontrivial solutions in $Z[i]$.

By step (3), a nontrivial solution of the equation $x^4 - y^4 = pz^2$ leads to a nontrivial solution of the equation $x^4 - p^2 y^4 = z^2$. Thus the equation $x^4 - y^4 = pz^2$ does not have nontrivial solutions in $Z[i]$. This completes the proof.

We may describe the combinatorial content of our argument in the following way. We consider a directed graph $\Gamma$ whose vertices and edges are labeled. To a nontrivial solution $(x_0, y_0, z_0)$ of the equation $E : ax^4 + by^4 = cz^2$, we assign a type $T$ depending on the residues of $x_0$, $y_0$, $z_0$ modulo $\omega$. (There are only a limited number of possibilities for $T$.) The nodes of $\Gamma$ are the pairs $(E, T)$, where $E$ is an equation and $T$ is a possible solution type. We label the node $(E, T)$ impossible if the equation $E$ has no nontrivial solution of type $T$. We draw an arrow from the node $(E_1, T_1)$ to the node $(E_2, T_2)$ if a nontrivial solution of $E_1$ with type $T_1$ gives rise to a nontrivial solution of $E_2$ with type $T_2$. We label an arrow with a $-$ sign if a quantity associated with a solution decreases. If each path starting with a node $(E, T_i)$ terminates at a node labeled impossible or eventually reaches $(E, T_i)$ again but the edges are labeled with $-$ signs, then the equation cannot have nontrivial solutions.

## 4. The equation $x^4 + my^4 = z^2$

If $(x_0, y_0, z_0)$ is a nontrivial solution either one of the equations

$$x^4 + 4y^4 = z^2, \qquad x^4 - 4y^4 = z^2, \qquad x^4 - 8y^4 = z^2,$$

then $(x_0, \omega y_0, z_0)$ is a nontrivial solution one of the equations

$$x^4 - y^4 = z^2, \qquad x^4 + y^4 = z^2, \qquad x^4 + 2y^4 = z^2,$$

respectively, as the first three equations can be written in the forms

$$x^4 - \omega^4 y^4 = z^2, \qquad x^4 + \omega^4 y^4 = z^2, \qquad x^4 + 2\omega^4 y^4 = z^2,$$

respectively. Thus, it will be enough to prove that the equation $x^4 + 2y^4 = z^2$ has only trivial solutions in $Z[i]$.

**Theorem 3.** The equation $x^4 + 2y^4 = z^2$ has only trivial solutions in $Z[i]$.

*Proof.* We divide the proof into 5 steps.

(1) If $(x_0, y_0, z_0)$ is a nontrivial solution of the equation $x^4 + 2y^4 = z^2$, then we may assume that $x_0$, $y_0$, $z_0$ are pairwise relatively prime.

(1.a) Choose a nontrivial solution $(x_0, y_0, z_0)$ of the equation $x^4 + 2y^4 = z^2$ with minimal height. Suppose first that $x_0$ and $y_0$ are not relatively prime and let $g$ be a greatest common divisor of $x_0$ and $y_0$. The left hand side of the equation $(x_0/q)^4 + 2(y_0/g)^4 = (z_0/g^2)^2$ is an element of $Z[i]$ and so the right hand side of the equation is an element of $Z[i]$. Hence $(x_0/g, y_0/g, z_0/g^2)$ is a nontrivial solution of $x^4 + 2y^4 = z^2$. By the minimality of the height we may assume that $x_0$ and $y_0$ are relatively prime.

Assume next that there is a prime $q$ in $Z[i]$ such that $q|x_0$, $q|z_0$. It follows that $q|x_0$. This is a contradiction since $x_0$, $y_0$ are relatively prime.

Finally suppose there is a prime $q$ in $Z[i]$ such that $q|x_0$, $q|z_0$. We get that $q^2|2y_0$. If $q|y_0$ we get the contradiction that $x_0$ and $y_0$ are not relatively prime. Thus $q \not| y_0$ and consequently $q^2|2$. We get that $q$ is an associate of $\omega$. Set $x_0 = \omega x_1$, $z_0 = \omega z_1$. From $\omega^4 x_1^4 + 2y_0^4 = \omega^2 z_1^2$ we get

$$\begin{aligned} \omega^4 x_1^4 - i\omega^2 y_0^4 &= \omega^2 z_1^2, \\ \omega^2 x_1^4 - i y_0^4 &= z_1^2, \\ (-i)\omega^2 x_1^4 + y_0^4 &= (-i) z_1^2, \\ y_0^4 + 2x_1^4 &= -i z_1^2. \end{aligned}$$

Therefore $(y_0, x_1, iz_1)$ is a nontrivial solution of the equation $x^4 + 2y^4 = iz^2$.

(1.b) Pick a nontrivial solution of $(x_0, y_0, z_0)$ of the equation $x^4 + 2y^4 = iz^2$ with minimal height. Using the argument we have seen in step (1.a) we may assume that $x_0$ and $y_0$ are relatively prime. The assumption that there is a prime $q$ with $q|y_0$, $q|z_0$ gives the contradiction that $x_0$ and $y_0$ are not relatively prime.

Finally suppose that there is a prime $q$ such that $q|x_0$, $q|z_0$. We get that $q^2|2y_0^4$. Here $q|y_0$ leads to the contradiction that $x_0$ and $y_0$ are not relatively prime. Thus $q^2|2$ and so $q$ and $\omega$

are associates. Set $x_0 = \omega x_1$, $z_0 = \omega z_1$. From $\omega^4 x_1^4 + 2y_0^4 = i\omega^2 z_1^2$ we get

$$\begin{aligned} \omega^4 x_1^4 - i\omega^2 y_0^4 &= i\omega^2 z_1^2, \\ \omega^2 x_1^4 - iy_0^4 &= iz_1^2, \\ (-i)\omega^2 x_1^4 - y_0^4 &= iz_1^2, \\ 2x_1^4 - y_0^4 &= iz^2 \\ y_0^4 - 2x_1^4 &= -iz_1^2. \end{aligned}$$

Hence $(y_0, x_1, iz_1)$ is a nontrivial solution of the equation $x^4 - 2y^4 = iz^2$.

(1.c) Choose a nontrivial solution $(x_0, y_0, z_0)$ of the equation $x^4 - 2y^4 = iz^2$ with minimal height. As before we may assume that $x_0$ and $y_0$ are relatively prime. If there is a prime $q$ with $q|y_0$, $q|z_0$ we get the contradiction that $x_0$ and $y_0$ are not relatively prime.

Finally consider the case when there is a prime $q$ with $q|x_0$, $q|z_0$. It follows that $q^2|2y_0^4$. If $q|y_0$, then we get that $x_0$ and $y_0$ are not relatively prime. This is not the case. So $q^2|2$ and we get that $q$ and $\omega$ are associates. Setting $x_0 = \omega x_1$, $z_0 = \omega z_1$ from $\omega^4 - 2y_0^4 = i\omega^2 z_1^2$ we get

$$\begin{aligned} \omega^4 x_1^4 + i\omega^2 y_0^4 &= i\omega^2 z_1^2, \\ \omega^2 x_1^4 + iy_0^4 &= iz_1^2, \\ (-i)\omega^2 x_1^4 + y_0^4 &= z_1^2, \\ y_0^4 + 2x_1^4 &= z_1^2. \end{aligned}$$

Hence $(y_0, x_1, z_1)$ is a nontrivial solution of $x^4 + 2y^4 = z^2$. The minimality of the height in (1.a) gives that we may assume that $x_0$, $y_0$, $z_0$ are relatively prime in (1.a).

(2) Let $(x_0, y_0, z_0)$ be a nontrivial solution of the equation $x^4 + 2y^4 = z^2$ in $Z[i]$ such that $x_0$, $y_0$, $z_0$ are pairwise relatively prime. Note that at most one of $x_0$, $y_0$, $z_0$ can be congruent to 0 modulo $\omega$. We consider the four cases listed in Table 1.

In cases 2 and 4, the equation $x_0^4 + 2y_0^4 = z_0^2$ leads to the contradictions $0 + 0 \equiv 1 \pmod{\omega}$ and $1 + 0 \equiv 0 \pmod{\omega}$, respectively.

We next show that case 1 is not possible either. To do this, write $z_0$ in the form $z_0 = k\omega^2 + l$, $k, l \in Z[i]$. Computing $z_0^2$, $z_0^2 = k^2\omega^4 + 2k\omega^2 l + l^2$ shows that $z_0^2 \equiv l^2 \pmod{\omega^4}$. As 0, 1, $i$, $1+i$ is a complete set of representatives modulo $\omega^2$, it follows that $l$ can be chosen to be 1 or $i$. From the equation $x_0^4 + 2y_0^4 = z_0^2$ we get that $1 + 2 \equiv l^2 \pmod{\omega^4}$. In the case $l = 1$ this leads to the contradiction $1 + 2 \equiv 1 \pmod{\omega^4}$ and so we left with the choice $l = i$. Now writing $z_0$ in the form $z_0 = r\omega^4 + s$, $r, s \in Z[i]$ and computing $z_0^2$, $z_0^2 = r^2\omega^8 + 2r\omega^4 s + s^2$ gives that $z_0^2 \equiv s^2 \pmod{\omega^6}$. From $z_0 \equiv i \pmod{\omega^2}$, it follows that we can choose $s$ and $s^2$ in the way summarized by Table 2. From the equation $x_0^4 + 2y_0^4 = z_0^2$, we get the contradictions

$$3 \equiv -1 \pmod{\omega^6}, \qquad 3 \equiv 3 + 4i \pmod{\omega^6}.$$

(3) In case 3, let $(x_1, \omega^r y_1, z_1)$ be a solution of the equation $x^4 + 2y^4 = z^2$, where $r \geq 1$, $x_1 \equiv y_1 \equiv z_1 \equiv 1 \pmod{\omega}$, and $x_1$, $y_1$, $z_1$ are pairwise relatively prime. We will show that $z_1 \equiv 1 \pmod{\omega^2}$.

In order to prove this claim, write $z_1$ in the form $z_1 = k\omega^2 + l$, $k, l \in Z[i]$, and compute $z_1^2$.

$$z_1^2 = k^2\omega^4 + 2k\omega^2 l + l^2.$$

From this, it follows that $z_1^2 \equiv l^2 \pmod{\omega^4}$ Since the elements 0, 1, $i$, $1+i$ form a complete set of representatives modulo $\omega^2$, and since $z_1 \equiv 1 \pmod{\omega}$, we may choose $l$ to be 1 or $i$. Consequently, $z_1^2$ is congruent to 1 or $-1$ modulo $\omega^4$. The equation $x_1^4 + 2\omega^{4r} y_1^4 = z_1^2$ gives that $1 \equiv z_1^2 \pmod{\omega^4}$, and so $z_1 \equiv 1 \pmod{\omega^2}$.

(4) We will show that there are pairwise relatively prime elements $x_2$, $y_2$, $z_2$ of $Z[i]$, such that $x_2 \equiv y_2 \equiv z_2 \equiv 1 \pmod{\omega}$, and $(x_2, \omega^{r-1} y_2, z_2)$ is a solution of the equation $x^4 + 2y^4 = z^2$.

In order to verify the claim, write the equation $x_1^4 + 2\omega^{4r} y_1^4 = z_1^2$ in the form $2\omega^{4r} y_1^4 = (z_1 - x_1^2)(z_1 + x_1^2)$, and compute the greatest common divisor of $(z_1 - x_1^2)$ and $(z_1 + x_1^2)$. Let $g$ be this greatest common divisor. As $g | \omega^{4r} y_1^4$, it follows that $g \neq 0$. Since $g | (z_1 - x_1^2)$, $g | (z_1 + x_1^2)$ we get that $g | 2x_1^2$, $g | 2z_1$. If $q$ is a prime divisor of $g$ with $q \not| \omega$, we then get $q | x_1$, $q | z_1$. But we know that this is not the case as $x_1$ and $z_1$ are relatively prime. Thus, $g = \omega^s$ and $0 \leq s \leq 2$ since $g | 2$. By step (3) $z_1 \equiv 1 \pmod{\omega^2}$. This together with $x_1^2 \equiv 1 \pmod{\omega^2}$, gives that $(z_1 - x_1^2) \equiv (z_1 + x_1^2) \equiv 0 \pmod{\omega^2}$. Therefore $g = \omega^2$. The unique factorization property in $Z[i]$ implies that there are relatively prime elements $a, b \in Z[i]$ such that

$$z_1 - x_1^2 = \omega^2 a, \qquad z_1 + x_1^2 = \omega^2 b.$$

Let $a = \omega^u a_1$, $b = \omega^v b_1$. So $(-i)\omega^{4r+2} y_1^4 = \omega^{u+v+4} a_1 b_1$. By the unique factorization property in $Z[i]$, there are elements $a_2$, $b_2$ and units $\varepsilon$, $\mu$ in $Z[i]$ for which

$$z_1 - x_1^2 = \omega^{u+2}\varepsilon a_2^4, \qquad z_1 + x_1^2 = \omega^{v+2}\mu b_2^4,$$

$$4r + 2 = u + v + 4, \qquad a_2^4 b_2^4 = y_1^4, \qquad \varepsilon\mu = -i.$$

Here, $a_2$, $b_2$ are prime to $\omega$. It follows that $a_2 \equiv b_2 \equiv 1 \pmod{\omega}$. By addition, we get

$$2x_1^2 = \omega^{v+2}\mu b_2^4 - \omega^{u+2}\varepsilon a_2^4.$$

After dividing it by $\omega^2$, we get

$$(-i)x_1^2 = \omega^v \mu b_2^4 - \omega^u \varepsilon a_2^4.$$

We distinguish two cases depending on whether $u = 0$, $v = 4r - 2$, or $v = 0$, $u = 4r - 2$. When $u = 0$, $v = 4r - 2$, we get

$$(-i)x_1^2 = \omega^{4r-2}\mu b_2^4 - \varepsilon a_2^4.$$

This reduces to $(-i) \equiv -\varepsilon \pmod{\omega^2}$. From this, it follows that $\varepsilon = \pm i$, that is, either $\varepsilon = i$, $\mu = -1$, or $\varepsilon = -i$, $\mu = 1$. In the first case, we get

$$\begin{aligned}(-i)x_1^2 &= \omega^{4r-2}(-1)b_2^4 - (i)a_2^4,\\ x_1^2 &= (-i)\omega^2\omega^{4r-4}b_2^4 + a_2^4,\\ x_1^2 &= 2\omega^{4r-4}b_2^4 + a_2^4.\end{aligned}$$

Thus, $(a_2, \omega^{r-1}b_2, x_1)$ is a nontrivial solution of the equation $x^4 + 2y^4 = z^2$. In the second case, we get

$$\begin{aligned}(-i)x_1^2 &= \omega^{4r-2}(-1)b_2^4 - (i)a_2^4,\\ -x_1^2 &= (-i)\omega^2\omega^{4r-4}b_2^4 + a_2^4,\\ -x_1^2 &= 2\omega^{4r-4}b_2^4 + a_2^4.\end{aligned}$$

Therefore, $(a_2, \omega^{r-1}b_2, ix_1)$ is a nontrivial solution of the equation $x^4 + 2y^4 = z^2$.

When $v = 0$, $u = 4r - 2$, we get

$$(-i)x_1^2 = \mu b_2^4 - \omega^{4r-2}\varepsilon a_2^4.$$

This reduces to $(-i) \equiv \mu \pmod{\omega^2}$. From this, it follows that $\mu = \pm i$, that is, either $\varepsilon = -1$, $\mu = i$, or $\varepsilon = 1$, $\mu = -i$. In the first case, we get

$$\begin{aligned}(-i)x_1^2 &= (i)b_2^4 - \omega^{4r-2}(-1)a_2^4,\\ -x_1^2 &= b_2^4 + (-i)\omega^2\omega^{4r-4}a_2^4,\\ -x_1^2 &= b_2^4 + 2\omega^{4r-4}a_2^4.\end{aligned}$$

Thus, $(b_2, \omega^{r-1}a_2, ix_1)$ is a nontrivial solution of the equation $x^4 + 2y^4 = z^2$. In the second case, we get

$$\begin{aligned}(-i)x_1^2 &= (-i)b_2^4 - \omega^{4r-2}a_2^4,\\ x_1^2 &= b_2^4 + (-i)\omega^2\omega^{4r-4}a_2^4,\\ x_1^2 &= b_2^4 + 2\omega^{4r-4}a_2^4,\end{aligned}$$

and so $(b_2, \omega^{r-1}a_2, x_1)$ is a nontrivial solution of the equation $x^4 + 2y^4 = z^2$.

(5) Let $(x_0, y_0, z_0)$ be a nontrivial solution of the equation $x^4 + 2y^4 = z^2$ in $Z[i]$. By step (2), there is a solution $(x_1, \omega^r y_1, z_1)$ with $x_1, y_1, z_1 \equiv 1 \pmod{\omega}$, $r \geq 1$. Choose a solution for which $r$ is minimal. According to step (4), there is a solution $(x_2, \omega^{r-1}y_2, z_2)$, where $x_2, y_2, z_2 \equiv 1 \pmod{\omega}$, $r \geq 2$. This contradicts the choice of $r$ and so completes the proof.

## References

1. J. T. Cross, In the Gaussian integers $\alpha^4 + \beta^4 \neq \gamma^4$, *Math. Magazine*, **66** (1993), 105–108.

2. D. Hilbert, Jahresbericht d. Deutschen Math.-Vereinigung, 4, 1894-1895, 517–525.

3. L. J. Mordell, *Diophantine Equations*, Academic Press, New York, 1969, 116–117.

4. T. Nagell, *Introduction to Number Theory*, Chelsea Publ. Comp., New York, 1981.