PARRY EXPANSIONS OF POLYNOMIAL SEQUENCES [1]

Wolfgang Steiner

Institut für Geometrie, Technische Universität Wien, Wiedner Hauptstraße 8-10, A-1040 Wien, Austria

steiner@geometrie.tuwien.ac.at

Received: 4/19/02, Revised: 12/17/02, Accepted: 12/19/02, Published: 12/26/02

Abstract

We prove that the sum-of-digits function with respect to certain digital expansions (which are related to linear recurrences) and similarly defined functions evaluated on polynomial sequences of positive integers or primes satisfy a central limit theorem. These digital expansions are special cases of numeration systems associated to primitive substitutions on finite alphabets, the digits of which form Markov chains and induce Markov partitions of the torus $\mathbb{T}^d$. We provide an algorithm to determine the (fractal) boundary of these partitions.

1. Introduction

Let the sequence $G = (G_k)_{k\geq 0}$ be defined by the linear recurrence

$$G_k = a_1 G_{k-1} + a_2 G_{k-2} + \cdots + a_d G_{k-d} \text{ for } k \geq d$$

and

$$G_k = a_1 G_{k-1} + a_2 G_{k-2} + \cdots + a_k G_0 \text{ for } 1 \leq k < d,\ G_0 = 1,$$

with non-negative integers a_i which satisfy the relations

$$(a_j, a_{j+1}, \ldots, a_d) \leq (a_1, a_2, \ldots, a_{d-j+1}) \text{ for } j = 2, \ldots, d$$

(where "$<$" denotes the lexicographical order) and $a_d > 0$.

Then every non-negative integer n has a unique (greedy) G-ary digital expansion

$$n = \sum_{k\geq 0} \epsilon_k(n) G_k$$

with integer digits $\epsilon_k(n) \geq 0$ satisfying

$$(\epsilon_k(n), \epsilon_{k-1}(n), \ldots, \epsilon_{k-d+1}(n)) < (a_1, a_2, \ldots, a_d) \text{ for all } k \geq 0. \tag{1}$$

[1] This research was supported by the Austrian Science Foundation FWF, grant S8302-MAT.

Furthermore, let

$$\chi(x) = x^d - a_1 x^{d-1} - \cdots - a_{d-1}x - a_d$$

be the characteristic polynomial of the linear recurrence. It is easy to show that it has a unique dominant root $\alpha \in \mathbb{R}^+$ (e.g. consider its (primitive) companion matrix and apply the Perron-Frobenius theorem). If $\chi(x)$ is irreducible over $\mathbb{Z}$, denote by $\alpha_2, \ldots, \alpha_d$ the (distinct) algebraic conjugates of α. Then we have, for some constants $c_1, \ldots, c_d$,

$$G_k = c_1\alpha^k + c_2\alpha_2^k + \cdots + c_d\alpha_d^k. \tag{2}$$

($c_1 = \frac{\alpha^d-1}{\alpha-1}\frac{1}{\prod_{j>1}(\alpha-\alpha_j)}$ will be calculated in Section 4 and, for reasons of symmetry, we have $c_j = \frac{\alpha_j^d-1}{\alpha_j-1}\frac{1}{\prod_{i\neq j}(\alpha_j-\alpha_i)}$ for all $j \geq 1$, where $\alpha_1 = \alpha$.)

(1) and (2) show that these G-ary expansions of integers have essentially the same properties as α-expansions of real numbers (where α is a simple β-number), which where first considered by Rényi [24]. The relevant characterization of α-expansions (and the notion β-number) is due to Parry [21]. Therefore we call these G-ary expansions Parry expansions. For additional properties, we refer to Grabner and Tichy [16].

We want to study the distribution of G-additive functions f, i.e.

$$f(n) = \sum_{k\geq 0} f(\epsilon_k(n)G_k) = \sum_{k\geq 0} f_k(\epsilon_k(n)) \text{ for all } n \in \mathbb{N},\ f(0) = 0,$$

on polynomial sequences of non-negative integers and primes.

Drmota and the author proved in [8] the following theorem for sequences G of the above type with $d = 2, a_2 = 1$:

Theorem 1. Let the sequence $G = (G_k)_{k\geq 0}$ be defined by

$$G_k = aG_{k-1} + G_{k-2} \text{ for } k \geq 2,\ G_0 = 1,\ G_1 = a+1$$

for some integer $a \geq 1$, α be the dominant root of $x^2 - ax - 1$ and f a G-additive function such that $f_k(e) = \mathcal{O}(1)$ as $k \to \infty$ for all $e = 0, \ldots, a$. Then, for all $\eta > 0$, the expected value of $f(n)$, $0 \leq n < N$, is given by

$$E_N = \frac{1}{N}\sum_{n<N} f(n) = M(N) + \mathcal{O}\left((\log N)^\eta\right),$$

where

$$M(N) = \sum_{k=0}^{[\log_\alpha N]} \mu_k \quad \text{with} \quad \mu_k = \frac{\alpha}{\alpha^2+1}\sum_{e=1}^{a-1} f_k(e) + \frac{1}{\alpha^2+1} f_k(a).$$

Furthermore, set

$$D(N)^2 = \sum_{k,k'=0}^{[\log_\alpha N]} \sigma_{k,k'}^{(2)} \quad \text{with} \quad \sigma_{k,k'}^{(2)} = \begin{cases} \frac{\alpha}{\alpha^2+1}\sum\limits_{e=1}^{a-1} f_k(e)^2 + \frac{1}{\alpha^2+1} f_k(a)^2 - \mu_k^2 & \text{if } k = k' \\ \left(-\frac{1}{\alpha^2}\right)^{|k-k'|} \mu_{\min(k,k')}\overline{\mu}_{\max(k,k')} & \text{if } k \neq k', \end{cases}$$

where

$$\overline{\mu}_k = -\frac{\alpha}{\alpha^2+1}\sum_{e=1}^{a-1} f_k(e) + \frac{\alpha^2}{\alpha^2+1} f_k(a)$$

and assume that there exists a constant $c > 0$ such that $\sigma_{k,k}^{(2)} \geq c$ for all $k \geq 0$. Then we have, as $N \to \infty$,

$$\frac{1}{N}\sum_{n<N}(f(n) - E_N)^2 \sim D(N)^2,$$

and, for polynomials $P(n)$ of degree r with positive leading term,

$$\frac{1}{N}\#\left\{n < N \,\middle|\, \frac{f(P(n)) - M(N^r)}{D(N^r)} < x\right\} \to \Phi(x)$$

and

$$\frac{1}{\pi(N)}\#\left\{p \in \mathbb{P}, p < N \,\middle|\, \frac{f(P(p)) - M(N^r)}{D(N^r)} < x\right\} \to \Phi(x).$$

(Here and in the sequel $\Phi(x)$ denotes the distribution function of the normal law.)

The strategy of the proof is based on a paper by Bassily and Kátai [3] who studied q-additive functions (i.e. G-additive functions with $G_k = q^k$). Gittenberger and Thuswaldner [15] used this strategy to prove a similar theorem for b-additive functions on the Gaussian integers. More generally, the distribution of q-additive functions has been discussed by several authors (see Drmota [7] for a list of references).

Our aim is to prove a similar theorem for G-additive functions with sequences G defined by linear recurrences of higher degree. As in [8], we will first prove the following theorem on the distribution of the sequence $f(n), 0 \leq n < N$.

Theorem 2. Let G be as in the first paragraph, f a G-additive function such that $f_k(e) = \mathcal{O}(1)$ as $k \to \infty$ for all $e = 0, \ldots, a_1$. Then, for all $\eta > 0$, the expected value of $f(n)$, $0 \leq n < N$, is given by

$$E_N = \frac{1}{N}\sum_{n<N} f(n) = M(N) + \mathcal{O}\left((\log N)^\eta\right), \tag{3}$$

where

$$M(N) = \sum_{k=0}^{[\log_\alpha N]} \mu_k \quad \text{with} \quad \mu_k = \sum_{e=1}^{a_1} p_e f_k(e)$$

and the constants p_e are the asymptotic probabilities of the digits e, the values of which are determined by equation (11).

Furthermore, set

$$D(N)^2 = \sum_{k,k'=0}^{[\log_\alpha N]} \sigma_{k,k'}^{(2)}$$

with

$$\sigma_{k,k'}^{(2)} = \begin{cases} \sum_{e=1}^{a_1} p_e f_k(e)^2 - \mu_k^2 & \text{if } k = k' \\ \sum_{i=2}^{d} \left(\frac{\alpha_i}{\alpha}\right)^{|k-k'|} \sum_{e=1}^{a_1} \sum_{e'=1}^{a_1} p_{e,e'}^{(i)} f_{\min(k,k')}(e) f_{\max(k,k')}(e') & \text{if } k \neq k' \end{cases}$$

and constants $p_{e,e'}^{(i)}$ described on page 10 and assume that there exists a constant $c > 0$ such that $\sigma_{k,k}^{(2)} \geq c$ for all $k \geq 0$. Then we have, as $N \to \infty$,

$$\frac{1}{N} \sum_{n<N} (f(n) - E_N)^2 \sim D(N)^2, \tag{4}$$

$$\frac{1}{N} \# \left\{ n < N \,\middle|\, \frac{f(n) - M(N)}{D(N)} < x \right\} \to \Phi(x), \tag{5}$$

$$\frac{1}{N} \sum_{n<N} \left(\frac{f(n) - M(N)}{D(N)} \right)^h \to \int_{-\infty}^{\infty} x^h \, d\Phi(x) \tag{6}$$

for all integers $h \geq 0$.

The proof of Theorem 1 (in [8]) relies on the fact that the digits of the possible G-ary expansions can be represented by random variables which form a Markov chain (of order 1). In our more general case, this Markov chain would be of order $d-1$. By using a representation of the digital expansions in terms of substitutions, like Dumont and Thomas [11] (who studied strongly G-additive functions, i.e. $f(n) = \sum_{k\geq 0} f(\epsilon_k(n))$), we obtain a Markov chain of order 1 (see Section 2). Furthermore, this approach permits to consider more general numeration systems associated to primitive substitutions on finite alphabets, which is not done in this paper for the sake of readability. As in [8], we will use Theorem 2 and a method similar to that of Bassily and Kátai to prove Theorem 3.

Unfortunately we have to make some restrictions on the sequence G: α has to be a Pisot unit with minimal polynomial $\chi(x)$, i.e. $|\alpha_i| < 1$ for $i = 2, \ldots, d$ and $a_d = 1$, and

$$\mathrm{Fin}(\alpha) = \mathbb{Z}[\alpha^{-1}] \cap \mathbb{R}^+, \tag{F}$$

where $\mathrm{Fin}(\alpha)$ denotes the set of non-negative real numbers with finite α-expansion, i.e. $\{x \in \mathbb{R}^+ : x = \sum_{k=-L}^{M} \epsilon_k \alpha^k \text{ with } (\epsilon_j, \ldots, \epsilon_{j-d+1}) < (a_1, \ldots, a_d) \text{ for all } j \leq M\}$.

Theorem 3. Let G be as in the first paragraph with irreducible characteristic polynomial $\chi(x)$ and its dominant root α a Pisot unit which satisfies (F). Let f, M, D be as in Theorem 2 and $P(n)$ a polynomial of degree r with integer coefficients and positive leading term. Then, as $N \to \infty$,

$$\frac{1}{N} \# \left\{ n < N \,\middle|\, \frac{f(P(n)) - M(N^r)}{D(N^r)} < x \right\} \to \Phi(x)$$

and

$$\frac{1}{\pi(N)} \# \left\{ p \in \mathbb{P}, p < N \,\middle|\, \frac{f(P(p)) - M(N^r)}{D(N^r)} < x \right\} \to \Phi(x).$$

A plan of the proof of this theorem is given in Section 3. The difficult part is Lemma 8, which is proved in Section 6 with the help of Sections 4 and 5. As in [8], the proof uses tilings of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ in order to determine the value of a digit $\epsilon_k(n)$ without using the greedy algorithm (Section 4). Whereas these tilings consist of rectangles for $d = 2$, the involved sets have fractal boundary for $d > 2$. The restrictions on G are due to the problem of finding such tilings. Note that tilings with fractal boundary appear also in the case of digital expansions of the Gaussian integers (cf. [15]).

For $a_1 \geq a_2 \geq \cdots \geq a_d > 0$, we know from Brauer [4] that α is a Pisot number with minimal polynomial $\chi(x)$. Since (F) has been shown in this case by Frougny and Solomyak [12], Theorem 3 holds for these sequences. For $d = 3, a_2 = 0$ (and $a_3 = 1$), α_2 and α_3 are complex numbers and have therefore absolute value $1/\sqrt{\alpha}$. For these a_i, (F) was shown by Akiyama [1]. Thus Theorem 3 holds for these sequences too and the only restriction for $d = 3$ is $a_3 = 1$.

Remark. α may not be a Pisot number (e.g. the dominant root of $x^6 - x^5 - 1$). There are also α which are Pisot units, but do not satisfy (F): let α be the dominant root of $x^4 - x^3 - 1$. Then the α-expansion of 2 is $10.010(00001)^\infty$.

2. Proof of Theorem 2

First we recall the notion of digital expansions associated to substitutions (cf. Dumont and Thomas [10, 11]). Let σ be the substitution on $\mathcal{A} = \{1, \ldots, d\}$ defined by

$$\begin{aligned} \sigma : i &\to 1^{a_i}(i+1) \quad \text{for } i = 1, \ldots, d-1 \\ d &\to 1^{a_d} \end{aligned}$$

and let σ also stand for its extension on the set of words $\mathcal{A}^* = \bigcup_{i=1}^\infty \mathcal{A}^i \cup \{\Lambda\}$ with Λ being the empty word. Denote by $|m|$ the length of the word m and write $m' < m$ if m' is a strict prefix of m.

A sequence of words $m_{j-1}m_{j-2}\ldots m_0$ is said to be b-admissible, if there exist (unique) letters $b_j = b, b_{j-1}, \ldots, b_0$ such that $m_k b_k \leq \sigma(b_{k+1})$. The admissible representation of an integer $n \geq 1$ is the (unique) 1-admissible sequence $m_{j-1}(n)m_{j-2}(n)\ldots m_0(n)$, with $m_{j-1}(n) \neq \Lambda$, such that

$$n = \left|\sigma^{j-1}(m_{j-1}(n))\right| + \cdots + \left|\sigma^0(m_0(n))\right|.$$

Denote by $b_k(n)$ the letter b_k corresponding to this 1-admissible sequence. It is easy to show (by induction) that the numbers $\left|\sigma^k(1)\right|$ are just the G_k defined by the linear recurrence in the Introduction and $m_k(n) = 1^{\epsilon_k(n)}$.

The matrix of the substitution

$$M = \Big(\#\{\text{occurrences of } b \text{ in } \sigma(b')\}\Big)_{b,b' \in \mathcal{A}} = \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_d \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

is the companion matrix of the characteristic polynomial of the linear recurrence.

Our aim is to study the distribution behaviour of $f(n), 0 \le n < N$, i.e. the random variable Y_N defined by

$$\mathbf{Pr}[Y_N \le x] = \frac{1}{N}\#\{n < N \mid f(n) \le x\}.$$

If we define $Y_{k,N}$ by

$$\mathbf{Pr}[Y_{k,N} \le x] = \frac{1}{N}\#\{n < N \mid f_k(\epsilon_k(n)) \le x\},$$

$\xi_{k,N}$ by

$$\mathbf{Pr}[\xi_{k,N} = (m,b)] = \frac{1}{N}\#\{n < N \mid (m_k(n), b_k(n)) = (m,b)\},$$

and $f(m,b) = f(|m|)$, we have

$$Y_N = \sum_{k \ge 0} Y_{k,N} = \sum_{k \ge 0} f_k(\xi_{k,N}),$$

i.e. Y_N is a weighted sum of the $\xi_{k,N}$. Therefore we first have a detailed look at the $\xi_{k,N}$.

Dumont and Thomas [11] showed that, for fixed j, the sequence $(\xi_{j-1,G_j}, \xi_{j-2,G_j}, \dots, \xi_{0,G_j})$ constitutes a Markov chain with transition probabilities

$$\begin{aligned} \mathbf{Pr}[\xi_{k,G_j} = (m,b) | \xi_{k+1,G_j} = (m',b')] &= \mathbf{Pr}[\xi_{k,G_j} = (m,b) | \xi_{k+1,G_j} = (.,b')] \\ &= \begin{cases} \frac{|\sigma^k(b)|}{|\sigma^{k+1}(b')|} = p_{(.,b'),(m,b)} + o(\rho^k) & \text{if } mb \le \sigma(b') \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $(.,b)$ denotes the set of states $\{(m,b) \mid m \in \mathcal{A}^*\}$, $p_{(.,b'),(m,b)} = \frac{\nu_b}{\nu_{b'}\alpha}$,

$$(\nu_1, \dots, \nu_d) = (1, \alpha - a_1, \alpha^2 - a_1\alpha - a_2, \dots, \alpha^{d-1} - a_1\alpha^{d-2} - \cdots - a_{d-1})$$

is a left eigenvector of M to the eigenvalue α, and $\rho < 1$ a constant such that all roots of $\chi(x)$ except α have modulus less then $\alpha\rho$. (For Pisot numbers α, we can set $\rho = \alpha^{-1}$.)

Furthermore, denote by $P_{k,j}$ the matrix of transition probabilities $\mathbf{Pr}[\xi_{k,G_j} = (.,b)|\xi_{k+1,G_j} = (.,b')]$. Then we have $P_{k,j} = P + \mathcal{O}\left(\rho^k\right)$ with

$$P = \left(p_{(.,b'),(.,b)}\right)_{b',b\in\mathcal{A}} = \begin{pmatrix} \frac{a_1}{\alpha} & \frac{a_2}{\alpha^2-a_1\alpha} & \cdots & \frac{a_{d-1}}{\alpha^{d-1}-a_1\alpha^{d-2}-\cdots-a_{d-2}\alpha} & 1 \\ \frac{\alpha-a_1}{\alpha} & 0 & \cdots & \cdots & 0 \\ 0 & \frac{\alpha^2-a_1\alpha-a_2}{\alpha^2-a_1\alpha} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\alpha^{d-1}-a_1\alpha^{d-2}-\cdots-a_{d-1}}{\alpha^{d-1}-a_1\alpha^{d-2}-\cdots-a_{d-2}\alpha} & 0 \end{pmatrix}.$$

P is similar to

$$\begin{pmatrix} \frac{a_1}{\alpha} & \frac{a_2}{\alpha^2} & \cdots & \cdots & \frac{a_d}{\alpha^d} \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

and its eigenvalues are therefore $1, \alpha_2/\alpha, \ldots, \alpha_d/\alpha$. Hence we have

$$\mathbf{Pr}[\xi_{k,G_j} = (.,b)] = p_{(.,b)} + \mathcal{O}\left(\rho^{\min(k,j-k)}\right), \tag{7}$$

where the probability vector $(p_{(.,1)}, \ldots, p_{(.,d)})^t$ is the right eigenvector of P to the eigenvalue 1 with $\sum_{i=1}^d p_{(.,i)} = 1$:

$$\begin{pmatrix} p_{(.,1)} \\ \vdots \\ p_{(.,d)} \end{pmatrix} = \frac{1}{\chi'(\alpha)}\left(\alpha^{d-1}, \alpha^{d-1} - a_1\alpha^{d-2}, \alpha^{d-1} - a_1\alpha^{d-2} - a_2\alpha^{d-3}, \ldots, \frac{a_d}{\alpha}\right)^t$$

Thus

$$\begin{aligned} \mathbf{Pr}[\xi_{k,G_j} = (m,b)] &= \sum_{b':mb\le\sigma(b')} \mathbf{Pr}[\xi_{k,G_j} = (m,b), \xi_{k+1,G_j} = (.,b')] \\ &= p_{(m,b)} + \mathcal{O}\left(\rho^{\min(k,j-k)}\right) \end{aligned}$$

with $p_{(m,b)} = \sum_{b':mb\le\sigma(b')} p_{(.,b'),(m,b)} p_{(.,b')}$.

This suggests to approximate the digital distribution by a stationary Markov chain $(X_k, k \ge 0)$, with the stationary probability distribution $\mathbf{Pr}[X_k = (m,b)] = p_{(m,b)}$ and the transition probabilities $\mathbf{Pr}[X_k = (m,b)|X_{k+1} = (.,b')] = p_{(.,b'),(m,b)}$. The next lemma shows how this approximation can be quantified for finite-dimensional distributions.

Lemma 1. For every $h \ge 1$ and all integers $0 \le k_1 < k_2 < \cdots < k_h < j$, we have

$$\mathbf{Pr}[\xi_{k_1,G_j} = (.,b_1), \ldots, \xi_{k_h,G_j} = (.,b_h)] = q_{k_1,\ldots,k_h,(.,b_1),\ldots,(.,b_h)} + \mathcal{O}\left(\rho^{\min(k_1,j-k_h)}\right),$$

where

$$q_{k_1,\ldots,k_h,(.,b_1),\ldots,(.,b_h)} = \mathbf{Pr}[X_{k_1} = (.,b_1), \ldots, X_{k_h} = (.,b_h)].$$

Proof. For $0 \leq k < k' < j$, we have

$$P_{k,j}P_{k+1,j}\cdots P_{k'-1,j} = P^{k'-k} + \mathcal{O}\left(\rho^k\right)$$

and consequently

$$\mathbf{Pr}[\xi_{k,G_j} = (.,b)|\xi_{k',G_j} = (.,b')] = \mathbf{Pr}[X_k = (.,b)|X_{k'} = (.,b')] + \mathcal{O}\left(\rho^k\right). \tag{8}$$

Because of

$$\begin{aligned}
&\mathbf{Pr}[\xi_{k_1,G_j} = (.,b_1),\ldots,\xi_{k_h,G_j} = (.,b_h)] \\
&\quad = \mathbf{Pr}[\xi_{k_1,G_j} = (.,b_1)|\xi_{k_2,G_j} = (.,b_2)]\mathbf{Pr}[\xi_{k_2,G_j} = (.,b_2)|\xi_{k_3,G_j} = (.,b_3)]\cdots \\
&\qquad \cdots\mathbf{Pr}[\xi_{k_{h-1},G_j} = (.,b_{h-1})|\xi_{k_h,G_j} = (.,b_h)]\mathbf{Pr}[\xi_{k_h,G_j} = (.,b_h)],
\end{aligned}$$

it suffices to apply (7) and (8) and the lemma follows.

Hence we have

$$\mathbf{Pr}[\xi_{k,G_j} = (m,b),\xi_{k',G_j} = (m',b')] = q_{k,k',(m,b),(m',b')} + \mathcal{O}\left(\rho^{\min(k,j-k')}\right)$$

$(0 \leq k < k' < j)$ with

$$q_{k,k',(m,b),(m',b')} = \sum_{c:mb\leq\sigma(c)} \frac{p_{(m',b')}}{p_{(.,b')}} q_{k+1,k',(.,c),(.,b')}p_{(.,c),(m,b)} \tag{9}$$

because of

$$\begin{aligned}
&\mathbf{Pr}[\xi_{k,G_j} = (m,b)|\xi_{k',G_j} = (m',b')] \\
&\quad = \sum_{c:mb\leq\sigma(c)} \mathbf{Pr}[\xi_{k,G_j} = (m,b)|\xi_{k+1,G_j} = (.,c)]\mathbf{Pr}[\xi_{k+1,G_j} = (.,c)|\xi_{k',G_j} = (.,b')].
\end{aligned}$$

For the finite-dimensional distributions, we obtain

$$\mathbf{Pr}[\xi_{k_1,G_j} = (m_1,b_1),\ldots,\xi_{k_h,G_j} = (m_h,b_h)] = q_{k_1,\ldots,k_h,(m_1,b_1),\ldots,(m_h,b_h)} + \mathcal{O}\left(\rho^{\min(k_1,j-k_h)}\right), \tag{10}$$

where the $q_{k_1,\ldots,k_h,(m_1,b_1),\ldots,(m_h,b_h)}$ are defined similarly to (9).

The next lemma shows that, for general N, $\xi_{k,N}$ is similar to ξ_{k,G_j} where G_j is the largest element of G not exceeding N ($j \approx [\log_\alpha N]$).

Lemma 2. The probability distribution of $\xi_{k,N}$ for $G_j \leq N < G_{j+1}$ with $k < j$ is given by

$$\mathbf{Pr}[\xi_{k,N} = (m,b)] = \mathbf{Pr}[\xi_{k,G_j} = (m,b)] + \mathcal{O}\left(\rho^{(j-k)/2}\right).$$

The joint distribution for $0 \le k_1 < k_2 < \cdots < k_h < j$ is given by

$$\begin{aligned}\mathbf{Pr}[\xi_{k_1,N} = (m_1,b_1),\ldots,\xi_{k_h,N} = (m_h,b_h)] \\ = \mathbf{Pr}[\xi_{k_1,G_j} = (m_1,b_1),\ldots,\xi_{k_h,G_j} = (m_h,b_h)] + \mathcal{O}\left(\rho^{(j-k_h)/2}\right)\end{aligned}$$

The proof is similar to that of Lemma 3 in [8] and therefore omitted.

The above calculations indicate that, in order to obtain uniform estimates, we have to concentrate on the digits $\epsilon_k(n)$ with $A(N) \le k \le B(N)$, where $A(N) = [(\log N)^\eta]$, $B(N) = [\log_\alpha N] - [(\log N)^\eta]$.

Lemma 3. For every $h \ge 1$ and for every $\lambda > 0$, we have

$$\frac{1}{N}\#\{n < N \mid \epsilon_{k_1}(n) = e_1,\ldots,\epsilon_{k_h}(n) = e_h\} = q_{k_1,\ldots,k_h,e_1,\ldots,e_h} + \mathcal{O}\left((\log N)^{-\lambda}\right)$$

uniformly for

$$A(N) \le k_1, k_2, \cdots, k_h \le B(N),$$

where

$$q_{k_1,\ldots,k_h,e_1,\ldots,e_h} = \sum_{(m_i,b_i):|m_i|=e_i} q_{k_1,\ldots,k_h,(m_1,b_1),\ldots,(m_h,b_h)}.$$

This lemma is a direct consequence of Lemma 2 and (10). Note that it is not necessary that $k_1,\ldots,k_h$ are ordered and that they are distinct.

Now, we turn to the derivation of $E_N = \mathbf{E}\,Y_N$, i.e. to the proof of (3). We have

$$\mathbf{E}\,Y_{k,N} = \sum_{m,b} \mathbf{Pr}[\xi_{k,N} = (m,b)] f_k(|m|) = \sum_{e=1}^{a_1} p_e f_k(e) + \mathcal{O}\left(\rho^{\min(k,(j-k)/2)}\right),$$

where

$$p_e = \sum_{m,b:|m|=e} p_{(m,b)}. \tag{11}$$

Since $f_k(e)$ is bounded, we get

$$E_N = \sum_{k=0}^{[\log_\alpha N]} \mathbf{E}\,Y_{k,N} = \sum_{k=A(N)}^{B(N)} \mathbf{E}\,Y_{k,N} + \mathcal{O}\left((\log N)^\eta\right) = M(N) + \mathcal{O}\left((\log N)^\eta\right).$$

The variance is given by

$$\mathbf{Var}\left(\sum_{k=0}^{[\log_\alpha N]} f_k(X_k)\right) = \sum_{k,k'=0}^{[\log_\alpha N]} \Big(\mathbf{E}\big(f_k(X_k) f_{k'}(X_{k'})\big) - \mathbf{E}\,f_k(X_k)\mathbf{E}\,f_{k'}(X_{k'})\Big)$$

and

$$\mathbf{E}\big(f_k(X_k)f_{k'}(X_{k'})\big) - \mathbf{E}\, f_k(X_k)\mathbf{E}\, f_{k'}(X_{k'}) = \sum_{e,e'=1}^{a_1} (q_{k,k',e,e'} - p_e p_{e'}) f_k(e) f_{k'}(e').$$

Since the eigenvalues of M are $\alpha_1/\alpha, \ldots, \alpha_d/\alpha$ (with $\alpha_1 = \alpha$), we have, for $k < k'$,

$$q_{k,k',(.,b),(.,b')} = \sum_{i=1}^{d} r^{(i)}_{(.,b),(.,b')} \left(\frac{\alpha_i}{\alpha}\right)^{k'-k}$$

with (easily determined) constants $r^{(i)}_{(.,b),(.,b')}$ and $r^{(1)}_{(.,b),(.,b')} = p_{(.,b)}p_{(.,b')}$. Since the $q_{k,k',e,e'}$ are (weighted) sums of $q_{k,k',(.,b),(.,b')}$, we have

$$q_{k,k',e,e'} = \sum_{i=1}^{d} p^{(i)}_{e,e'} \left(\frac{\alpha_i}{\alpha}\right)^{k-j},$$

where the constants $p^{(i)}_{e,e'}$ are the respective sums of $r^{(i)}_{(.,b),(.,b')}$. Note that $p^{(1)}_{e,e'} = p_e p_{e'}$. With these $p^{(i)}_{e,e'}$, we get $D(N)^2 = \mathbf{Var}\left(\sum_{k=0}^{[\log_\alpha N]} f_k(X_k)\right)$.

In Lemma 5, we will need $D(N)/(\log N)^\eta \to \infty$. We show that this lower bound holds for $\eta < 1/2$ if the variances of $f_k(X_k)$ have a uniform lower bound.

Lemma 4. Suppose that there exists a constant $c > 0$ such that $\sigma^{(2)}_{k,k} \geq c$ for all $k \geq 0$. Then we have a constant w such that

$$\mathbf{Var}\left(\sum_{k=s}^{s'} f_k(X_k)\right) \geq w(s' - s + 1) \tag{12}$$

for all $s, s' \geq 0$ with $s' - s \geq 3d$.

Proof. By Dobrušin [5], we have $\mathbf{Var}\left(\sum_{k=s}^{s'} f_k(X_k)\right) \geq c(s' - s + 1)\beta/100$, where β is the ergodicity coefficient

$$\beta = 1 - \sup_{m,b,m',b',\mathcal{A}} \big|\mathbf{Pr}[X_k \in \mathcal{A} | X_{k+1} = (m,b)] - \mathbf{Pr}[X_k \in \mathcal{A} | X_{k+1} = (m',b')]\big|$$

(which does not depend on k). Hence the lemma is proved, if we have $\beta > 0$.

If all a_i are non-zero, we have $p_{(m,b),(\Lambda,1)} > 0$ for all possible (m,b). Therefore, if $(\Lambda, 1) \in \mathcal{A}$, we have $\mathbf{Pr}[X_k \in \mathcal{A} | X_{k+1} = (m,b)] > 0$ for all (m,b) and the difference cannot be 1. If $(\Lambda, 1) \notin \mathcal{A}$, the difference cannot be 1 because we have $\mathbf{Pr}[X_k \in \mathcal{A} | X_{k+1} = (m,b)] < 1$ for all (m,b). Since the transition probabilities attain just finitely many values, we have $\beta > 0$.

If $a_b = 0$ for some b $(1 < b < d)$, then $\mathbf{Pr}[X_k = (\Lambda, b+1) | X_{k+1} = (1^{a_{b-1}}, b)] = 1$ and $\mathbf{Pr}[X_k = (\Lambda, b+1) | X_{k+1} = (\Lambda, 1)] = 0$, hence $\beta = 0$. Nevertheless we have

$\mathbf{Pr}[X_k = (m,b)|X_{k+d-1} = (m',b')] > 0$ for all possible $(m,b),(m',b')$ and Corollary 6 in Giesbrecht [13] concludes the lemma (after a tedious consideration of his notation).

Immediately, we get the following corollary.

Corollary 1. Suppose that there exists a constant $c > 0$ such that $\sigma^{(2)}_{k,k} \geq c$ for all $k \geq 0$. Then we have

$$D(N)^2 \gg \log N.$$

In order to prove (6), it suffices, because of the following lemma, to show that the moments

$$\frac{1}{N}\sum_{n<N}\left(\frac{\overline{f}(n)-\overline{M}(N)}{\overline{D}(N)}\right)^h$$

with

$$\overline{f}(n) = \sum_{k=A(N)}^{B(N)} f_k(\epsilon_k(n)) = f(n) + \mathcal{O}\left((\log N)^\eta\right),$$

$$\overline{M}(N) = \sum_{k=A(N)}^{B(N)} \mu_k \quad \text{and} \quad \overline{D}(N)^2 = \sum_{k,k'=A(N)}^{B(N)} \sigma^{(2)}_{k,k'}$$

converge to the corresponding moments of the normal law. This implies

$$\frac{1}{N}\#\left\{n<N \,\middle|\, \frac{\overline{f}(n)-\overline{M}(N)}{\overline{D}(N)} < x\right\} \to \Phi(x),$$

and, again by the following lemma, (5).

Lemma 5 (cf. Lemma 5 in [8]). Suppose that $D(N)/(\log N)^\eta \to \infty$ for some $\eta > 0$. Then we have

$$\frac{1}{N}\#\left\{n<N \,\middle|\, \frac{f(n)-M(N)}{D(N)} < x\right\} \to \Phi(x)$$

for all $x \in \mathbb{R}$ if and only if

$$\frac{1}{N}\#\left\{n<N \,\middle|\, \frac{\overline{f}(n)-\overline{M}(N)}{\overline{D}(N)} < x\right\} \to \Phi(x)$$

for all $x \in \mathbb{R}$.

Furthermore, if for all $h \geq 0$

$$\frac{1}{N}\sum_{n<N}\left(\frac{\overline{f}(n)-\overline{M}(N)}{\overline{D}(N)}\right)^h \to \int_{-\infty}^{\infty} x^h\, d\Phi(x),$$

then we also have

$$\frac{1}{N}\sum_{n<N}\left(\frac{f(n)-M(N)}{D(N)}\right)^h \to \int_{-\infty}^{\infty} x^h\, d\Phi(x)$$

and conversely.

The proof is the same as that of Lemma 5 in [8].

For the convergence of the moments of $\overline{f}(n)$, we first prove a central limit theorem (with convergence of moments) for the exact Markov chain X_k and compare $\overline{f}(n)$ to $\sum_{k=A(N)}^{B(N)} f_k(X_k)$ afterwards.

Lemma 6. Suppose that there exists a constant $c > 0$ such that $\sigma_{k,k}^{(2)} \geq c$ for all $k \geq 0$. Then the sums of the random variables $f_k(X_k)$ satisfy a central limit theorem. More precisely,

$$\frac{\sum_{k=A(N)}^{B(N)} f_k(X_k) - \overline{M}(N)}{\overline{D}(N)} \Rightarrow \mathcal{N}(0,1)$$

and, for all $h \geq 0$, we have, as $N \to \infty$,

$$\mathbf{E}\left(\frac{\sum_{k=A(N)}^{B(N)} f_k(X_k) - \overline{M}(N)}{\overline{D}(N)}\right)^h \to \int_{-\infty}^{\infty} x^h \, d\Phi(x).$$

Proof. If all a_i are non-zero, then $\beta > 0$ (see the proof of Lemma 4) and the lemma can be proved with the help of Theorem 4 of Lifšic [18], as in [8]. If $\beta = 0$, we have to adapt this theorem.

An inspection of Lifšic' proof and Dobrušin's paper [5] (which is used by Lifšic) shows that we get the same result if we replace the ergodicity coefficient β by a constant $\theta > 0$ that satisfies

$$\gamma_j = \frac{1}{2} \sup_{m,b} \sum_{m',b'} \left|\mathbf{Pr}[X_k = (m',b')|X_{k+j} = (m,b)] - \mathbf{Pr}[X_k = (m',b')]\right| \leq (1-\theta)^j \tag{13}$$

for all $j \geq 1$ and

$$\mathbf{Var}\left(\sum_{k=s}^{s'} f_k(X_k)\right) \geq c(s'-s+1)\theta \tag{14}$$

for all $s, s' \geq 0$ with $s' - s \geq s_0$ for some constant s_0.

We have $\gamma_j > 0$ for all $j \geq 1$ since the sum in (13) is always less than 1 and the number of states (m,b) is finite. Dobrušin [5] proved $\gamma_j \leq 1 - \beta_j$, where

$$\beta_j = 1 - \sup_{m,b,m',b',\mathcal{A}} \left|\mathbf{Pr}[X_k \in \mathcal{A}|X_{k+j} = (m,b)] - \mathbf{Pr}[X_k \in \mathcal{A}|X_{k+j} = (m',b')]\right|.$$

For some j_0 with $1 < j_0 < d$, we have $\mathbf{Pr}[X_k = (\Lambda,1)|X_{k+j} = (m,b)] > 0$ for all possible (m,b) and all $j \geq j_0$, which implies $\beta_j > 0$ for all $j \geq j_0$. Set

$$\theta = \min\left(1 - \max_{1 \leq k < j_0} \gamma_k^{1/k}, 1 - \max_{j_0 \leq k < 2j_0} (1-\beta_k)^{1/k}, \frac{w}{c}\right).$$

Then (14) holds because of (12) and (13) holds for $j < 2j_0$ because of $\gamma_j \le 1 - \beta_j$. For $j \ge 2j_0$, we apply the inequality $1 - \beta_{i+j} \le (1 - \beta_i)(1 - \beta_j)$ (see [5]) and get, by induction on q,

$$1 - \beta_{qj_0+t} \le (1 - \beta_{j_0})(1 - \beta_{(q-1)j_0+t}) \le (1-\theta)^{j_0}(1-\theta)^{(q-1)j_0+t} = (1-\theta)^{qj_0+t}$$

for $q \ge 2$, $t < j_0$. Hence θ satisfies the required properties, we can apply the (adapted) theorem of Lifšic and the lemma is proved.

The next lemma concludes the proof of Theorem 2. In particular, for $h = 2$, the equation implies together with Lemma 5 and (3) the asymptotics for the variance (4).

Lemma 7 (cf. Lemma 7 in [8]). For every $h \ge 1$ and every $\lambda > 0$, we have

$$\frac{1}{N}\sum_{n<N}\left(\frac{\overline{f}(n) - \overline{M}(N)}{\overline{D}(N)}\right)^h = \mathbf{E}\left(\frac{\sum_{k=A(N)}^{B(N)} f_k(X_k) - \overline{M}(N)}{\overline{D}(N)}\right)^h + \mathcal{O}\left((\log N)^{-\lambda}\right).$$

The proof is the same as that of Lemma 7 in [8].

3. Plan of the Proof of Theorem 3

Set $\overline{f}$ as in Section 2 with $A(N^r)$ and $B(N^r)$ instead of $A(N)$ and $B(N)$. Then, similarly to Theorem 2, it is enough to prove

$$\frac{1}{N}\#\left\{n < N \,\middle|\, \frac{\overline{f}(P(n)) - \overline{M}(N^r)}{\overline{D}(N^r)} < x\right\} \to \Phi(x) \tag{15}$$

and

$$\frac{1}{\pi(N)}\#\left\{p < N \,\middle|\, \frac{\overline{f}(P(p)) - \overline{M}(N^r)}{\overline{D}(N^r)} < x\right\} \to \Phi(x). \tag{16}$$

We show

$$\frac{1}{N}\sum_{n<N}\left(\frac{\overline{f}(P(n)) - \overline{M}(N^r)}{\overline{D}(N^r)}\right)^h - \frac{1}{N^r}\sum_{n<N^r}\left(\frac{\overline{f}(n) - \overline{M}(N^r)}{\overline{D}(N^r)}\right)^h \to 0$$

and

$$\frac{1}{\pi(N)}\sum_{p<N}\left(\frac{\overline{f}(P(p)) - \overline{M}(N^r)}{\overline{D}(N^r)}\right)^h - \frac{1}{N^r}\sum_{n<N^r}\left(\frac{\overline{f}(n) - \overline{M}(N^r)}{\overline{D}(N^r)}\right)^h \to 0$$

as $N \to \infty$ by the following lemma. With the results of the previous section and the Fréchet-Shohat theorem, this proves (15), (16) and thus Theorem 3.

Lemma 8 (Main Lemma). Let $P(n)$ be an integer polynomial with degree $r \geq 1$ and positive leading term. Then, for every $h \geq 1$ and for every $\lambda > 0$, we have

$$\frac{1}{N}\#\{n < N \mid \epsilon_{k_1}(P(n)) = e_1, \ldots, \epsilon_{k_h}(P(n)) = e_h\} = q_{k_1,\ldots,k_h,e_1,\ldots,e_h} + \mathcal{O}\left((\log N)^{-\lambda}\right)$$

and

$$\frac{1}{\pi(N)}\#\{p < N \mid \epsilon_{k_1}(P(p)) = e_1, \ldots, \epsilon_{k_h}(P(p)) = e_h\} = q_{k_1,\ldots,k_h,e_1,\ldots,e_h} + \mathcal{O}\left((\log N)^{-\lambda}\right)$$

uniformly for all integers

$$(\log N^r)^\eta \leq k_1, k_2, \ldots, k_h \leq \log_\alpha N^r - (\log N^r)^\eta$$

and $e_1, e_2, \ldots, e_h \in \{0, 1, \ldots, a_1\}$. (The $q_{k_1,\ldots,k_h,e_1,\ldots,e_h}$ are as in Lemma 3.)

The proof of this lemma (Section 6) requires tilings of the torus and exponential sums which are treated in the next sections.

4. Tilings

For q-ary expansions we have

$$\epsilon_k(n) = e \Longleftrightarrow \left\{\frac{n}{q^{k+1}}\right\} \in \left[\frac{e}{q}, \frac{e+1}{q}\right),$$

where $\{x\}$ denotes the fractional part of x.

In order to get an analogue to this for our expansions, we need a tiling of the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, i.e. a family of sets $(\Omega_e)_{e\in\{0,\ldots,a_1\}}$ such that

- $\bigcup_{e=0}^{a_1} \Omega_e = \mathbb{T}^d$,
- each of the Ω_e is the closure of its interior,
- the intersection of two different Ω_e has Lebesgue measure zero,

and vectors $\mathbf{v}(n,k) \in \mathbb{T}^d$ such that we have (at least in most cases)

$$\epsilon_k(n) = e \Longleftrightarrow \mathbf{v}(n,k) \in \Omega_e.$$

Proposition 1. Let G be as in Theorem 3 and

$$\mathbf{v}(n,k) = \frac{n}{\alpha^k}\frac{\alpha-1}{\alpha^d-1}\left(\alpha^{d-1}, \ldots, \alpha, 1\right)^t \in \mathbb{T}^d.$$

Then we have a tiling $(\Omega_e)_{e\in\{0,\dots,a_1\}}$ of $\mathbb{T}^d$ with

$$d(\mathbf{v}(n,k),\Omega_{\epsilon_k(n)}) = \mathcal{O}\left(\alpha^{-k}\right) \text{ for all } k,n\in\mathbb{N}, \tag{17}$$

where $d(\mathbf{x},S) = \inf_{\mathbf{y}\in S}\|\mathbf{x}-\mathbf{y}\|_\infty$.

Remark. The distance $d(\mathbf{v}(n,k),\Omega_{\epsilon_k(n)})$ is positive and, equivalently, $\mathbf{v}(n,k)\notin\Omega_{\epsilon_k(n)}$ only for a small number of n and k (see Lemma 15). The error term $\mathcal{O}\left(\alpha^{-k}\right)$ is not surprising because we have $\frac{1}{G_j}\#\{n<G_j \mid \epsilon_k(n)=e\} = p_e + \mathcal{O}\left(\alpha^{-\min(k,j-k)}\right)$ for all $j>k$, whereas q-ary expansions satisfy exactly $\frac{1}{q^j}\#\{n<q^j \mid \epsilon_k(n)=e\} = \frac{1}{q}$. The Lebesgue measure of the tilings is $\lambda_d(\Omega_e)=p_e$.

Proof. We regard the linear map

$$\phi = \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_d \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathrm{GL}(d,\mathbb{Z})$$

with eigenvalues $\alpha,\alpha_2,\dots,\alpha_d$. Since α is a Pisot unit, ϕ is a hyperbolic toral automorphism and we have a ϕ-invariant decomposition of $\mathbb{R}^d$ into the unstable eigenspace $E_u = \mathbb{R}(\alpha^{d-1},\dots,\alpha,1)^t$ and the stable eigenspace E_s (of dimension $d-1$). Let $\mathbf{e}_u = \pi_u((1,0,\dots,0)^t)$ and $\mathbf{e}_s = \pi_s((1,0,\dots,0)^t)$ where $\pi_u:\mathbb{R}^d\to E_u$ is the projection along E_s to E_u and $\pi_s:\mathbb{R}^d\to E_s$ the projection along E_u to E_s. Set $\mathbf{e}_u = c_1'(\alpha^{d-1},\dots,\alpha,1)^t$.

Then the sequence $(G_j')_{j\ge0}$ defined by the linear recurrence

$$G_j' = a_1G_{j-1}' + \cdots + a_dG_{j-d}' \text{ for } j\ge d$$

with initial values $G_0'=0,\dots,G_{d-2}'=0, G_{d-1}'=1$ satisfies

$$G_j' = c_1'\alpha^j + c_2'\alpha_2^j + \cdots + c_d'\alpha_d^j$$

for some constants $c_2',\dots,c_d'$. By induction on j, the equation

$$G_j = G_j' + G_{j+1}' + \cdots + G_{j+d-1}'$$

can be easily proved. Since $G_j\to c_1\alpha^j$ and $G_j'\to c_1'\alpha^j$ for $j\to\infty$, we have

$$c_1 = c_1'(1+\alpha+\cdots+\alpha^{d-1}).$$

With

$$n = c_1\sum_{j=0}^{\infty}\epsilon_j(n)\alpha^j + \mathcal{O}\left(1\right),$$

we obtain

$$\mathbf{v}(n,k) = \frac{n}{c_1\alpha^k}\mathbf{e}_u = \sum_{j=0}^{\infty}\epsilon_j(n)\alpha^{j-k}\mathbf{e}_u + \mathcal{O}\left(\alpha^{-k}\right) = \sum_{j=0}^{\infty}\epsilon_j(n)\phi^{j-k}(\mathbf{e}_u) + \mathcal{O}\left(\alpha^{-k}\right).$$

Clearly we have

$$\phi^j(\mathbf{e}_u) + \phi^j(\mathbf{e}_s) = \phi^j((1,0,\dots,0)^t) \in \mathbb{Z}^d \text{ for all } j \geq 0$$

and hence

$$\mathbf{v}(n,k) \equiv \underbrace{\sum_{j=0}^{k-1}\epsilon_j(n)\phi^{j-k}(\mathbf{e}_u) - \sum_{j=k}^{\infty}\epsilon_j(n)\phi^{j-k}(\mathbf{e}_s)}_{\mathbf{v}'(n,k)} + \mathcal{O}\left(\alpha^{-k}\right) \bmod \mathbb{Z}^d. \tag{18}$$

Set

$$\Omega_e = \mathrm{Clos}\{\mathbf{v}'(n,k) : k,n \in \mathbb{N} \text{ with } \epsilon_k(n) = e\}.$$

Then we know by Praggastis [22] that $(\Omega_e)_{e\in\{0,\dots,a_1\}}$ is a tiling of $\mathbb{T}^d$ if $\mathrm{Fin}(\alpha) = \mathbb{Z}[\alpha]\cap\mathbb{R}_+$. Since α is an algebraic integer and a unit, we have $\mathbb{Z}[\alpha] = \mathbb{Z}[\alpha^{-1}]$. Thus (F) implies that $(\Omega_e)_{e\in\{0,\dots,a_1\}}$ is a tiling and (17) holds because of (18).

Example. Figure 1 shows the sets Ω_e for the Tribonacci expansion ($d = 3$, $a_1 = a_2 = a_3 = 1$). Ω_0 is the largest of the three prisms and Ω_1 is the union of the two smaller ones. $\pi_s(\Omega_0)$ is the Rauzy fractal (for details on the Rauzy fractal see Messaoudi [19, 20] and Rauzy [23] for the original work). Figure 2 illustrates how (Ω_0, Ω_1) tiles $\mathbb{R}^3$. These figures were drawn by Siegel, who obtained in [25] tilings for substitutions of Pisot type by different methods than Praggastis.

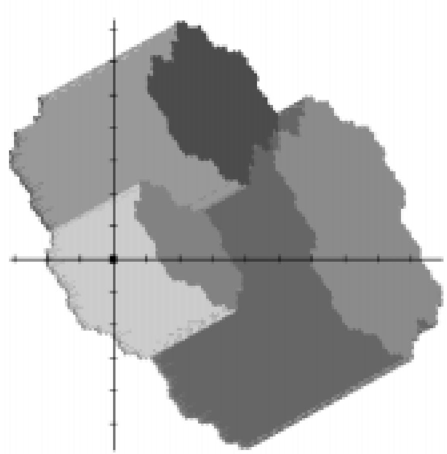

Figure 1.
Ω_0, Ω_1 for the Tribonacci expansion

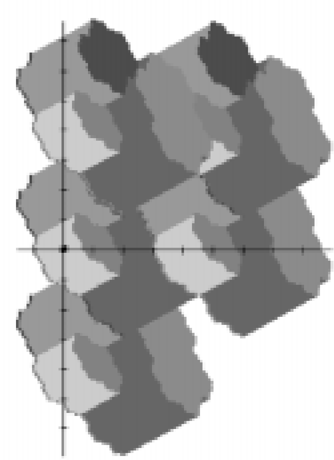

Figure 2.
Tiling of $\mathbb{R}^3$ for the Tribonacci expansion

Remark. Akiyama [2] obtained tilings of $\mathbb{R}^{d-1}$ for Pisot units α satisfying the condition

$$\mathbb{Z}[\alpha^{-1}] \cap \mathbb{R}^+ \subset \mathrm{Fin}(\alpha) - \mathrm{Fin}(\alpha) \cap [0, \varepsilon) \text{ for all } \varepsilon > 0, \tag{W}$$

which is weaker then (F) and conjectured to hold for all Pisot numbers α, but this improvement is not important for this work because we need (F) for the proof of Proposition 2 too.

In Section 6, we will need a covering of Ω_e and its boundary by convex sets. Since the boundary of Ω_e has fractal structure for $d > 2$, we approximate it by parallelepipeds.

Each Ω_e is the union of sets

$$\Omega_{e_0,\ldots,e_{d-2}} = \mathrm{Clos}\left\{\mathbf{v}'(n,k) : k, n \in \mathbb{N} \text{ with } (\epsilon_k(n), \ldots, \epsilon_{k+d-2}(n)) = (e_0, \ldots, e_{d-2})\right\}$$

(with $e_0 = e$) which are prisms:

$$\Omega_{e_0,\ldots,e_{d-2}} = \pi_s(\Omega_{e_0,\ldots,e_{d-2}}) \oplus [0, \sup_{k,n \text{ as above}} \sum_{j=0}^{k-1} \epsilon_j(n)\alpha^{j-k}]\mathbf{e}_u.$$

Therefore we study the boundary of $\pi_s(\Omega_{e_0,\ldots,e_{d-2}})$.

The problem of determining all points on the boundary is equivalent to determining all points with more than one ϕ-representation, which can be done with the help of a finite automaton. This method is adapted from Messaoudi [20] who examined the Rauzy fractal. Siegel [25] studied similar problems with similar automata.

Let $\mathcal{N}$ be the set of sequences $(b_j)_{j\in\mathbb{Z}}$ with

$$(b_j, b_{j-1}, \ldots, b_{j-d+1}) < (a_1, a_2, \ldots, a_d) \text{ for all } j \in \mathbb{Z}$$

and an integer K such that $b_j = 0$ for $j \geq K$. Let $\mathcal{N}_f$ be the set of sequences $(b_j)_{j\in\mathbb{Z}} \in \mathcal{N}$ with an integer J so that $b_j = 0$ for $j \leq J$. With $\mathcal{E} = \left\{\sum_{j=1}^{\infty} \epsilon_j \phi^j(\mathbf{e}_s) : (\epsilon_j)_{j\geq 1} \in \mathcal{N}_f\right\}$, we get the following proposition.

Proposition 2 (cf. Théorème 1 in Messaoudi [20]). Let $\mathbf{x} = \sum_{j=-L}^{\infty} b_j \phi^j(\mathbf{e}_s)$ and $\mathbf{y} = \sum_{j=-L}^{\infty} b'_j \phi^j(\mathbf{e}_s)$, where $(b_j)_{j\geq -L} \in \mathcal{N}$ and $(b'_j)_{j\geq -L} \in \mathcal{N}$, then $\mathbf{x} = \mathbf{y}$ if and only if we have, for all $i \geq -L$,

$$\mathbf{x}_i - \mathbf{y}_i \in \mathcal{S}$$

where $\mathbf{x}_i = \phi^{-i}\left(\sum_{j=-L}^{i} b_j \phi^j(\mathbf{e}_s)\right)$, $\mathbf{y}_i = \phi^{-i}\left(\sum_{j=-L}^{i} b'_j \phi^j(\mathbf{e}_s)\right)$ and

$$\mathcal{S} = \left\{ \pm \sum_{j=-s}^{0} \epsilon_j \phi^j(\mathbf{e}_s) \,\middle|\, (\epsilon_j)_{-s\leq j\leq 0} \in \mathcal{N}_f, \mathcal{E} \cap \left(\mathcal{E} \pm \sum_{j=-s}^{0} \epsilon_j \phi^j(\mathbf{e}_s)\right) \neq \emptyset \right\}.$$

for some (fixed) integer s.

For the proof of Proposition 2, two small lemmata are used.

Lemma 9. For all integers $j \geq d-1$, we have

$$\alpha^j = \alpha^{d-1}G'_j + \alpha^{d-2}(a_2G'_{j-1} + a_3G'_{j-2} + \cdots + a_dG'_{j-d+1}) \\ + \cdots + \alpha(a_{d-1}G'_{j-1} + a_dG'_{j-2}) + a_dG'_{j-1}, \quad (19)$$

where the sequence $(G'_j)_{j\geq 0}$ is defined in the proof of Proposition 1.

Proof. Induction on j.

Lemma 10. Define the linear map

$$\kappa : \left\{ \pm \sum_{j=-\infty}^{\infty} \epsilon_j \phi^j(\mathbf{e}_s) \middle| (\epsilon_j)_{j\in\mathbb{Z}} \in \mathcal{N}_f \right\} \to \pm\mathrm{Fin}(\alpha)$$

by $\kappa(\phi^j(\mathbf{e}_s)) = \alpha^j$ for all $j \in \mathbb{Z}$. Then κ is well defined and a bijection.

Proof. We have to show that all elements on the left side are distinct. Suppose that two representations $\varepsilon \sum_{j=-\infty}^{\infty} \epsilon_j \phi^j(\mathbf{e}_s)$ and $\varepsilon' \sum_{j=-\infty}^{\infty} \epsilon'_j \phi^j(\mathbf{e}_s)$ with $(\epsilon_j)_{j\in\mathbb{Z}}, (\epsilon'_j)_{j\in\mathbb{Z}} \in \mathcal{N}_f$ and $\varepsilon, \varepsilon' \in \{\pm 1\}$ represent the same vector. Hence we have $Q(\phi)(\mathbf{e}_s) = \mathbf{0}$ for some polynomial $Q = q_m x^m + \cdots + q_1 x + q_0 \not\equiv 0$.

The proof of Proposition 1 shows $\phi^j(\mathbf{e}_s) = \sum_{i=2}^{d} c'_i \alpha_i^j (\alpha_i^{d-1}, \ldots, \alpha_i, 1)^t$. Hence $\sum_{j=0}^{m} q_j \sum_{i=2}^{d} c'_i \alpha_i^j (\alpha_i^{d-1}, \ldots, \alpha_i, 1)^t = \mathbf{0}$. By easy calculations (solution of a linear equation system), we obtain $c'_i = \left(\prod_{k\neq i}(\alpha_i - \alpha_k)\right)^{-1} \neq 0$. If $\alpha_i \in \mathbb{R}$ for all $i = 2, \ldots, d$, then the $(\alpha_i^{d-1}, \ldots, \alpha_i, 1)^t$ are linearly independent vectors of $\mathbb{R}^d$ and we must have $Q(\alpha_i) = 0$ for all $i = 2, \ldots, d$. For $\alpha_i \notin \mathbb{R}$, we obtain $Q(\alpha_i) = 0$ similarly.

This implies $Q(\alpha) = 0$ and $\varepsilon \sum_{j=-\infty}^{\infty} \epsilon_j \alpha^j = \varepsilon' \sum_{j=-\infty}^{\infty} \epsilon'_j \alpha^j$. Therefore $\varepsilon = \varepsilon'$ and, since finite α-representations are unique, $(\epsilon_j)_{j\in\mathbb{Z}} = (\epsilon'_j)_{j\in\mathbb{Z}}$.

Thus, κ is well defined and clearly bijective.

Proof of Proposition 2. Since $\phi|_{E_s}$ is contracting, we have

$$\mathbf{x} - \mathbf{y} = \lim_{i\to\infty} \phi^{i-d+1}(\mathbf{x}_i - \mathbf{y}_i) = \mathbf{0},$$

if $\mathbf{x}_i - \mathbf{y}_i \in \mathcal{S}$.

Now, suppose $\mathbf{x} = \mathbf{y}$. Hence $\phi^{-i}(\mathbf{x}) = \phi^{-i}(\mathbf{y})$ and

$$\mathbf{x}_i - \mathbf{y}_i = \sum_{j=i+1}^{\infty} (b'_j - b_j)\phi^{j-i}(\mathbf{e}_s) = \sum_{j=1}^{\infty} (b'_{j+i} - b_{j+i})\phi^j(\mathbf{e}_s).$$

On the other hand, we have

$$\mathbf{x}_i - \mathbf{y}_i = \phi^{-i}\left(\sum_{j=-L}^{i} (b_j - b'_j)\phi^j(\mathbf{e}_s)\right) = \phi^{-L-i-d+1}\left(\sum_{j=d-1}^{L+i+d-1} g_j \phi^j(\mathbf{e}_s)\right),$$

where $g_j = b_{j-L-d+1} - b'_{j-L-d+1}$. We apply κ and get, by (19),

$$\kappa(\mathbf{x}_i - \mathbf{y}_i) = \alpha^{-L-i-d+1}\left(g'_{d-1}\alpha^{d-1} + \cdots + g'_1\alpha + g'_0\right)$$

with integers g'_j which are easily seen to be all positive if

$$(b_i, b_{i-1}, \ldots, b_{-L}) > (b'_i, b'_{i-1}, \ldots, b'_{-L})$$

and all negative for "<". Hence we have $\kappa(\mathbf{x}_i - \mathbf{y}_i) \in \mathbb{Z}_+[\alpha^{-1}]$ and $\kappa(\mathbf{x}_i - \mathbf{y}_i) \in \mathbb{Z}_-[\alpha^{-1}]$ respectively. (F) implies

$$\kappa(\mathbf{x}_i - \mathbf{y}_i) = \pm \sum_{j=-s}^{m} \epsilon_j \alpha^j \text{ with } (\epsilon_j)_{-s \le j \le m} \in \mathcal{N}_f. \tag{20}$$

Assume, w.l.o.g., $\kappa(\mathbf{x}_i) = \kappa(\mathbf{y}_i) + \sum_{j=-s}^{m} \epsilon_j \alpha^j$. Again, (F) implies

$$\kappa(\mathbf{x}_i) = \sum_{j=-s'}^{m'} \epsilon'_j \alpha^j \text{ with } (\epsilon'_j)_{-s' \le j \le m'} \in \mathcal{N}_f \text{ and } m' \ge m.$$

Since $\kappa(\mathbf{x}_i) = \sum_{j=-L}^{i} b_j \alpha^{j-i}$ and finite α-expansions are unique, we have $m' = 0$ and thus $m \le 0$.

By applying κ^{-1} to (20), we obtain

$$\sum_{j=1}^{\infty} (b'_{j+i} - b_{j+i}) \phi^j(\mathbf{e}_s) = \pm \sum_{j=-s}^{0} \epsilon_j \phi^j(\mathbf{e}_s)$$

and

$$\sum_{j=1}^{\infty} b_{j+i} \phi^j(\mathbf{e}_s) \in \mathcal{E} \cap \left(\mathcal{E} \pm \sum_{j=-s}^{0} \epsilon_j \phi^j(\mathbf{e}_s)\right).$$

Lemma 2.10 of Praggastis [22] shows that there is an integer s such that $\left(\mathcal{E} \pm \sum_{j=-\infty}^{0} \epsilon_j \phi^j(\mathbf{e}_s)\right) = \emptyset$ if $\epsilon_j \neq 0$ for some $j < -s$. This concludes the proof of the proposition.

If we set $\mathbf{z}_i = \mathbf{x}_i - \mathbf{y}_i$, then

$$\mathbf{z}_{i+1} = \phi^{-1}(\mathbf{z}_i) + (b_{i+1} - b'_{i+1})\mathbf{e}_s.$$

Therefore the points with two representations are determined by a finite automaton, the states of which are the elements of $\mathcal{S}$ and two states $\mathbf{z}, \mathbf{z}'$ are connected by an edge labeled by (b, b') if $\mathbf{z}' = \phi^{-1}(\mathbf{z}) + (b - b')\mathbf{e}_s$ or, equivalently, $\kappa(\mathbf{z}') = \kappa(\mathbf{z})/\alpha + b - b'$. (The starting point is $\mathbf{0}$.)

As Gilbert [14] for the twin dragon, we obtain a ν-th approximation to the boundary by determining all paths of length ν in the automaton and drawing for each such path p a parallelepiped that contains the image of all paths which start with p. This is the idea of the following lemma.

Lemma 11. For all $\nu \in \mathbb{N}$ and $e = 0, \ldots, a_1$, the boundary of Ω_e is contained in sets $U_{e,\nu}$ which are the union of $\mathcal{O}(\gamma^\nu)$ parallelepipeds of size $c\alpha^{-\nu}$ for some constants $\gamma < \alpha$ and c, with edges parallel to $\mathbf{a}_1, \ldots, \mathbf{a}_d$, where $\mathbf{a}_i = (\alpha_i^{d-1}, \ldots, \alpha_i, 1)^t$ for the real eigenvalues α_i ($\alpha_1 = \alpha$) and $\mathbf{a}_i = (\Re\alpha_i^{d-1}, \ldots, \Re\alpha_i, 1)^t$, $\mathbf{a}_{i+1} = (\Im\alpha_i^{d-1}, \ldots, \Im\alpha_i, 0)^t$ for the pairs of complex eigenvalues $(\alpha_i, \alpha_{i+1} = \overline{\alpha_i})$.

Proof. A point can be on the boundary of Ω_e if its π_s-image has at least two ϕ-representations $\sum_{j=0}^\infty b_j\phi^j(\mathbf{e}_s) = \sum_{j=-L}^\infty b'_j\phi^j(\mathbf{e}_s)$ with $(b_0, \ldots, b_{d-2}) \neq (b'_0, \ldots, b'_{d-2})$, $b_0 = e$ and j_0 the smallest integer with $b_{j_0} \neq b'_{j_0}$. Denote by B_ν the number of different initial sequences $(b_0, \ldots, b_\nu)$ of points on the boundary. We show that these sequences cannot have $2s+2$ subsequent zeros.

Suppose on the contrary that $(b_{j_1+1}, \ldots, b_{j_1+2s+2}) = (0, \ldots, 0)$ for some $j_1 \geq j_0$ and set $\mathbf{z}_i = \sum_{j=j_0}^i (b_j - b'_j)\phi^{j-i}(\mathbf{e}_s)$. We have $\mathbf{z}_{j_0} \neq \mathbf{0}$ by definition and $\mathbf{z}_i \neq \mathbf{0}$ for all $i > j_0$, because $\kappa(\mathbf{z}_i) = 0$ would imply that two different finite α-expansions are equal.

Assume $\kappa(\mathbf{z}_{j_1}) < 0$. Then we have $\kappa(\mathbf{z}_i) < 0$ for all $i = j_1+1, \ldots, j_1+2s+2$ too and the uniqueness of finite α-expansions implies $\mathbf{z}_i \notin \mathcal{S}$ for some $i \in \{j_1+1, \ldots, j_1+s+1\}$, which contradicts Proposition 2. If $\kappa(\mathbf{z}_{j_1}) > 0$, then we get either $\kappa(\mathbf{z}_i) < 0$ or $\mathbf{z}_i \notin \mathcal{S}$ for some $i \in \{j_1+1, \ldots, j_1+s+1\}$ and, as above, $\kappa(\mathbf{z}_i) < 0$ implies $\mathbf{z}_{i'} \notin \mathcal{S}$ for some $i' \leq i+s+1$.

Therefore $2s+2$ subsequent zeros are not possible and $B_\nu = \mathcal{O}(\gamma^\nu)$ for some $\gamma < \alpha$.

The $\mathbf{a}_i$ are the real eigenvectors of ϕ and the real and imaginary parts of the complex eigenvectors respectively. Let c be the size of the parallelepiped that covers $\mathcal{E}$ and all its images of rotations in the planes spanned by the complex eigenvectors. ($\mathcal{E}$ is a bounded set.) Then all points on the boundary with same initial sequence $(b_0, \ldots, b_\nu)$ are covered by a parallelepiped of size $c|\alpha_2|^\nu \ldots |\alpha_d|^\nu = c\alpha^{-\nu}$ and we have B_ν of these parallelepipeds. This concludes the proof.

5. Exponential sums

In order to prove Lemma 8, we have to study exponential sums of the form

$$\frac{1}{N}\sum_{n<N} e\left(S\frac{\alpha-1}{\alpha^d-1}P(n)\right),$$

with

$$\begin{aligned} S &= \frac{m_1^{(1)}\alpha^{d-1}+\cdots+m_d^{(1)}}{\alpha^{k_1}}+\cdots+\frac{m_1^{(h)}\alpha^{d-1}+\cdots+m_d^{(h)}}{\alpha^{k_h}} \\ &= \frac{\alpha^{k_h-k_1+d-1}m_1^{(1)}+\cdots+m_d^{(1)}\alpha^{k_h-k_1}+\cdots+m_1^{(h)}\alpha^{d-1}+\cdots+m_d^{(h)}}{\alpha^{k_h}}. \end{aligned} \tag{21}$$

We use the following well known lemma in order to get bounds for S.

Lemma 12. Suppose $1, \beta_1, \beta_2, \ldots, \beta_v$ are linearly independent over $\mathbb{Q}$, and they generate an algebraic number field of degree d. Then

$$|\beta_1 q_1 + \cdots + \beta_v q_v - p| > cq^{-d+1}$$

for arbitrary integers $q_1, \ldots, q_v, p$ having $q = \max(|q_1|, \ldots, |q_v|) > 0$ and some constant c.

Lemma 13. Let $|m_j^{(i)}| \le (\log N)^{2\delta}$ for all i, j,

$$(\log N^r)^\eta \le k_1 < k_2 < \cdots < k_h \le \log_\alpha N^r - (\log N^r)^\eta$$

and arbitrary constants $\delta > 0, \eta > 0$. Then S defined by (21) satisfies

$$S = 0 \quad \text{or} \quad \frac{\alpha^{(\log N)^{\eta'}}}{N^r} \ll |S| \ll \frac{1}{\alpha^{(\log N)^{\eta'}}}$$

for all $\eta' < \eta$.

Proof. Assume $S \ne 0$. Because of Lemma 9, we have

$$S = \frac{\hat{m}_1\alpha^{d-1} + \cdots + \hat{m}_{d-1}\alpha + \hat{m}_d}{\alpha^{k_h}}$$

with integers $\hat{m}_j$ which satisfy

$$|\hat{m}_j| \ll (\log N)^{2\delta}\alpha^{k_h-k_1} \quad (1 \le j \le d).$$

Therefore we have

$$|S| \ll \frac{(\log N)^{2\delta}\alpha^{k_h-k_1}}{\alpha^{k_h}} \le \frac{(\log N)^{2\delta}}{\alpha^{(\log N)^\eta}} \ll \frac{1}{\alpha^{(\log N)^{\eta'}}}.$$

To obtain the lower bound, start by setting $\varepsilon = \eta/h$. Then there exists an integer K, $0 \le K \le h-1$, such that

$$k_{i+1} - k_i \notin \left[(\log N)^{K\varepsilon}, (\log N)^{(K+1)\varepsilon}\right)$$

for all i. Fix K with this property.

If $k_{i+1} - k_i \leq (\log N)^{K\varepsilon}$ for all j, we apply Lemma 12 and get

$$|S| \gg \frac{1}{\max\limits_{j\in\{1,\dots,d\}} |\hat{m}_j|^{d-1}\alpha^{k_h}} \gg \frac{1}{(\log N)^{2(d-1)\delta}\alpha^{k_h+(d-1)(h-1)(\log N)^{K\varepsilon}}}$$
$$\gg \frac{\alpha^{(\log N)^{\eta}-(d-1)(h-1)(\log N)^{\frac{h\eta}{h+1}}}}{N^r(\log N)^{2(d-1)\delta}} \gg \frac{\alpha^{(\log N)^{\eta'}}}{N^r}.$$

Otherwise, there is an $i < h$ such that $k_{i+1} - k_i \geq (\log N)^{(K+1)\varepsilon}$ and $k_i - k_1 \leq (i-1)(\log N)^{K\varepsilon}$. Then split up the sum into two terms

$$\begin{aligned} S &= \frac{\alpha^{k_i-k_1+d-1}m_1^{(1)} + \cdots + m_d^{(1)}\alpha^{k_i-k_1} + \cdots + m_1^{(i)}\alpha^{d-1} + \cdots + m_d^{(i)}}{\alpha^{k_i}} \\ &\quad + \frac{\alpha^{k_h-k_{i+1}+d-1}m_1^{(i+1)} + \cdots + m_d^{(i+1)}\alpha^{k_h-k_{i+1}} + \cdots + m_1^{(h)}\alpha^{d-1} + \cdots + m_d^{(h)}}{\alpha^{k_h}} \\ &= S_1 + S_2. \end{aligned}$$

If $S_1 = 0$, then $S = S_2$ and we are concerned with a problem containing less terms. By using induction on h (which is not made explicit here), we may assume that this case has already been treated. Otherwise, we have

$$|S_1| \gg \frac{1}{(\log N)^{2(d-1)\delta}\alpha^{k_i+(d-1)(i-1)(\log N)^{K\varepsilon}}},$$

whereas

$$|S_2| \ll \frac{(\log N)^{2\delta}\alpha^{k_h-k_{i+1}}}{\alpha^{k_h}} \leq \frac{(\log N)^{2\delta}}{\alpha^{k_i+(\log N)^{(K+1)\varepsilon}}}.$$

Hence

$$|S| \gg \frac{\alpha^{(\log N)^{\eta}-(i-1)(d-1)(\log N)^{K\varepsilon}}}{N^r(\log N)^{2\delta}} \gg \frac{\alpha^{(\log N)^{\eta'}}}{N^r}$$

and the lemma is proved.

The next lemma on exponential sums, which can also be found in [8], contains adapted versions of results due to Hua and Vinogradov.

Lemma 14 (cf. Lemma 6.2 and Theorem 10 in Hua [17]). *Let $P(n)$ be a polynomial of degree r with leading coefficient β. For every $\tau_0 > 0$, we have a $\tau > 0$ such that*

$$N^{-r}(\log N)^{\tau} < \beta < (\log N)^{-\tau}$$

implies

$$\frac{1}{N}\sum_{n<N} e(P(n)) = \mathcal{O}\left((\log N)^{-\tau_0}\right)$$
$$\frac{1}{\pi(N)}\sum_{p<N} e(P(p)) = \mathcal{O}\left((\log N)^{-\tau_0}\right)$$

as $N \to \infty$.

6. Proof of the Main Lemma

Denote by $U_{e,\nu}$ the union of parallelepipeds of Lemma 11 containing the boundary of Ω_e. Let $\mathbf{1}_{\Omega_e \cup U_{e,\nu}}$ the characteristic function of $\Omega_e \cup U_{e,\nu}$ on the torus $\mathbb{T}^d$ and $\sum_{\mathbf{m}\in\mathbb{Z}^d} c_{\mathbf{m},e,\nu} e(\mathbf{m}\cdot\mathbf{x})$ its Fourier expansion. Clearly we have $c_{\mathbf{0},e,\nu} = \lambda_d(\Omega_e \cup U_{e,\nu})$. In order to calculate the other $c_{\mathbf{m},e,\nu}$, split up $\Omega_e \cup U_{e,\nu}$ into parallelepipeds with edges parallel to $\mathbf{a}_1, \ldots, \mathbf{a}_d$. The $\mathbf{m}$-th Fourier coefficient of such a parallelepiped is, by Lemma 1 of Drmota [6],

$$\sum_{\mathbf{x}\in V} \frac{\left|\det(\mathbf{x}-\mathbf{y})_{\mathbf{y}\in\Gamma(\mathbf{x})}\right|}{\prod_{\mathbf{y}\in\Gamma(\mathbf{x})}(-2\pi i)\mathbf{m}\cdot(\mathbf{x}-\mathbf{y})} e(-\mathbf{m}\cdot\mathbf{x}) = \sum_{\mathbf{x}\in V} \frac{\left|\det(\pm\mathbf{a}_j)_{1\le j\le d}\right|}{\prod_{j=1}^{d}(-2\pi i)\mathbf{m}\cdot(\pm\mathbf{a}_j)} e(-\mathbf{m}\cdot\mathbf{x}),$$

where V denotes the set of vertices of the parallelepiped and $\Gamma(\mathbf{x})$ the set of vertices adjacent to $\mathbf{x}$. As in Gittenberger and Thuswaldner [15], the contributions of the inner parallelepipeds cancel out and only the $\mathcal{O}(\gamma^\nu)$ corners of the boundary of $\Omega_e \cup U_{e,\nu}$ play a role. By Lemma 2 of Drmota [6], the contribution of a corner can be bounded by

$$\left|\frac{\left|\det(\pm\mathbf{a}_j)_{1\le j\le d}\right|}{\prod_{j=1}^{d}(-2\pi i)\mathbf{m}\cdot(\pm\mathbf{a}_j)}\right| \ll \prod_{j=1}^{d} \frac{1}{(1+|\mathbf{m}\cdot\mathbf{a}_j|)^2}$$

uniformly for all $\mathbf{m}$. Hence we define $\tilde{m}_j = \mathbf{m}\cdot\mathbf{a}_j$ and obtain

$$|c_{\mathbf{m},e,\nu}| \ll \gamma^\nu \prod_{j=1}^{d} \min\left(1, \frac{1}{|\tilde{m}_j|}\right).$$

Now, consider the function

$$\psi_{e,\nu,\Delta}(\mathbf{x}) = \frac{1}{\Delta^d} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \cdots \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \mathbf{1}_{\Omega_e\cup U_{e,\nu}}(\mathbf{x}+z_1\mathbf{a}_1+\cdots+z_d\mathbf{a}_d)\,dz_1\ldots dz_d.$$

By enlarging the parallelepipeds of $U_{e,\nu}$, we obtain sets $Q_{e,\nu}$ which are again unions of $\mathcal{O}(\gamma^\nu)$ parallelepipeds with $\lambda_d(Q_{e,\nu}) = \mathcal{O}\left(\left(\frac{\gamma}{\alpha}\right)^\nu\right)$ such that

$$\psi_{e,\nu,\Delta}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega_e \setminus Q_{e,\nu} \\ 0 & \text{if } \mathbf{x} \notin \Omega_e \cup Q_{e,\nu} \end{cases}$$

for $\Delta < \alpha^{-\nu}$. For the Fourier expansion $\psi_{e,\nu,\Delta}(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^d} d_{\mathbf{m},e,\nu,\Delta} e(\mathbf{m}\cdot\mathbf{x})$, we get

$$d_{\mathbf{m},e,\nu,\Delta} = c_{\mathbf{m},e,\nu} \prod_{j=1}^{d} \frac{e\left(\frac{\tilde{m}_j\Delta}{2}\right) - e\left(-\frac{\tilde{m}_j\Delta}{2}\right)}{-2\pi i \tilde{m}_j \Delta}, \tag{22}$$

$$|d_{\mathbf{m},e,\nu,\Delta}| \ll \gamma^\nu \prod_{j=1}^{d} \min\left(1, \frac{1}{|\tilde{m}_j|}, \frac{1}{\Delta\tilde{m}_j^2}\right).$$

If we set
$$t(n) = \psi_{e_1,\nu,\Delta}(\mathbf{v}(n,k_1)) \dots \psi_{e_h,\nu,\Delta}(\mathbf{v}(n,k_h)),$$
then $t(n) = 1$ if $\mathbf{v}(n,k_i) \in \Omega_{e_i} \backslash Q_{e_i,\nu}$ for all $i = 1, \dots, h$ and $t(n) = 0$ if $\mathbf{v}(n,k_i) \notin \Omega_{e_i} \cup Q_{e_i,\nu}$ for some i. Therefore we estimate the number of integers with $\mathbf{v}(n,k_i) \in Q_{e_i,\nu}$ by the following lemma.

Lemma 15. Let
$$E_{k,e,\nu} = \# \{ n < N \,| \mathbf{v}(P(n),k) \in Q_{e,\nu} \} \,, \; F_{k,e,\nu} = \# \{ p < N \,| \mathbf{v}(P(p),k) \in Q_{e,\nu} \}$$
and λ an arbitrary positive constant. Then, uniformly in k for $(\log N^r)^\eta \le k \le \log_\alpha N^r - (\log N^r)^\eta$, we have
$$E_{k,e,\nu} \ll \left(\frac{\gamma}{\alpha}\right)^\nu N + N(\log N)^{-\lambda}, \; F_{k,e,\nu} \ll \left(\frac{\gamma}{\alpha}\right)^\nu \pi(N) + N(\log N)^{-\lambda}.$$

The lemma can be proved similarly to Lemma 13 in [8] by the help of the isotropic discrepancy and the Erdős-Turán-Koksma inequality (see e.g. Drmota and Tichy [9]).

Define
$$\Sigma_1 = \#\{ n < N \mid \epsilon_{k_1}(P(n)) = e_1, \dots, \epsilon_{k_h}(P(n)) = e_h \},$$
$$\Sigma_2 = \#\{ n < N \mid \epsilon_{k_1}(P(n)) = e_1, \dots, \epsilon_{k_h}(P(n)) = e_h \}.$$
Since, for $\nu \ll \log\log N$ and $(\log N^r)^\eta \le k \le \log_\alpha N^r - (\log N^r)^\eta$, the error term $\mathcal{O}\left(\alpha^{-k}\right)$ of Proposition 1 is negligible compared to the size of each parallelepiped in $Q_{e,\nu}$, we have
$$\left| \Sigma_1 - \sum_{n<N} t(P(n)) \right| \le E_{k_1,e_1,\nu} + \dots + E_{k_h,e_h,\nu},$$
$$\left| \Sigma_2 - \sum_{n<N} t(P(p)) \right| \le F_{k_1,e_1,\nu} + \dots + F_{k_h,e_h,\nu}.$$
We will consider only Σ_1, since Σ_2 can be treated in the same way.

Let $\mathcal{M}$ be the set of vectors $\mathbf{M} = (\mathbf{m}_1, \dots, \mathbf{m}_h)$ with integer vectors $\mathbf{m}_i = (m_1^{(i)}, \dots, m_d^{(i)})$. Then we have
$$\sum_{n<N} t(P(n)) = \sum_{\mathbf{M}\in\mathcal{M}} T_{\mathbf{M},\nu,\Delta} \sum_{n<N} e\Big(\big(\mathbf{m}_1 \cdot \mathbf{v}(1,k_1) + \dots + \mathbf{m}_h \cdot \mathbf{v}(1,k_h)\big) P(n)\Big),$$
with
$$T_{\mathbf{M},\nu,\Delta} = d_{\mathbf{m}_1,e_1,\nu,\Delta} \cdots d_{\mathbf{m}_h,e_h,\nu,\Delta}.$$

Since the $\mathbf{a}_i$ form a basis of $\mathbb{R}^d$, we have
$$\frac{1}{|m_1|} \cdots \frac{1}{|m_d|} \ll \frac{1}{|\mathbf{a}_1 \cdot \mathbf{m}|} \cdots \frac{1}{|\mathbf{a}_d \cdot \mathbf{m}|} \ll \frac{1}{|m_1|} \cdots \frac{1}{|m_d|}$$

and therefore

$$\begin{aligned}\sum_{\mathbf{M}\in\mathcal{M}} |T_{\mathbf{M},\nu,\Delta}| &\ll \left(\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_d=-\infty}^{\infty}\gamma^\nu\prod_{j=1}^{d}\min\left(1,\frac{1}{|\tilde{m}_j|},\frac{1}{\tilde{m}_j^2\Delta}\right)\right)^h\\ &\ll \left(\sum_{m_1=-\infty}^{\infty}\cdots\sum_{m_d=-\infty}^{\infty}\gamma^\nu\prod_{j=1}^{d}\min\left(1,\frac{1}{|m_j|},\frac{1}{m_j^2\Delta}\right)\right)^h\\ &\ll \gamma^{h\nu}\left(\log\frac{1}{\Delta}\right)^{dh}\end{aligned}$$

If $|m_j^{(i)}| > (\log N)^{2\delta}$ for some i, j, then

$$\begin{aligned}\sum_{\exists i,j \text{ with } |m_j^{(i)}|>(\log N)^{2\delta}} |T_{\mathbf{M},\nu,\Delta}| &\ll \gamma^{h\nu}\left(\sum_{m=[(\log N)^{2\delta}]}^{\infty}\frac{1}{m^2\Delta}\right)\left(\sum_{m=1}^{\infty}\min\left(\frac{1}{|m|},\frac{1}{m^2\Delta}\right)\right)^{dh-1}\\ &\ll \gamma^{h\nu}\frac{\left(\log(\log N)^\delta\right)^{dh-1}}{(\log N)^\delta},\end{aligned}$$

if we set $\Delta = (\log N)^{-\delta}$. For $\mathbf{M}$ with $|m_j^{(i)}| \le (\log N)^{2\delta}$ for all i, j, the exponential sums can be estimated by Lemma 14 with the help of Lemma 13 if $\sum_{i=1}^h \mathbf{m}_i\cdot\mathbf{v}(1,k_i)\neq 0$.

Hence we have

$$\Sigma_1 = \sum_{\mathbf{M}\in\mathcal{M}:\sum\mathbf{m}_i\cdot\mathbf{v}(1,k_i)=0} T_{\mathbf{M},\nu,\Delta} + \mathcal{O}\left(\gamma^{h\nu}N(\log N)^{-\tau_0} + \gamma^{h\nu}N(\log N)^{-\delta/2} + N\left(\frac{\gamma}{\alpha}\right)^\nu\right).$$

Set

$$T'_{\mathbf{M},\nu} = c_{\mathbf{m}_1,e_1,\nu}\cdots c_{\mathbf{m}_h,e_h,\nu}$$

and compare $T_{\mathbf{M},\nu,\Delta}$ to $T'_{\mathbf{M},\nu}$. (22) implies

$$T_{\mathbf{M},\nu,\Delta} = T'_{\mathbf{M},\nu} + \mathcal{O}\left(\gamma^\nu\max_{i,j}\left|\tilde{m}_j^{(i)}\right|\Delta\right)$$

and

$$\sum_{\mathbf{M}\in\mathcal{M}:|\tilde{m}_j^{(i)}|<(\log N)^{\frac{\delta}{2dh}} \text{ for all } i,j} \left|T_{\mathbf{M},\nu,\Delta} - T'_{\mathbf{M},\nu}\right| \ll \gamma^\nu(\log N)^{-\delta/3}.$$

For the other $\mathbf{M}$ which satisfy $\sum_{i=1}^h \mathbf{m}_i\cdot\mathbf{v}(1,k_i) = 0$, we obtain by the same methods as in Lemma 14 in [8],

$$\sum_{\mathbf{M}\in\mathcal{M}:\sum\mathbf{m}_i\cdot\mathbf{v}(1,k_i)=0,|\tilde{m}_j^{(i)}|\ge(\log N)^{\frac{\delta}{2dh}} \text{ for some } i,j} T'_{\mathbf{M},\nu} \ll (\log N)^{-\frac{\delta}{2dh(dh-1)}}.$$

If we set

$$q'_{k_1,\ldots,k_h,e_1,\ldots,e_h,\nu} = \sum_{\mathbf{M}\in\mathcal{M}:\sum \mathbf{m}_i\cdot\mathbf{v}(1,k_i)=0} T'_{\mathbf{M},\nu},$$

we get

$$\Sigma_1 = Nq'_{k_1,\ldots,k_h,e_1,\ldots,e_h,\nu} + \mathcal{O}\left(\gamma^\nu N(\log N)^{-\frac{\delta}{2dh(dh-1)}}\right) + \mathcal{O}\left(N\left(\frac{\gamma}{\alpha}\right)^\nu\right)$$

Remark. In case $h = 1$, we have $\mathbf{m}\cdot\mathbf{v}(1,k_1) = 0$ only for $\mathbf{m} = \mathbf{0}$. Hence $q'_{k,e,\nu} = c_{\mathbf{0},e,\nu} \to \lambda_d(\Omega_e) = p_e = q_{k,e}$ as $\nu\to\infty$.

Set $\nu = [C\log\log N]$ for some constant C which satisfies $\left(\frac{\gamma}{\alpha}\right)^\nu \ll (\log N)^{-\lambda}$, choose δ such that $(\log N)^{-\frac{\delta}{2dh(dh-1)}} \ll \alpha^{-\nu}$ and τ_0 large enough. Then

$$\Sigma_1 = Nq'_{k_1,\ldots,k_h,e_1,\ldots,e_h,[C\log\log N]} + \mathcal{O}\left(N(\log N)^{-\lambda}\right).$$

For $P(n) = n$ and $(\log N)^\eta \le k_1,\ldots,k_h \le \log_\alpha N - (\log N)^\eta$, Lemma 3 implies

$$\Sigma_1 = Nq_{k_1,\ldots,k_h,e_1,\ldots,e_h} + \mathcal{O}\left(N(\log N)^{-\lambda}\right)$$

and therefore

$$q'_{k_1,\ldots,k_h,e_1,\ldots,e_h,[C\log\log N]} = q_{k_1,\ldots,k_h,e_1,\ldots,e_h} + \mathcal{O}\left((\log N)^{-\lambda}\right).$$

For $(\log N^r)^\eta \le k_1,\ldots,k_h \le \log_\alpha N^r - (\log N^r)^\eta$, we obtain this result by considering Σ_1 for $P(n) = n$ and N^r.

As already noted, we get the corresponding result for primes by the same arguments. This concludes the proofs of Lemma 8 and Theorem 3.

References

[1] S. Akiyama, Cubic pisot units with finite beta expansions, in: *Algebraic Number Theory and Diophantine Analysis (F. Halter-Koch and R. F. Tichy eds)*, de Gruyter, Berlin, 2000, 11–26.

[2] S. Akiyama, On the boundary of self affine tilings generated by Pisot numbers, J. Math. Soc. Japan, **54** (2) (2002), 283–308.

[3] N. L. Bassily and I. Kátai, Distribution of the values of q-additive functions on polynomial sequences, *Acta Math. Hung.* **68** (1995), 353–361.

[4] A. Brauer, On algebraic equations with all but one root in the interior of the unit circle, *Math. Nach.* **4** (1951) 250–257.

[5] R. L. Dobrušin, Central limit theorem for nonstationary Markov chains II, *Theory Prob. Applications* **1** (1956), 329–383. (Translated from: *Teor. Vareojatnost. i Primenen.* **1** (1956), 365–425.)

[6] M. Drmota, Irregularities of Distributions with Respect to Polytopes, *Mathematika* **43** (1996) 108–119.

[7] M. Drmota, The joint distribution of q-additive functions, *Acta Arith.* **100** (1) (2001), 17-39.

[8] M. Drmota and W. Steiner, The Zeckendorf expansion of polynomial sequences, submitted, http://www.geometrie.tuwien.ac.at/steiner/ze.ps.gz

[9] M. Drmota and R. F. Tichy, *Sequences, Discrepancies and Applications*, volume 1651 of *Lecture Notes in Mathematics*, Springer, Berlin, 1998.

[10] J. M. Dumont and A. Thomas, Systèmes de numération et fonctions fractales relatifs aux substitutions, *J. Theoret. Comput. Sci.* **65** (1989), 153–169.

[11] J. M. Dumont and A. Thomas, Gaussian asymptotic properties of the sum-of-digits functions, *J. Number Th.* **62** (1997), 19–38.

[12] C. Frougny and B. Solomyak, Finite β-expansions, *Ergodic Theory and Dynamical Systems* **12** (4) (1992), 713–723.

[13] N. Giesbrecht, Bounds for sums of random variables over a Markov chain, *Stochastic Process. Appl.* **53** (2) (1994), 269–283.

[14] W. J. Gilbert, Fractal dimension of sets derived from complex bases, *Ann. Sci. Math. Quebec* **11** (1) (1987), 65–77.

[15] B. Gittenberger and J. M. Thuswaldner, Asymptotic normality of b-additive functions on polynomial sequences in the gaussian number field, J. Number Theory **84** (2000), 317–341.

[16] P. Grabner and R. F. Tichy, α-Expansions, linear recurrences, and the sum-of-digits function, *manuscripta math.* **70** (1991), 311–324.

[17] L. K. Hua, *Additive Theory of Prime Numbers*, Translations of Mathematical Monographs Vol. 13, Am. Math. Soc., Providence, 1965.

[18] B. A. Lifšic, On the convergence of moments in the central limit theorem for non-homogeneous Markov chains, *Theory Prob. Applications* **20** (1975), 741-758. (Translated from: *Teor. Vareojatnost. i Primenen.* **20** (1975), 755-772.)

[19] A. Messaoudi, Autour du fractal de Rauzy, *Thèse de doctorat*, Université de la Méditerranée, 1996.

[20] A. Messaoudi, Propriétés arithmétiques et dynmamiques du fractal de Rauzy, *J. Théor. Nombres Bordx.* **10** (1998), 135–162.

[21] W. Parry, On the β-expansion of real numbers, *Acta Math. Acad. Sci. Hung.* **11** (1960), 401-416.

[22] B. Praggastis, Numeration Systems and Markov Partitions from Self Similar Tilings, *Trans. Am. Math. Soc.* **351** (8) (1999), 3315–3349.

[23] G. Rauzy, Nombres algébriques et substitutions, *Bull. Soc. Math. Fr.* **110** (2) (1982), 147–178.

[24] A. Rényi, Representations for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hung.* **8** (1957), 477–493.

[25] A. Siegel, Représentations géometrique, combinatoire et arithmétique des systèmes susbstitutifs de type Pisot, *Thèse de doctorat*, Université de la Méditerranée, 2000.