On strings of consecutive economical numbers of arbitrary length

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§1. Introduction

In 1995, Bernardo Recamán Santos [4] defined a number n to be *equidigital* if the prime factorization of n requires the same number of decimal digits as n, and *economical* if its prime factorization requires no more digits. He asked whether there are arbitrarily long sequences of consecutive economical numbers. In 1998, Richard Pinch [2] gave an affirmative answer to this question assuming the *prime k-tuple conjecture* stated by L.E. Dickson [1] in 1904. He also exhibited one such sequence of length nine starting with the 19-digit number 1034429177995381247 and conjectured that such a sequence of arbitrary length always exists.

In this paper, we give an unconditional proof of Pinch's conjecture – in fact, for any base $B \geq 2$ – and we prove other results concerning economical numbers.

§2. Preliminary results and notations

Let $B \geq 2$ be an integer. For any positive integer n whose factorization is $n = \prod_{p^{\alpha_p} \| n} p^{\alpha_p}$, we set

$$S_B(n) := \left\lfloor \frac{\log n}{\log B} \right\rfloor + 1 \qquad \text{and} \qquad T_B(n) := \sum_{p^{\alpha_p} \| n} \left(\left\lfloor \frac{\log p}{\log B} \right\rfloor + 1 \right) + \sum_{\substack{p^{\alpha_p} \| n \\ \alpha_p > 1}} \left(\left\lfloor \frac{\log \alpha_p}{\log B} \right\rfloor + 1 \right).$$

We then let $\mathcal{E}_B$ (resp. $\mathcal{E}'_B$) be the set of *economical numbers* in base B (resp. *stronlgy economical numbers* in base B), that is those positive integers n such that

$$S_B(n) \leq T_B(n) \qquad (\text{resp. } S_B(n) < T_B(n)).$$

Throughout this paper, $\omega(n)$ stands for the number of distinct prime factors of n. We shall write p_i to denote the i-th prime number. We also use the Vinogradov symbols $\gg$, $\ll$, $\asymp$ and the Landau symbols O and o with their usual meaning.

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Lemma 1. *For each integer* $n = \prod_{p^{\alpha_p} \| n} p^{\alpha_p} \geq 2$, *we have*

(i) $$S_B(n) - T_B(n) > \frac{1}{\log B} \cdot \log \left(\prod_{p^{\alpha_p} \| n} \frac{p^{\alpha_p - 1}}{\alpha_p} \right) - 2\omega(n),$$

(ii) $$\frac{p^{\alpha_p - 1}}{\alpha_p} \geq 1 \qquad (p \geq 2,\ \alpha_p \geq 1).$$

(iii) $$\frac{p^{\alpha_p - 1}}{\alpha_p} \geq \frac{1}{2} p^{\frac{\alpha_p - 1}{2}} \qquad (p \geq 2,\ \alpha_p \geq 1).$$

Proof. (i) follows from the two inequalities

$$S_B(n) > \frac{\log n}{\log B} \quad \text{and} \quad T_B(n) < 2\omega(n) + \sum_{p^{\alpha_p} \| n} \frac{\log(p\alpha_p)}{\log B}.$$

(ii) and (iii) are trivial.

Lemma 2. *Let n be a positive integer. Assume that there exist a prime q and a positive integer β such that $q^\beta | n$ with*

(1) $$q^{\beta - 1} > \beta B^{2\omega(n)}.$$

Then $n \in \mathcal{E}'_B$.

Proof. Using parts (ii) and (iii) of Lemma 1, and then (1), we get

(2) $$\prod_{p^{\alpha_p} \| n} \frac{p^{\alpha_p - 1}}{\alpha_p} \geq \frac{q^{\beta - 1}}{\beta} > B^{2\omega(n)}.$$

Hence, using part (i) of Lemma 1, we obtain that

$$S_B(n) - T_B(n) > \frac{1}{\log B} \cdot \log B^{2\omega(n)} - 2\omega(n) = 0,$$

thus completing the proof of Lemma 2.

Corollary. *Only a finite number of powerful numbers are not in* $\mathcal{E}'_B$.

Proof. It follows from (2) that if for a certain integer $n = \prod_{p^{\alpha_p} \| n} p^{\alpha_p} \geq 2$ we have

(3) $$\prod_{p^{\alpha_p} \| n} \frac{p^{\alpha_p - 1}}{\alpha_p} > B^{2\omega(n)},$$

then $n \in \mathcal{E}'_B$. Hence, observing that for any prime number p, the function $f(x) = \dfrac{p^{x-1}}{x}$ is increasing for all $x \geq 2$, it follows that if n is a powerful number, in order for (3) to hold, it is sufficient that

$$\prod_{p|n} \frac{p}{2} > B^{2\omega(n)},$$

that is

$$\prod_{p|n} p > (2B^2)^{\omega(n)}, \tag{4}$$

or similarly, by taking logarithms,

$$\sum_{p|n} \log p > \omega(n) \log(2B^2). \tag{5}$$

Since it follows from the Prime Number Theorem that

$$\sum_{p|n} \log p \geq \sum_{i=1}^{\omega(n)} \log p_i = (1 + o(1))\omega(n) \log \omega(n),$$

it is clear that (5) will hold provided $\omega(n) > C_1$, where C_1 is a constant depending only on B.

On the other hand, that is if $\omega(n) \leq C_1$ and if we set $C_2 := (2B^2)^{C_1}$ and let $C_3 > 1$ be such that $\frac{2^{C_3-1}}{C_3} > B^{2C_1}$, then there are three possibilities:

1. there exists a prime p dividing n such that $p > C_2$;
2. all primes p dividing n satisfy $p \leq C_2$ with corresponding $\alpha_p \leq C_3$;
3. there exists a prime q and a positive integer β such that $q^\beta | n$ with $\beta > C_3$.

In the first case, inequality (4) is satisfied anyway, so that in this case $n \in \mathcal{E}'_B$. In the second case, there can only exist a finite number of such powerful integers n, a case which fits the conclusion of the Corollary. Finally, in the third case, the conditions of Lemma 2 are fulfilled because

$$\frac{q^{\beta-1}}{\beta} \geq \frac{2^{\beta-1}}{\beta} > \frac{2^{C_3-1}}{C_3} > B^{2C_1} \geq B^{2\omega(n)},$$

in which case $n \in \mathcal{E}'_B$. The proof of the Corollary is thus complete.

§3. The main result

Theorem. *Given $\varepsilon > 0$, there exist infinitely many positive integers n such that $n + j \in \mathcal{E}'_B$ for each $j = 1, 2, \ldots, \ell$, where $\ell = \left\lfloor \dfrac{\log\log n}{(2+\varepsilon)\log B} \right\rfloor$.*

Proof. Let $\eta = \varepsilon/20$, $r = \lfloor \eta^{-1} \rfloor$. Pick a large number X and put

$$R = \left\lfloor (1+\eta) \frac{\log\log 2X}{(2+\varepsilon)\log B} \right\rfloor.$$

The number of r-th powers of primes between X and $2X$ is

$$\sim \frac{rX^{1/r}}{\log X}\left(2^{1/r}-1\right) > R$$

assuming that X is sufficiently large. Pick R of these prime powers: $p_1^r, \ldots, p_R^r$. The product of these prime powers, say P, lies between X^R and $(2X)^R$. By the Chinese Remainder Theorem, there is some positive integer $n \le P - R$ such that, for $j = 1, \ldots, R$, each number $n + j - 1$ is divisible by p_j^r. Now if $m = n + j - 1$, we have

$$\omega(m) \le (1+\eta)\frac{\log P}{\log\log P} < (1+\eta)\frac{R\log 2X}{\log\log 2X}.$$

Hence, it follows that

$$\log(p_j^{r-1}) \ge (1 - 1/r)\log X, \tag{6}$$

while

$$\log\left(rB^{2\omega(m)}\right) < \log r + (1+\eta)(\log B)\frac{2R\log 2X}{\log\log 2X}. \tag{7}$$

Comparing (6) and (7) gives (1) for all large X in view of our choice for R. Now

$$\log\log m \le \log\log 2X + \log R < (1+\eta)\log\log 2X$$

for all sufficiently large X, and this completes the proof of the Theorem.

§4. Numerical data

For each positive integer k, let $e(k)$ (resp. $e'(k)$) stand for the smallest integer n such that $n + i \in \mathcal{E}_{10}$ (resp. $n + i \in \mathcal{E}'_{10}$) for $0 \le i \le k - 1$.

A computer search allows one to obtain the following tables:

k	2	3	4	5	6	7	8	9	10
$e(k)$	1	1	13	13	157	157	1169312	10990399	1016258233

The above value of $e(10)$ provides a much smaller number than the 19-digit number obtained by Pinch (see section 1), and furthermore it leads to a longer string of consecutive economical numbers.

k	2	3	4
$e'(k)$	4374	1097873	179210312

Moreover, again using a basic computer search, one can check that $e(k) > 5\times 10^9$ for $k \ge 11$ and that $e'(k) > 5\times 10^9$ for $k \ge 5$. Hence, in order to find longer strings of consecutive economical (or of strongly economical) numbers, one needs to try another method. For instance, using the idea of the proof given in section 3, one can find large strings, say up to $k = 12$ in the case of economical numbers and at least up to $k = 10$ in the case of strongly economical numbers. For instance, in order to find a string of 10 consecutive elements of $\mathcal{E}'_{10}$, we consider a set of consecutive integers $n + j - 1$, $j = 1, \ldots, 10$, each divisible by p_j^4, where the p_j's are 10 numbers picked at random amongst the primes $11, 13, \ldots, 43$. Doing so, we find that for the 55-digit number $n_0 = 1187615078125922863258960810793892104104920690716348821$, we have $n_0 + i \in \mathcal{E}'_{10}$ for $i = 1, 2, \ldots, 10$. Clearly, the exact value of $e'(10)$ should be much smaller than n_0.

Proceeding in a similar manner, one finds that:

- with $n = 13893190253813562840755283778863436828514163286$, the numbers $n + i$, with $i = 1, 2, \ldots, 11$, are all in $\mathcal{E}_{10}$;
- with $n = 12800357478746692178414328391814503664216763232307 1$, the numbers $n + i$, with $i = 1, 2, \ldots, 12$, are all in $\mathcal{E}_{10}$.

Clearly, each of these two numbers is not the smallest with the given property, and it would be interesting to identify the exact value of $e(k)$ for any given $k \geq 11$ and similarly for $e'(k)$ with $k \geq 5$.

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