A CLASS OF QUADRINOMIAL GARSIA NUMBERS

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Abstract

Real algebraic integers larger than 1 whose minimal polynomials are certain quadrinomials of degree at least 5 with constant term ± 2 and all roots outside the closed unit disk are determined and some of their properties are mentioned.

1. Introduction

It has been customary to describe some important kinds of algebraic integers by specifying the sizes of their conjugates, e.g., Pisot, Salem or Perron numbers. Here we are concerned with Garsia numbers[1] which can be defined as follows: An algebraic integer is called a Garsia number if it is real and larger than one, if all conjugates have modulus larger than one and if its norm has modulus two.

Examples of Garsia numbers are given in [6, 5], and exhaustive calculations were performed by A. Kovács [7] and by P. Burcsi and A. Kovács [4]. A characterization of totally real and trinomial Garsia numbers and a list of Garsia numbers of degrees up to four can be found in [3]. The author is indebted to Ch. van de Woestijne for informing him on his extensive calculations of expanding and Pisot polynomials [10].

Here we deal with quadrinomial Garsia numbers, and in view of the aforementioned results we restrict to degrees at least five. More specifically, we classify monic quadrinomials with symmetric exponent sequence which are minimal polynomials of Garsia numbers (see Theorem 1 for details), and we specify their signatures. It turns out that these Garsia numbers are rather small, and we exploit this fact to generate a dense subset of the reals (see Proposition 2). Further we show that certain other polynomials cannot have Garsia numbers among its roots (see Theorem 3 and Lemma 7).

[1] The name of these particular numbers seems to have first appeared in a publication of D.-J. Feng [5] and used by the present author in the spelling Garcia [3]. However, D. Boyd [2] pointed out that these numbers were first considered by A. Garsia [6].

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2. Certain Quadrinomial Garsia Numbers

We select a class of certain quadrinomial Garsia numbers which can easily be dealt with because their minimal polynomials are closely linked to trinomials. We first state our main result and postpone its proof to Section 3.

Theorem 1 Let $n, k \in \mathbb{N},\ n \geq 5,\ 0 < 2k < n$ and $b, c, d \in \mathbb{Z} \setminus \{0\}$. Then the following statements hold.

(i) The polynomial

$$X^n + bX^{n-k} + cX^k + d \tag{1}$$

is the minimal polynomial of a Garsia number if and only if $bc = -1$ and $d = -2$.

(ii) Let γ be a Garsia number with minimal polynomial

$$X^n - X^{n-k} + X^k - 2 \qquad \text{or} \qquad X^n + X^{n-k} - X^k - 2\,. \tag{2}$$

Then

$$\gamma < 2^{1/\max\{2,k\}} \qquad \text{and} \qquad \overline{|\gamma|} < \sqrt{2}\,.$$

Moreover, if n is odd then all other conjugates of γ are nonreal, and if n is even then γ has exactly one other real conjugate, and this conjugate is negative.

Remark Let γ_k be the Garsia number with minimal polynomial $X^{2k+1} - X^{k+1} + X^k - 2$. Then our theorem yields $\lim_{k\to\infty} \gamma_k = 1$, thereby illustrating the known fact that the set of all Garsia numbers is not closed.

We mention that for every Garsia number γ described by a polynomial of type (2) the set

$$\left\{ \sum_{i=0}^{n} \varepsilon_i \gamma^i \ :\ \varepsilon_0, \ldots, \varepsilon_n \in \{1, -1\}\,,\ n \in \mathbb{N} \right\} \tag{3}$$

is dense in $\mathbb{R}$ as the following Proposition shows.

Proposition 2 Let γ be a Garsia number. If $\gamma < \sqrt{2}$ then the set (3) is a dense subset of the reals.

Proof. As the norm of γ has modulus 2 there does not exist a polynomial

$$f \in \{-1, 0, 1\}\,[X] \setminus \{0\}$$

with $f(\gamma^2) = 0$. Now, [8, Proposition 1.2] implies our statement. □

As a byproduct and illustration of our method we deduce the following result on Garsia numbers.

Theorem 3 A Garsia number cannot have a minimal polynomial of the form

$$X^n + cX^{n-k} + bX^{n-m} + bdX^m + cdX^k + d \qquad (0 < k < m < n, \quad b, c, d \in \mathbb{Z} \setminus \{0\}).$$

Proof. Let $|d| \geq 2$. We show more generally that a polynomial of the given form has a root inside the closed unit disk. In view of Cohn's Rule 1 [9, Theorem 11.5.3] it suffices to prove that the polynomial

$$F = d(X^n + cX^{n-k} + bX^{n-m} + bdX^m + cdX^k + d) \\ -(1 + cX^k + bX^m + bdX^{n-m} + cdX^{n-k} + dX^n)$$

has a root inside the closed unit disk. Now, $F = (d^2 - 1) \cdot f$, where

$$f = bX^m + cX^k + 1$$

does indeed have a root of modulus at most 1: This is trivial for $m + k < 4$, and for $m + k \geq 4$ we apply Bohl's method [1] as follows.

`Case 1.` $|c| = |b|$. We look at the triangle with side lengths $|b|, |b|, 1$ and denote by β the size of the angle opposite a side of length $|b|$, thus $\cos\beta = 1/(2|b|)$. The length of the interval

$$\left(\frac{m(\arg(c)+\pi) - k(\arg(b)+\pi)}{2\pi} - \frac{(m+k)\beta}{2\pi}, \ \frac{m(\arg(c)+\pi) - k(\arg(b)+\pi)}{2\pi} + \frac{(m+k)\beta}{2\pi}\right)$$

exceeds 1 because

$$2 \cdot \frac{(m+k)\beta}{2\pi} = \frac{m+k}{\pi} \arccos \frac{1}{2|b|} \geq \frac{m+k}{\pi} \arccos \frac{1}{2} > \frac{m+k}{\pi} \geq \frac{4}{\pi} > 1,$$

hence this interval contains at least one integer. Thus by the first part of [1] the trinomial f has a root inside the unit disk.

`Case 2.` $|c| < |b|$. Now, by the second part of [1] the number of roots of f inside the unit disk is positive.

`Case 3.` $|c| > |b|$. Again by the second part of [1] f has at least one root inside the unit disk. Note that also for $k = 1$ no exceptional case in the sense of Bohl occurs because otherwise we would have $|c| = |b| + 1$ and

$$1 \leq \frac{k|c|}{m|b|} = \frac{|b|+1}{m|b|}$$

which would imply the contradiction $3|b| \leq m|b| \leq |b| + 1$. □

3. Proof of Theorem 1

We start with some preparation. Here our first lemma plays a central role: In its first part the connection between trinomials and the polynomials defined above is exhibited.

Lemma 4 Let $n, k \in \mathbb{N},\ n \geq 5,\ 0 < 2k < n,\ b, c, d \in \mathbb{Z} \setminus \{0\},\ \ |d| \geq 2$ and $P = X^n + bX^{n-k} + cX^k + d$. Set

$$A = bd - c,\ B = cd - b\ . \tag{4}$$

(i) P has a zero inside the closed unit disk if and only if the polynomial

$$F = AX^{n-k} + BX^k + d^2 - 1 \tag{5}$$

has a zero inside the closed unit disk.

(ii) Let k and n be coprime, $|d| = 2$ and assume that P does not vanish inside the closed unit disk. Then we have the following statements.

(a) $0 < |A| < |B| + 3$ and $0 < |B| < |A| + 3$.

(b) If $|A| + |B| > 3$ then $\max\{\sigma, \tau\} > 4/5$ where we set

$$\sigma = \frac{c^2 - b^2 + 3}{2\,|cd - b|} \quad \text{and} \quad \tau = \frac{b^2 - c^2 + 3}{2\,|bd - c|}\ . \tag{6}$$

Proof. (i) This follows from Cohn's Rule 1 [9, Theorem 11.5.3].
(ii) If $A = 0$ then $F = 3(bX^k + 1)$ which has a root inside the closed unit disk contradicting (i). Thus $A \neq 0$, and similarly we deduce $B \neq 0$.

Assume $|A| \geq |B| + 3$. By the second part of [1] F has at least one root inside the unit disk. Note that for the same reason the exceptional case in the sense of Bohl cannot occur. Thus we have shown $0 < |A| < |B| + 3$.

Again by the second part of [1] we cannot have $|B| > |A| + 3$ because then F would have k roots inside the unit disk. Let us assume $|B| = |A| + 3$. By the second part of [1] this would imply $k = 1$ and

$$1 \leq \frac{k\,|B|}{(n-k)\,|A|} = \frac{|A| + 3}{(n-1)\,|A|}\ .$$

As $n \geq 5$ this would yield $|A| = 1, n = 5$ and then $|B| = 4$, hence F would have a root in the interval $[-1, 1]$ which we excluded.

Therefore we must have $|B| < |A| + 3$.

We easily check

$$\sigma = \frac{9 - |A|^2 + |B|^2}{6\,|B|} \quad \text{and} \quad \tau = \frac{9 - |B|^2 + |A|^2}{6\,|A|}\ .$$

We have $|A|^2 < (|B|+3)^2$, hence $-6|B| < 9 - |A|^2 + |B|^2$ and therefore $-1 < \sigma$. Further, $-|A| < 3 - |B| < |A|$ implies $\sigma < 1$. Analogously we see $\tau \in (-1, 1)$.

From now on we assume $|A| + |B| > 3$. It suffices to establish

$$(n-k)\arccos\sigma + k\arccos\tau \le \pi \ , \tag{7}$$

because then the assumption $\max\{\sigma, \tau\} \le 4/5$ would yield the contradiction

$$\pi \ge (n-k)\arccos\frac{4}{5} + k\arccos\frac{4}{5} = n\arccos\frac{4}{5} \ge 5\arccos\frac{4}{5} > 3.2\ .$$

We look at the (non-degenerate) triangle with side lengths $|A|, |B|$ and 3 and denote by α (β, resp.) the size of the angle opposite the side of length $|A|$ ($|B|$, resp.). Elementary considerations yield

$$\cos\alpha = \sigma \quad \text{and} \quad \cos\beta = \tau\ .$$

If (7) would not hold then the length of the open interval I (cf. the proof of Theorem 3) defined by

$$\frac{(n-k)(\arg(B)+\pi) - k(\arg(A)+\pi)}{2\pi} \pm \frac{(n-k)\alpha + k\beta}{2\pi}$$

would exceed 1 because

$$2 \cdot \frac{(n-k)\alpha + k\beta}{2\pi} = \frac{1}{\pi}\Big((n-k)\arccos\sigma + k\arccos\tau\Big) > \frac{1}{\pi}\cdot\pi = 1\ ,$$

hence I would contain at least one integer which contradicts the first part of [1]. □

The proof of the next lemma is obvious.

Lemma 5 Let β be a nonreal conjugate of a Garsia number. Then $|\beta| < \sqrt{2}$.

We denote by $\mathcal{P}_n$ the set of minimal polynomials of Garsia numbers of degree n.

Lemma 6 Let d be the greatest common divisor of $i_1, \ldots, i_m \in \mathbb{N},\ 0 < i_1 < \ldots < i_m = n$ and $a_0, a_{i_1}, \ldots, a_{i_m} \in \mathbb{Z},\ a_n = 1$. Then $g = \sum_{k=1}^{m} a_{i_k} X^{i_k/d} + a_0 \in \mathcal{P}_{n/d}$ if and only if $f = \sum_{k=1}^{m} a_{i_k} X^{i_k} + a_0 \in \mathcal{P}_n$.

Proof. If $f \in \mathcal{P}_n$ then $g \in \mathcal{P}_{n/d}$ by [3, Proposition 2.8 (i)]. Conversely, if $g \in \mathcal{P}_{n/d}$ then

$$f(X) = g(X^d) \in \mathcal{P}_{d\cdot(n/d)} = \mathcal{P}_n$$

by [3, Proposition 2.8 (ii)]. □

Our next result shows that Garsia quadrinomials often admit only negative constant terms.

Lemma 7 Let $k, m, n \in \mathbb{N},\ 0 < k < m < n,\ n \ge m+k$ and $b, c \in \{-1, 1\}$. Then

$$f = X^n + bX^m + cX^k + 2 \notin \mathcal{P}_n.$$

Proof. Let us assume $f \in \mathcal{P}_n$. Then f has a real positive root, hence $b < 0$ or $c < 0$.

Let $b > 0$. Then $f = X^n + X^m - X^k + 2$, $f(1) = 3$ and $f'(t) > 0$ for $t \geq 1$, hence f does not have a root larger than 1: Contradiction.

Now, let $b < 0$. Then $f(1) = c + 2 > 0$ and f is increasing on the interval $[1, \infty)$ which yields a contradiction as above. $\square$

Now we turn to the proof of Theorem 1 and let P (F, resp.) be the polynomial given by (1) ((5), resp.), A, B be defined by (4) and σ, τ by (6).

First, let $bc = -1$ and $d = -2$. Take $b = -1$, hence $c = A = 1$ and $B = -1$. The polynomial $F = X^{n-k} - X^k + 3$ obviously has no root inside the closed unit disk, thus the same holds for P by Lemma 4 (i). Therefore, P is irreducible (see for instance [3, Lemma 2.3]). Using

$$P' = X^{k-1}\Big(X^{n-2k}\big(nX^k - (n-k)\big) + k\Big) \tag{8}$$

we see that P is strictly increasing on the interval $[1, \infty)$. It has exactly one positive root γ, and this root lies in the interval $(1, \sqrt{2})$ because $P(1) < 0$ and $P(\sqrt{2}) > 0$ in view of $n \geq 5$. Thus we have shown $P \in \mathcal{P}_n$ with largest real root $\gamma < \sqrt{2}$. We check $P(2^{1/k}) > 0$, hence we also have $\gamma < 2^{1/k}$.

Now we turn the the behavior of P on the negative real axis. Clearly, $P(-1) < 0$.

Let n be odd. By (8) we have $P'(t) > 0$ for $t \leq -1$, hence P is strictly increasing on $(-\infty, -1]$, thus it does not vanish in this interval. Together with Lemma 5 we thus have shown $\overline{|\gamma|} < \sqrt{2}$ in this case.

Now let n be even. Analogously as above we find that P is strictly decreasing on the interval $(-\infty, -1]$, thus P has exactly one negative real root. We check $P(-\sqrt{2}) > 0$, hence this root is larger than $-\sqrt{2}$, and again we find $\overline{|\gamma|} < \sqrt{2}$.

The case $b = 1$ can be treated analogously, and we omit the details here.

Second, we let $P \in \mathcal{P}_n$. In view of Lemma 6 we may restrict to the case that k and n are coprime.

We first show that d is negative. Let us suppose to the contrary that $d = 2$, hence

$$s = b + c \geq -2 \tag{9}$$

by [3, Lemma 2.6 (vii)].

`Case 1.` $b > 0$. As P has a positive root we then must have $c < 0$. Therefore $A \geq 3, B \leq -3$, $|A| + |B| > 3$ and $-2 \leq s \leq 2$ where the right inequality follows from Lemma 4 (ii). Our assertion is proved by showing that Lemma 4 (ii) is violated in all five cases: If $s = -2$ then

$$\sigma = \frac{4b+7}{2(3b+4)} < \frac{4}{5}, \qquad \tau = \frac{-4b-1}{2(3b+2)} < 0,$$

and similarly for $s = -1, 0$. If $s = 1$ then $b \geq 2$ and

$$\sigma = \frac{2-b}{3b-2} \leq 0, \qquad \tau = \frac{b+1}{3b-1} < \frac{4}{5},$$

and finally for $s = 2$ we have $b \geq 3$ and $\sigma < 0,\ \tau < 4/5$.

`Case 2.` $b < 0$. Then $c \geq -1$ by (9). The assumption $c = -1$ would yield $A < 0$ and by Lemma 4 (ii) $B = -1$ or $B > 0$. But $B = -1$ would imply $b = -1$ contradicting Lemma 7, and $B > 0$ would yield $b \leq -3$ on the one hand and $|A| < |B| + 3$ would imply $b > -2$ on the other hand. Therefore $c \geq 1$ and our assertion can be proved analogously as above.

Thus we have established $d = -2$. By [3, Lemma 2.6 (iv), (vi)] we have

$$|b| \leq |c| + 3, \quad |c| \leq |b| + 3, \quad b \leq -c\,. \tag{10}$$

`Case 1.` $b > 0$. We then have $c \leq -1, B > 0$, and by Lemma 4 (ii) $A < 0$ and $B < -A + 3$. Using the same Lemma we find

$$-2c - b = |B| \leq |A| + 2 = 2b + c + 2,$$

hence $-c < b + 1$ and therefore $b = -c$ by (10). Again by Lemma 4 (ii) we have $b = 1$ because otherwise $|A| + B = 2b > 3$ and

$$\sigma = \tau = \frac{3}{2b} < \frac{4}{5}\,.$$

`Case 2.` $b < 0$. We first show that c must be positive. Assume to the contrary that $c \leq 0$. Then $A, B \geq 3$, hence $A + B > 3$ and $-c \leq -b + 2$ by Lemma 4 (ii). But then

$$\left(-c - \frac{8}{5}\right)^2 < \left(-b + \frac{4}{5}\right)^2 - \frac{27}{25}$$

which is equivalent to $\sigma < 4/5$. Similarly we find $\tau < 4/5$ which contradicts Lemma 4 (ii).

Therefore, from now on we have

$$1 \leq c \leq 3 - b, \quad -b \leq c + 3 \tag{11}$$

by (10).

`Case 2.1.` $b = -1$. Then (11) and Lemma 4 imply $c = 1$.

`Case 2.2.` $b = -2$. Then (11) and Lemma 4 exclude the remaining cases $c \in \{2, 3, 4, 5\}$.

`Case 2.3.` $b < -2$. We continue to use (11) and Lemma 4 repeatedly: We first find $A > 0$ and $B < 0$, then reduce to $-5 \leq b \leq -3$ and finally exclude these three remaining cases.

The proof of Theorem 1 is now completed.

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