MORE ON POINTS AND ARCS

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Abstract

Let $z_1, \dots, z_N$ be complex numbers, situated on the unit circle $|z| = 1$ so that any open arc of length $\varphi \in (0, \pi]$ of the circle contains at most n of them. Write $S := z_1 + \dots + z_N$. Complementing our earlier result, we show that

$$|S| \le n \, \frac{\sin(\varphi N/2n)}{\sin(\varphi/2)}.$$

Consequently, given that $|S| \ge \alpha N$ with $\alpha \in (0, 1]$, there exists an open arc of length φ containing at least

$$\frac{\varphi/2}{g^{-1}(\alpha g(\varphi/2))} \, N$$

of the numbers $z_1, \dots, z_N$; here $g(x) = \sin x/x$ and g^{-1} is the function, inverse to g on the interval $0 < x \le \pi$.

Let $\mathbb{U} := \{z \in \mathbb{C} \colon |z| = 1\}$ and suppose that for integers $N \ge n \ge 1$ and real $\varphi \in (0, \pi]$, the numbers $z_1, \dots, z_N \in \mathbb{U}$ have the property that any open arc of $\mathbb{U}$ of length φ contains at most n of them. Extending a well-known lemma of Freiman [F62, Lemma 1], we showed in [L05] that, writing $S := z_1 + \dots + z_N$, one has

$$|S| \le 2n - N + 2(N - n)\cos(\varphi/2); \tag{1}$$

thus, if $|S| \ge \alpha N$ with $\alpha \in [0, 1]$, then there is an open arc of $\mathbb{U}$ of length φ containing at least

$$\frac{\alpha + 1 - 2\cos(\varphi/2)}{2(1 - \cos(\varphi/2))} \, N \tag{2}$$

of the numbers $z_1, \dots, z_N$. Estimate (1) is sharp in the range $N/2 \le n \le N$: equality is attained, for instance, if $2n - N$ of the numbers $z_1, \dots, z_N$ equal 1, and the remaining $2N - 2n$ of them are evenly split between $\exp(i\varphi/2)$ and its conjugate $\exp(-i\varphi/2)$. Accordingly, the bound (2) is sharp if $\alpha \ge \cos(\varphi/2)$. Indeed, if in this case n is the smallest integer, greater than or equal to the expression in (2), then $N/2 \le n \le N$ and the configuration just described provides an example of $z_1, \dots, z_N$ with $|S| \ge \alpha N$ (as it follows from a brief computation) and no open arc of length φ containing more than n of the numbers $z_1, \dots, z_N$.

On the other hand, (1) and (2) can be far from sharp if $n < N/2$ and $\alpha < \cos(\varphi/2)$, respectively. For instance, straightforward averaging shows that there is an arc of length φ, containing at least $(\varphi/2\pi)N$ of $z_1, \dots, z_N$. This nearly trivial bound is better, than (2), if

$$\alpha < 1 - (2 - \varphi/\pi)(1 - \cos(\varphi/2));$$

that is, when both φ and α are small. Below we establish an estimate which remains reasonably sharp for small values of n and α and, in particular, is better than the trivial estimate for the whole range of parameters.

Theorem 1. *Let N and n be positive integers and let $\varphi \in (0, \pi]$. Suppose that the numbers $z_1, \dots, z_N \in \mathbb{U}$ have the property that any open arc of $\mathbb{U}$ of length φ contains at most n of them. Then, writing $S := z_1 + \cdots + z_N$, we have*

$$|S| \le n \, \frac{\sin(\varphi N/2n)}{\sin(\varphi/2)}.$$

For the rest of the note, we write $g(x) := \sin x/x$ and denote by g^{-1} the function, inverse to g on the interval $0 < x \le \pi$. Notice, that g^{-1} is defined and monotonically decreasing on $[0, 1)$.

Corollary 1. *Let N be a positive integer and let $\varphi \in (0, \pi]$. Suppose that $z_1, \dots, z_N \in \mathbb{U}$ and write $S := z_1 + \cdots + z_N$. If $|S| \ge \alpha N$ with $\alpha \in (0, 1]$, then there is an open arc of $\mathbb{U}$ of length φ containing at least*

$$\frac{\varphi/2}{g^{-1}(\alpha g(\varphi/2))} \, N$$

of the numbers $z_1, \dots, z_N$.

To deduce Corollary 1 from Theorem 1 observe that if $N, \varphi, z_1, \dots, z_N, S$, and α are as in the corollary, and if n is the largest number of points among $z_1, \dots, z_N$ on an open arc of length φ, then $\alpha N \le n \, \frac{\sin(\varphi N/2n)}{\sin(\varphi/2)}$ by Theorem 1, whence $\alpha \, \frac{\sin(\varphi/2)}{\varphi/2} \le \frac{\sin(\varphi N/2n)}{\varphi N/2n}$. Equivalently, $\alpha g(\varphi/2) \le g(\varphi N/2n)$, and the assertion follows by applying g^{-1} to both sides.

Note that the bound of Corollary 1 is attained if $\alpha = \sin(d\varphi/2)/(d \sin(\varphi/2))$, where $1 \le d \le 2\pi/\varphi$ is an integer and N is divisible by d. For, set in this case $n := N/d$ and consider a d-term geometric progression with the ratio $\exp(i\varphi)$, situated on $\mathbb{U}$. Placing exactly n points at each term of this progression, we obtain a system of N complex numbers such that no open arc of $\mathbb{U}$ of length φ contains more than $n = (\varphi/2)N/g^{-1}(\alpha g(\varphi/2))$ of them, while their sum equals αN in absolute value.

Finally, we notice that the bound of Corollary 1 is better than (2) for all α and φ such that $\alpha \le \cos(\varphi/2)$; we omit the (rather tedious) verification.

The remainder of the note is devoted to the proof of Theorem 1. We start with a lemma.

Lemma 1. *Suppose that the function $f \in L^1[-\pi, \pi]$ attains values in the interval $[0, 1]$. If $\int_{-\pi}^{\pi} f(\theta) \, d\theta = 2c$ (with a real c), then*

$$\int_{-\pi}^{\pi} f(\theta) \cos\theta \, d\theta \le 2 \sin c.$$

Proof. Let I_c denote the indicator function of the interval $[-c, c]$. Then

$$f(\theta)(\cos\theta - \cos c) \le I_c(\theta)(\cos\theta - \cos c)$$

for all $\theta \in [-\pi, \pi]$, and it follows that

$$\begin{aligned}\int_{-\pi}^{\pi} f(\theta)\cos\theta\, d\theta &= \int_{-\pi}^{\pi} f(\theta)(\cos\theta - \cos c)\, d\theta + 2c\cos c \\ &\le \int_{-c}^{c} (\cos\theta - \cos c)\, d\theta + 2c\cos c \\ &= 2\sin c.\end{aligned}$$

□

Remark. The estimate of Lemma 1 is attained for $f = I_c$. Thus, what the lemma actually says is that for a given value of $\int_{-\pi}^{\pi} f(\theta)\, d\theta$, the integral $\int_{-\pi}^{\pi} f(\theta)\cos\theta\, d\theta$ is maximized if f is concentrated around 0 (where $\cos\theta$ is maximal).

Proof of Theorem 1. Without loss of generality we can assume that S is real.

For $\theta \in [-\pi, \pi]$, let $K(\theta)$ denote the number of those indices $j \in [1, N]$ such that there is a value of $\arg z_j$ which is within less than $\varphi/2$ from θ; with a little abuse of notation, we can write

$$K(\theta) := \#\{j \in [1, N] \colon |\arg z_j - \theta| < \varphi/2\}.$$

Notice that $K(\theta)$ is piecewise continuous and attains values in $[0, n]$. Furthermore, it is readily verified that

$$\int_{-\pi}^{\pi} K(\theta)\, d\theta = \varphi N,$$

and applying Lemma 1 to the function $f(\theta) := K(\theta)/n$ we conclude that

$$\int_{-\pi}^{\pi} K(\theta)\cos\theta\, d\theta \le 2n \sin\frac{\varphi N}{2n}.$$

To complete the proof we observe that the integral in the left-hand side is

$$\Re\Big(\sum_{j=1}^{N} \int_{\arg z_j - \varphi/2}^{\arg z_j + \varphi/2} \exp(i\theta)\, d\theta\Big) = \Re\Big(\sum_{j=1}^{N} z_j \cdot 2\sin(\varphi/2)\Big) = 2S\sin(\varphi/2).$$

□

References

[F62] G.A. Freiman, Inverse problems of additive number theory, VII. On addition of finite sets, IV, *Izv. Vyss. Ucebn. Zaved. Matematika* **6** (31) (1962), 131–144.

[L05] V.F. Lev, Distribution of points on arcs, *INTEGERS: The Electronic Journal of Combinatorial Number Theory* **5**(2) (2005), A11, 8 pp. (electronic).